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AN ANALOGUE OF THE PALEY-WIENER THEOREM FOR CERTAIN FUNCTION SPACES ON $SL(2, C)$

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The classical theorem of Paley-Wiener is concerned with characterizing Fourier transforms of C^∞ functions of compact support on the real line. It states that an entire holomorphic function F is the Fourier-Laplace transform of a C^∞ function on the real line $\boldsymbol{R}$ with support in $|x| \leqq R$ it and only if for given integer m, there exists a constant C_m such that

$$(1) \qquad |F(\xi + i\eta)| \leqq C_m(1 + |\xi + i\eta|)^{-m} \exp R|\eta| , \qquad \xi, \eta \in \boldsymbol{R} .$$

The purpose of this paper is to prove an analogue of this theorem for certain convolution subalgebras of C^∞ functions with compact support on the group $SL(2, \boldsymbol{C})$, by using Fourier transform involving elementary spherical functions of general type δ.

These subalgebras have been defined on locally compact group by R. Godement [4], in order to study the spherical trace function, cf. also G. Warner [8]. On this special group mentioned, by use the differential equations satisfied by the spherical functions, we derive a parametrization of such functions. These are in turn utilized to prove the Paley-Wiener theorem.

The analogous question on symmetric space of noncompact type was considered by S. Helgason [5] and R. Gangolli [3]. L. Ehrenpreis and F. I. Mautner [2] studied the Fourier transform on the group $SL(2, \boldsymbol{R})$ in detail, and theorem of the same kind was proved there. Results of this sort involving spherical functions of general type δ on some other groups have also been investigated, see e.g. Y. Shimizu [7].

2. **Preliminaries.** Throughout this paper, let G denote the complex semisimple Lie group $SL(2, C)$ and let K denote the maximal compact subgroup consisting of all unitary matrices in G. A basis of the real Lie algebra g_0 of G consists of

$$(2) \qquad \begin{aligned} R_1 &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad R_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad R_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ S_1 &= \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \end{aligned}$$

The set $\{R_1, R_2, R_3\}$ also forms a basis of the Lie algebra k_0 of K.

Elements of g_0 are viewed as left invariant vector fields on G, which generates the algebra $\mathfrak{G}$ of all left invariant differential operators on G. Let $a_{p_0} = \{tS_3: t \in \boldsymbol{R}\}$. The root system for (g_0, a_{p_0}) consists of $\{\rho, -\rho\}$, where $\rho(S_3) = 1$, and each has multiplicity two. Let $N_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $N_2 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$ and let $\mathfrak{n}_0$ be the subspace of g_0 spanned by $\{N_1, N_2\}$, then $\mathfrak{n}_0$ is the root space for ρ. Let $N = \exp \mathfrak{n}_0$ and $A_p = \{a_t = \exp tS_3: t \in \boldsymbol{R}\}$. Then $g_0 = k_0 + a_{p_0} + \mathfrak{n}_0$ and $G = KA_pN$ (Iwasawa decomposition). It is also known that $G = KA_p^+K$, $A_p^+ = \{a_t: t \geqq 0\}$. The Haar measure on G is normalized so that

$$\int_G f(x)dx = \int_K \int_{A_p} \int_N f(ka_tn)e^{2t}dkdtdn\ , \qquad f \in C_c(G)\ , \tag{3}$$

where dk is the normalized Haar measure on K, dt is the Lebesgue measure on $\boldsymbol{R}$ and $dn = d\xi_1 d\xi_2$ if $n = \exp(\xi_1 N_1 + \xi_2 N_2)$, is the Lebesgue measure on $\boldsymbol{R}^2$. Let $k \in K$, we can write $k = u_{\varphi_1} v_\theta u_{\varphi_2}$ with $u_\varphi = \exp \varphi R_3$, $v_\theta = \exp \theta R$, and $0 \leqq \varphi_1 \leqq 2\pi$, $0 \leqq \varphi_2 \leqq 4\pi$. Then

$$\int_K f(k)dk = \frac{1}{16\pi^2} \int_{\varphi_1=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{\varphi_2=0}^{4\pi} f(u_{\varphi_1} v_\theta u_{\varphi_2}) \sin\theta d\varphi_1 d\theta d\varphi_2\ , \qquad f \in C(K)\ . \tag{4}$$

For each nonnegative integer or half integer s, let D^s be the unique (up to equivalence) irreducible unitary representation of K on a $2s+1$ dimensional Hilbert space E_s. We can choose a basis $\{v_{-s}, v_{-s+1}, \cdots, v_s\}$ of E_s so that the matrix $(D^s_{j,q}(k))$, $j, q = -s, -s+1, \cdots, s$ has the following expression [see e.g. 6, p. 129].

$$\begin{aligned} D^s_{j,q}(u_\varphi) &= \delta_{j,q} e^{-iq\varphi} \\ D^s_{j,q}(v_\theta) &= (-1)^{j-q}\left(\frac{(s+j)!\,(s-j)!}{(s+q)!\,(s-q)!}\right)^{1/2} \\ &\quad \times \sum_{r=\max\{0,q-j\}}^{\min\{s-j,s+q\}} (-1)^r \binom{s+q}{r}\binom{s-q}{s-j-r} \cdot \\ &\quad \cos^{2s-j+q-2s}\frac{\theta}{2} \sin^{j-q+2r}\frac{\theta}{2}\ . \end{aligned} \tag{5}$$

The infinitesimal form for D^s has

$$\begin{aligned} D^s(R_1)v_j &= \frac{1}{2}(s+j)v_{j-1} - \frac{1}{2}(s-j)v_{j+1} \\ D^s(R_2)v_j &= \frac{1}{2}(s+j)v_{j-1} + \frac{1}{2}(s-j)v_{j+1} \\ D^s(R_3)v_j &= -ijv_j\ . \end{aligned} \tag{6}$$

Hence $D^s(R_1^2 + R_2^2 + R_3^2) = -s(s+1)I$.

Let $M = \{u_\theta = \exp \theta S_3 : \theta \in \boldsymbol{R}\}$. Then M is the centralizer of A_p in K, also it is a maximal torus in K. The set $\hat{M}$ of all characters of M is parametrized by half integers, i.e., for each p with $2p$ an integer, $u_\theta \to e^{-ip\theta}$ gives a character of M. Let $p \in \hat{M}$, and let $E^p = \{f \in L^2(K): f(ku_\theta) = e^{ip\theta}f(k),\ k \in K$ and $u_\theta \in M\}$, with $\|f^2\| = \int_K |f(k)|^2\, dk$. Let λ be a complex number and given $x \in G$, define $U^{p,\lambda}(x)$ by the prescription

$$(U^{p,\lambda}(x)f)(k) = \exp\left(-(i\lambda + 1)\rho(H(x^{-1}k)\right)f(k(x^{-1}k)) , \qquad f \in E^p \tag{7}$$

where $x = \kappa(x) \cdot \exp H(x) \cdot n(x)$ is the Iwasawa decomposition for x. Then $U^{p,\lambda}$ defines a continuous representation of G on the Banach space E^p, and every *TCI* Banach representation of G is equivalent to a subquotient of $U^{p,\lambda}$ for some p, λ. The restriction of $U^{p,\lambda}$ to K is just the unitary representation of K induced from the character $u_\theta \to e^{ip\theta}$ of M, hence D^s occurs in $U^{p,\lambda}$ exactly once if and only if $s = |p| + q$ for some nonnegative integer q.

$U^{p,\lambda}$ is unitary if λ is real, which constitutes the principal series representation induced from the characters of the group MA_pN. Define

$$U^{p,\lambda}(f) = \int_G f(x)U^{p,\lambda}(x)dx , \qquad f \in C_c^\infty(G) .$$

Then $U^{p,\lambda}(f)$ is of trace class and we have the inversion formula

$$f(x) = \frac{1}{4\pi^3} \sum_{2p \in Z} \int_{\lambda=-\infty}^{\infty} (p^2 + \lambda^2)\, \mathrm{Trace}\,(U^{p,\lambda}(x^{-1})U^{p,\lambda}(f))d\lambda \tag{8}$$

where Z is the set of all integers and $d\lambda$ is the usual Euclidean measure.

3. The spherical functions. Let $C_c^\infty(G)$ be the algebra of all C^∞ functions with compact support on G, with multiplication defined by convolution. The subalgebra $I_c(G)$ is formed by those functions f in $C_c^\infty(G)$ satisfying $f(kxk^{-1}) = f(x)$ for $x \in G$, $k \in K$. Define $\chi_s(k) = (2k+1)\,\mathrm{Trace}\,(D^s(k))$, $k \in K$ and $D^s \in \hat{K}$. Let $C_{c,s}(G) = \{f \in C_c^\infty(G): f * \chi_x = f = \chi_s^* f\}$ and $I_{c,s}(G) = I_c(G) \cap C_{c,s}(G)$. $I_{c,s}(G)$ is a subalgebra of $C_c^\infty(G)$ and the mapping $f \to f^{0*}\chi_s$, $f^0(x) = \int_K f(kxk^{-1})dk$, is the projection of $C_c^\infty(G)$ onto $I_{c,s}(G)$.

DEFINITION. Let $D^s \in \hat{K}$. By a spherical function Φ on G of type s we mean a quasi-bounded continuous function on G such that (i) $\Phi(kxk^{-1}) = \Phi(x)$, $x \in G$ and $k \in K$; (ii) $\Phi^*\chi_s = \Phi$; (iii) the map $f \to \int_G f(x)\Phi(x)dx$ is a nonzero homomorphism of the algebra $I_{c,s}(G)$ onto the complex numbers C.

Spherical functions of type s relates naturally to the TCI Banach representations of G. Suppose U is a TCI Banach representation of G on a space E such that D^s occurs in the restriction of U to K. Let $U(\chi_s) = \int_K U(k)\chi_s(k)dk$ and $E(s) = U(\chi_s)E$. The s-spherical function Ψ_s^U of U on G is defined by $\Psi_s^U(x) = U(\chi_s)U(x)U(\chi_s)$. Since D^s occurs in U exactly once, choose a basis for $E(s)$ so that $U(k) = D^s(k)$ on $E(s)$. Then clearly $\Psi_s^U(k_1xk_2) = D^s(k_1)\Psi_s^U(x)D^s(k_2)$. Let $\Psi_{s,K}^U(x) = \int_K \Psi_s^U(kxk^{-1})dk$. Then $\Psi_{s,K}^U(x)D^s(k) = D^s(k)\Psi_s^U(x)$, $x \in G$, $k \in K$, and we have $\Psi_{s,K}^U(x)$ is a scalar $\Phi_s^U(x)$ times identity operator. We recall the following facts, [cf. 8, Ch. 6].

PROPOSITION 3.1. (i) *Φ_s^U is a spherical function of type s and every spherical function of type s is of this form.*

(ii) *Let κ_U be the infinitesimal character of U defined on the center $\mathfrak{Z}$ of the algebra $\mathfrak{G}$, then $D\Phi_s^U = \kappa_U(D)\Phi_s^U$ and $D\Psi_s^U = \kappa_U(D)\Psi_s^U$, $D \in \mathfrak{Z}$.*

Consider the Banach representation $U^{p,\lambda}$ with $s = |p| + q$ for some nonnegative integer q, let $\Psi_s^{p,\lambda}$ and $\Phi_s^{p,\lambda}$ be the s-spherical function and the spherical function of type s respectively of the TCI Banach representation of G which occurs in $U^{p,\lambda}$ and has D^s occurs in it. Let $E^p(s) = U^{p,\lambda}(\chi_s)E^p$, then $\{D_{j,-p}^s : j = -s, -s+1, \cdots, s\}$ forms a basis for $E^p(s)$. Now

$$
\begin{aligned}
\Psi_s^{p,\lambda}(x)\cdot D_{j,-p}^s &= U^{p\lambda}(\chi_s)U^{p,\lambda}(x)U^{p,\lambda}(\chi_s)D_{j,-p}^s \\
&= (2s+1)\sum_{l=-s}^{s}\int_K \exp\left(-(i\lambda+1)\rho(H(x^{-1}k))\right) \\
&\qquad \times D_{j,-p}^s(\kappa(x^{-1}k))dk\cdot D_{l,-p}^s\,.
\end{aligned} \tag{9}
$$

But $\Phi_s^{p,\lambda}(x) = 1/(2s+1)\operatorname{Trace}(\Psi_{s,K}^{p,\lambda}(x)) = 1/(2s+1)\operatorname{Trace}(\Psi_s^{p,\lambda}(x))$, so

$$
\Phi_s^{p,\lambda}(x) = \int_K \exp\left(-(i\lambda+1)\rho(H(x^{-1}k)\right)D_{-p,-p}^s(k^{-1}\kappa(x^{-1}k))dk\,. \tag{10}
$$

Using this formula and the above proposition, we will set up a differential equation which enables us to get a complete parametrization of the spherical functions of type s.

LEMMA 3.2. $\Phi_s^{p,\lambda}(x) = \Phi_s^{-p,-\lambda}(x^{-1})$.

Proof. It suffices to show that

$$
\int_G f(x)\Phi_s^{p,-\lambda}(x)dx = \int_G f(x)\Phi_s^{-p,\lambda}(x^{-1})dx
$$

for all $f \in C_c^\infty(G)$. Since $\Phi_s^{p,\lambda}(kxk^{-1}) = \Phi_s^{p,\lambda}(x)$, $x \in G$, $k \in K$ and $\Phi_s^{p,\lambda} * \chi_s = \Phi_s^{p,\lambda}$, we only need to consider those f in $I_{c,s}(G)$. Thus let $f \in I_{c,s}(G)$, by (10)

$$\begin{aligned}\int_G f(x)\Phi_s^{p,\lambda}(x)dx &= \int_G f(x^{-1})\Phi_s^{p,\lambda}(x^{-1})dx \\ &= \int_G f(x^{-1}) \exp\left(-(i\lambda + 1)\rho(H(x))\right)D^s_{-p,-p}(\kappa(x))dx \\ &= \int_K \int_{A_p} \int_N f(n^{-1}a_t^{-1}k^{-1})e^{-(i\lambda+1)t}D^s_{-p,-p}(k)e^{2t}dkdtdn \\ &= \int_K \int_{A_p} \int_N f(kna_t)e^{(i\lambda-1)t}D^s_{-p,-p}(k^{-1})dkdtdn \\ &= \int_K \int_{A_p} \int_N f(ka_tn)e^{(i\lambda+1)t}D^s_{-p,-p}(k^{-1})dkdtdn \ .\end{aligned}$$

$$\begin{aligned}\int_G f(x)\Phi_s^{-p,-\lambda}(x^{-1})dx &= \int_G f(x) \exp(-(-i\lambda + 1)\rho(H(x)))D^s_{pp}(k(x))dx \\ &= \int_K \int_{A_p} \int_N f(ka_tn)e^{(i\lambda-1)t}D^s_{pp}(k)e^{2t}dkdtdn \\ &= \int_K \int_{A_p} \int_N f(ka_tn)e^{(i\lambda+1)t}D^s_{pp}(k)dkdtdn \ .\end{aligned}$$

But $D^s_{-p,-p}(k^{-1}) = D^s_{pp}(k)$ by (6), hence the lemma.

Let $w_1 = S_1^2 + S_2^2 + S_3^2 - R_1^2 - R_2^2 - R_3^2$ and $w_2 = R_1S_1 + R_2S_2 + R_3S_3$. Then $\{w_1, w_2\}$ generates the center $\mathfrak{Z}$. It is easy to see that $S_1 = R_2 - N_2$, $S_2 = N_1 - R_1$ and $N_1R_1 = R_1N_1 - S_3$, $N_2R_2 = R_2N_2 - S_3$, substitute into w_1, w_2 we get

$$w_1 = S_3^2 + 2S_3 - R_3^2 + N_1^2 + N_2^2 - 2(R_1N_1 + R_2N_2) \tag{11}$$

$$w_2 = R_3S_3 + R_3 - R_1N_2 + R_2N_1 \ . \tag{12}$$

Use the formula for $\Phi_s^{p,\lambda}(x)$ in the above lemma, a direct computation gives us

$$w_1\Phi_s^{p,\lambda}(1) = p^2 - \lambda^2 - 1 \ , \qquad w_2\Phi_s^{p,\lambda}(1) = p\lambda \ . \tag{13}$$

Now, $\Phi_s^{p,\lambda} = 1/(2s + 1)\,\mathrm{Trace}\,(\Psi_s^{p,\lambda})$, and for $x \in G$, we can write $x = k_1a_tk_2$, $k_1, k_2 \in K$, $a_t \in A_p^+$, so $\Psi_s^{p,\lambda}(x) = \Psi_s^{p,\lambda}(k_1a_tk_2) = D^s(k_1)\Psi_s^{p,\lambda}(a_t)D^s(k_2)$. Then this function determined by the restriction of $\Psi_s^{p,\lambda}$ to A_p^+. Let $t \neq 0$, define $\mathrm{Ad}\,(a_t^{-1})X = a_t^{-1}Xa_t$, $X \in \mathfrak{g}_0$; then we have

$$\begin{aligned}\mathrm{Ad}\,(a_t^{-1})R_1 &= \cosh t \cdot R_1 - \sinh t \cdot S_2 \ , \\ \mathrm{Ad}\,(a_t^{-1})R_2 &= \cosh t \cdot R_2 + \sinh t \cdot S_1 \ .\end{aligned} \tag{14}$$

By substitution, we get

$$\begin{aligned} w_1 = S_3^2 &+ 2\coth t \cdot S_3 + \coth^2 t \cdot (R_1^2 + R_2^2) \\ &+ \operatorname{csch}^2 t \cdot \operatorname{Ad}(a_t^{-1})(R_1^2 + R_2^2) \\ &- 2\coth t \operatorname{csch} t \cdot ((\operatorname{Ad}(a_t^{-1})R_1)R_2 \\ &+ ((\operatorname{Ad}(a_t^{-1})R_2)R_1) - (R_1^2 + R_2^2 + R_3^2) \end{aligned} \tag{15}$$

$$\begin{aligned} w_2 = S_3R_3 &+ \coth t \cdot R_3 - \operatorname{csch} t \cdot ((\operatorname{Ad}(a_t^{-1})R_1)R_2 \\ &- (\operatorname{Ad}(a_t^{-1})R_2)R_1) . \end{aligned} \tag{16}$$

Hence for $t > 0$, apply w_1, w_2 on $\Psi_s^{p,\lambda}(a_t)$, we get

$$\begin{aligned} &\frac{d^2}{dt^2}\Psi_s^{p,\lambda}(a_t) + 2\coth t \frac{d}{dt}\Psi_s^{p,\lambda}(a_t) \\ &\quad + (\coth^2 t - \operatorname{csch}^2 t)D^s(R_1^2 + R_2^2)\Psi_s^{p,\lambda}(a_t) \\ &\quad + \coth t \operatorname{csch} t(X\Psi_s^{p,\lambda}(a_t)Y + Y\Psi_s^{p,\lambda}(a_t)X) \\ &\quad + s(s+1)\Psi_s^{p,\lambda}(a_t) = (p^2 - \lambda^2 - 1)\Psi_s^{p,\lambda}(a_t) . \end{aligned} \tag{17}$$

$$\begin{aligned} &D^s(R_3)\frac{d}{dt}\Psi_s^{p,\lambda}(a_t) + \coth t D^s(R_3)\Psi_s^{p,\lambda}(a_t) \\ &\quad - \frac{1}{2}\operatorname{csch} t(X\Psi_s^{p,\lambda}(a_t)Y - Y\Psi_s^{p,\lambda}(a_t)X) = p\lambda\Psi_s^{p,\lambda}(a_t) \end{aligned} \tag{18}$$

where $X = D^s(R_1) - iD^s(R_2)$, $Y = -D^s(R_1) - iD^s(R_2)$. Since $u_\theta a_t = a_t u_\theta$, $u_\theta \in M$, $a_t \in A_p$, by (5) we see that $\Psi_s^{p,\lambda}(a_t)$ is a diagonal matrix, so let $\Psi_{s,j}^{p,\lambda}$ be the jth diagonal element, $j = -s, -s+1, \cdots, s$, we see from (18) and (6)

$$\begin{aligned} &-ij\frac{d}{dt}\Psi_{s,j}^{p,\lambda}(a_t) - ij\coth t\Psi_{s,j}^{p,\lambda}(a_t) \\ &\quad - \frac{i}{2}\operatorname{csch} t((s-j)(s+j+1)\Psi_{s,j+1}^{p,\lambda}(a_t) \\ &\quad - (s-j)(s-j+1)\Psi_{s,j-1}^{p,\lambda}(a_t)) = p\lambda\Psi_{s,j}^{p,\lambda}(a_t) . \end{aligned} \tag{19}$$

Hence for $j = s, s-1, s-2, \cdots, -s+1$, we get

$$\begin{aligned} &(s+j)(s-j+1)\operatorname{csch} t\Psi_{s,j-1}^{p,\lambda}(a_t) \\ &\quad = 2j\frac{d}{dt}\Psi_{s,j}^{p,\lambda}(a_t) + 2j\coth t\Psi_{s,j}^{p,\lambda}(a_t) \\ &\quad\quad - 2ip\lambda\Psi_{s,j}^{p,\lambda}(a_t) + (s-j)(s+j+1)\operatorname{csch} t\Psi_{s,j+1}^{p,\lambda}(a_t) . \end{aligned} \tag{20}$$

Therefore, $\Psi_s^{p,\lambda}(a_t)$ is determined by knowing $\Psi_{s,s}^{p,\lambda}(a_t)$, $t > 0$. Consider the sth diagonal element of (17) + $2i\coth t \cdot$(18), we find that $\Psi_{s,s}^{p,\lambda}(a_t)$ satisfies the following differential equation

$$\begin{aligned} &\varphi''(t) + 2(1+s)\coth t\varphi'(t) \\ &\quad + ((s+1)^2 - p^2 + \lambda^2 - 2ip\lambda\coth t)\varphi(t) = 0 . \end{aligned} \tag{21}$$

This is a differential equations with regular singular point at $t = 0$. The inditial equation $f(z) = z(z + 1 + s)$, so we have $z_1 = 0$ and $z_2 = -(1 + s)$ as roots for $f(z) = 0$. From the general theory of such differential equation [e.g. 1, Ch. 4] we have

PROPOSITION 3.3. *Two linearly independent solutions of* (21) *can be represented in the following form*

$$\varphi_1(t) = t^{z_1} U_1(t) = U_1(t) \tag{22}$$

$$\varphi_2(t) = t^{z_2} U_2(t) + \alpha \varphi_1(t) \ln t \tag{23}$$

here U_1 *and* U_2 *are analytic on* $[0, \infty)$ *with* $U_1(0) = U_2(0) = 1$ *and* α *is some constant.*

COROLLARY 1. *The function* $\Psi_{s,s}^{p,\lambda}(a_t) = \varphi_1(t)$.

Proof. The only solutions of (21) which are bounded at $t = 0$ are constant multiples of $\varphi_1(t)$ and we know that $\Psi_{s,s}^{p,\lambda}(1) = 1$.

Let $\varphi_1(t) = \sum_{j=0}^{\infty} c_j t^j$. We will compute the coefficients c_j more explicitly. Since $\lim_{t\to 0} t \coth t = 1$, we get

$$\coth h t = \frac{1}{t} + \sum_{j=0}^{\infty} a_j t^j \tag{24}$$

with $g(t) = \sum_{j=0}^{\infty} a_j t^j$ analytic at $t = 0$. Substitute $\varphi_1(t)$ into (21), we get

$$2(1 + s)c_1 - 2ip\lambda c_0 = 0 \tag{25}$$

and the recursion formula, $j = 2, 3, \cdots$

$$\begin{aligned} j(j + 1 + 2s)c_j = [p^2 - \lambda^2 - (s + 1)^2]c_{j-2} - 2(1 + s) \sum_{r=1}^{j-1} r c_r a_{j-1-r} \\ + 2ip\lambda \sum_{r=0}^{j-1} c_r a_{j-2-r} \, . \end{aligned} \tag{26}$$

COROLLARY 2. *Two spherical functions* $\Phi_s^{p_1,\lambda_1}$ *and* $\Phi_s^{p_2,\lambda_2}$ *of type* s *are equal if and only if* $(p_2, \lambda_2) = \pm (p_1, \lambda_1)$ *or* $(p_2, \lambda_2) = \pm i(\lambda_1, -p_1)$.

Proof. From earlier discussion, it suffices to consider the functions $\Psi_{s,s}^{p_1,\lambda_1}(a_t)$ and $\Psi_{s,s}^{p_2,\lambda_2}(a_t)$, hence their corresponding coefficients derived from (25) and (26). Clearly then it is equivalent to have $p_1\lambda_1 = p_2\lambda_2$ and $p_1^2 - \lambda_1^2 = p_2^2 - \lambda_2^2$ and the corollary follows.

PROPOSITION 3.4. $\Phi_s^{p,\lambda}$ *is bounded if* $\lambda = \sigma + ib$ *with* $\sigma, b \in \boldsymbol{R}$ *and* $|b| \leq 1$.

Proof. Let $x \in G$ and write $x = k_1 a_t k_2$ with $t \geqq 0$. Then

$$\begin{aligned}\Phi_s^{p,\lambda}(x^{-1}) &= \Phi_s^{p,\lambda}((k_1 a_t k_2)^{-1}) = \Phi_s^{p,\lambda}((k_2 k_1 a_t)^{-1}) \\ &= \int_K \exp\left(-(i\lambda + 1)\rho(H(a_t k))D^s_{-p,-p}(k^{-1}k_2 k_1 k(a_t k)\right)dk \; . \end{aligned} \tag{27}$$

Now, write $k = u_{\varphi_1} v_\theta u_{\varphi_2}$, then $a_t k = (u_{\varphi_1} v_{\theta'}, u_{\varphi_2}) a_{t'} n$, $n \in N$,

$$\begin{aligned} e^{t'} &= e^t \cos^2\frac{\theta}{2} + e^{-t}\sin^2\frac{\theta}{2} \\ \cos\frac{\theta'}{2} &= e^{(t-t')/2}\cos\frac{\theta}{2}\,, \quad \sin\frac{\theta'}{2} = e^{-(t+t')/2}\sin\frac{\theta}{2}\,, \quad 0 \leqq \theta' \leqq \pi \; . \end{aligned} \tag{28}$$

Thus by (4) and (5) we get

$$\Phi_s^{p,\lambda}(x^{-1}) = \frac{1}{2}\int_0^\pi \exp\left(-(i\lambda + 1)t'\right)D^s_{-p,-p}(v_\theta^{-1}k_2 k_1 v_{\theta'})\sin\theta d\theta \; . \tag{29}$$

If $t = 0$, then $t' = 0$ and the integral (29) bounds by 1. If $t > 0$, by (28) with change of variable gives

$$\Phi_s^{p,\lambda}(x^{-1}) = \frac{1}{2\sinh t}\sum_{j=s}^{s}\int_{-t}^{t} e^{-\lambda t'}D^s_{-p,j}(v_\theta^{-1})D^s_{j,j}(k_2 k_1)D^s_{j,-p}(v_{\theta'})dt'$$

and

$$|\Phi_s^{p,\lambda}(x^{-1})| \leqq \frac{1}{2\sinh t}\int_{-t}^{t} e^{bt'}dt = \frac{\sinh t}{b\sinh t} \leqq 1 \; .$$

4. The analogue of Paley-Wiener theorem. Let

$$B_s = \{(p, \lambda) \colon p = -s, -s+1, \cdots, s; \lambda \in C\} \; .$$

For each pair $(p, \lambda) \in B_s$, there corresponds a spherical functions $\Phi_s^{p,\lambda}$ of type s. Let $f \in I_{c,s}(G)$, the Fourier-Laplace transform $\hat{f}$ of f is a function defined on B_s by

$$\hat{f}(p, \lambda) = \int_G f(x)\Phi_s^{p,\lambda}(x)dx \; . \tag{30}$$

Given $f \in I_c(G)$. Let $B_f = \{a_t \in A_p \colon f(ka_t) \neq 0$ for some $k \in K\}$. We say that f has support in the ball of radius R if $\sup\{|t| \colon a_t \in B_f\} \leqq R$. Clearly f has compact support if and only if there exists an R which is finite. For each $D^s \in \hat{K}$, define

$$F_f^s(a_t) = e^t\int_K\int_N f(ka_t n)D^s(k^{-1})dkdn \; . \tag{31}$$

This gives a map of A_p to the space of linear operators $L(E_s)$ on E_s. It is easy to see that $F_f^s = F_{f_s}^s$, $f \in I_c(G)$ and $f_s = f * \chi_s$.

LEMMA 4.1. *Let* $n \in N$, $a_t \in A_p$ *and write* $a_t n = k_1 a_{t_1} k_2$ *for some* $k_1, k_2 \in K$. *Then* $|t_1| \geqq |t|$.

Proof. Let $n = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $z \in C$ and $k_j = \begin{pmatrix} \alpha_j & \beta_j \\ \bar{\beta}_j & \bar{\alpha}_j \end{pmatrix}$ with $|\alpha_j|^2 + |\beta_j|^2 = 1$, $j = 1, 2$. Equating the corresponding matrix coefficients from $a_t n k_2^{-1}$ yields $e^t + (1 + |z|^2)e^{-t} = e^{t_1} + e^{-t_1}$, i.e., $2\cosh t_1 + e^t |z|^2$. Thus $|t_1| \geqq |t|$.

PROPOSITION 4.2. *Let* $f \in I_{c,s}(G)$ *have support in the ball of radius* R, *then* F_f^s *is* C^∞ *with support in* $\{a_t : |t| \leqq R\}$.

Proof. Suppose $F_f^s(a_t) \neq 0$, then $f(ka_t n) \neq 0$ for some $k \in K$, $n \in N$. Now $ka_t n = k_1 a_{t_1} k_2$ for some $k_1, k_2 \in K$ and $a_{t_1} \in A_p$. Thus $a_t n = k^{-1} k_1 a_{t_1} k_2$ and $f(k_2 k_1 a_{t_1}) = f(k_1 a_{t_1} k_2) = f(ka_t n) \neq 0$. By the above lemma and the assumption we get $|t| \leqq |t_1| \leqq R$. Differentiability is clear.

PROPOSITION 4.3. *The map* $f \rightarrow F_f^s$ *is a one-to-one algebra homomorphism of* $I_{c,s}(G)$ *into* $C_c^\infty(A_p, L(E_s))$.

Proof. Let $f, g \in I_{c,s}(t)$, use Fubini's theorem repeatedly

$$\begin{aligned} F_{f*g}^s(a_t) &= e^t \int_K \int_N (f*g)(ka_t n) D^s(k^{-1}) dk dn \\ &= e^t \int_K \int_N \int_G f(ka_t n x^{-1}) g(x) D^s(k^{-1}) dx dk dn \\ &= e^t \int_K \int_N \int_K \int_{A_p} \int_N f(ka_t n n_1^{-1} a_{t_1}^{-1} k_1^{-1}) g(k_1 a_{t_1} n_1) \\ &\quad \times e^{2t_1} D^s(k^{-1}) dk_1 dt_1 dn_1 dk dn \\ &= \int_{A_p} \int_K \int_N \int_K \int_N f(ka_t a_{t_1}^{-1} n) g(k_1 a_{t_1} n_1) e^{t_1} \cdot e^{t-t_1} D^s(k^{-1}) \\ &\quad \times D^s(k_1^{-1}) dk_1 dn_1 dk dr dt_1 \\ &= \int_{A_p} F_f^s(a_t a_{t_1}^{-1}) F_g^s(a_{t_1}) dt_1 = F_f^s * F_g^s(a_t) \; . \end{aligned}$$

The linearity is trivial, hence it is algebra homomorphism. As for one-to-one, given $f \in I_{c,s}(G)$ and $F_f^s \equiv 0$, to show $f \equiv 0$. Note first that $F_f^s(a_t) D^s(u_\theta) = D^s(u_\theta) F_f^s(a_t)$, hence $F_f^s(a_t)$ is a diagonal matrix. From (10) and Lemma 3.2, we see that if $F_{f,p}^s(a_t)$ is the pth diagonal element of $F_f^s(a_t)$,

$$\int_{A_p} F_{f,p}^s(a_t) e^{-i\lambda t} dt = \int_G f(x) \Phi_s^{p,\lambda}(x) dx. \tag{32}$$

If $F_f^s \equiv 0$, then $F_{f,p}^s \equiv 0$ for all p, hence $\hat{f}(p, \lambda) = 0$ for all p, λ. Thus $U^{p,\lambda}(f) = 0$ for all p, λ. But the set $\{U^{p,\lambda}\}$ forms a complete set of representations on G, thus we get $f = 0$.

COROLLARY. *$I_{c,s}(G)$ is commutative.*

For each nonnegative real number R, let $H_s(R)$ be the set of functions g defined on B_s satisfying (i) g is entire holomorphic in λ; (ii) $g(p, \lambda) = g(-p, -\lambda)$, $(p, \lambda) \in B_s$; (iii) $g(p, \lambda) = g(i\lambda, -ip)$ if both (p, λ) and $(i\lambda, -ip)$ are in B_s; (iv) given a positive integer m, there exists a constant C_m such that $|g(p, \lambda)| \leqq C_m(1 + |\lambda|)^{-m} \exp R|\eta|$, $\lambda = \xi + i\eta \in \boldsymbol{R} + i\boldsymbol{R}$. Let H_s be the union of all the $H_s(R)$.

Given f in $I_{c,s}(G)$, by Corollary 2 of Proposition 3.3 we see the function $\hat{f}$ defined in (30) satisfies conditions (i), (ii), (iii) of the definition of H_s. By (32), $\hat{f}(p, \lambda)$ is just the usual Fourier transform of the function $F_{f,p}^s$ on the real line, which is C^∞ with compact support, hence $\hat{f}$ is holomorphic in λ. If f has support in the ball of radius R, so is F_f^s, hence the classical Paley-Wiener theorem asserts that $\hat{f} \in H_s(R)$. Thus we have a linear map $f \to \hat{f}$ of $I_{c,s}(G)$ into H_s such that if f has support in the ball of radius R, we get $\hat{f} \in H_s(R)$. We want to show that this map is also onto now.

In the inversion formula (8), when $f \in I_{c,s}(G)$, it is easy to see that Trace $(U^{p,\lambda}(x^{-1})U^{p,\lambda}(f)) = (2s+1)\hat{f}(p, \lambda)\Phi_s^{p,\lambda}(x^{-1})$ for $p = -s, -s + 1, \cdots, s$; and $U^{p,\lambda}(f) = 0$ otherwise. Thus we have

$$f(x) = \frac{2s+1}{4\pi^3} \sum_{p=-s}^{s} (p^2 + \lambda^2)\hat{f}(p, \lambda)\Phi_s^{p,\lambda}(x^{-1})d\lambda \ . \tag{33}$$

LEMMA 4.4. *Let $g \in H_s(R)$ and define*

$$f_1(x) = \sum_{p=-s}^{s} \int_{\lambda=-\infty}^{\infty} (p^2 + \lambda^2) g(p, \lambda)\Phi_s^{p,\lambda}(x^{-1})d\lambda \ . \tag{34}$$

Then $f_1 \in I_{c,s}(G)$ and f_1 has support in the ball of radius R.

Proof. Since $g(p, \lambda)$ decreases rapidly at infinity on λ and $\Phi_s^{p,\lambda}$ is C^∞ and bounded when λ is real, the integral converges absolutely and defines a C^∞ function on G. By the property of $\Phi_s^{p,\lambda}$, it is clear that $f_1(k \times k^{-1}) = f_1(x)$, $k \in K$, $x \in G$ and $f_1 * \chi_s = f_1$. It remains to show that f_1 has support in the ball of radius R. Thus let $x = k_1 a_t$ with $k_1 \in K$ and $t \neq 0$. Since $a_t \in B_{f_1}$ if and only if $a_{-t} \in B_{f_1}$, may assume that $t > 0$. Using the expression and notation in Proposition 3.4, we get

$$\begin{aligned} f_1(k_1 a_t) = \frac{1}{2\sinh t} \sum_{p,j=-s}^{s} D_{j,j}^s(k_1) \int_{\lambda=-\infty}^{\infty} \int_{t'=-t}^{t} (p^2 + \lambda^2) g(p, \lambda) e^{-i\lambda t'} \\ \times D_{-p,j}^s(v_\theta^{-1}) D_{j,-p}^s(v_{\theta'}) dt' d\lambda \ . \end{aligned} \tag{35}$$

For each $p, j = -s, -s+1, \cdots, s$, define

$$f_{p,j}(t) = \int_{\lambda=-\infty}^{\infty}\int_{t'=t}^{t}(p^2+\lambda^2)g(p,\lambda)e^{-i\lambda t'}D^s_{-p,j}(v_\theta^{-1})D^s_{j,-p}(v_{\theta'})dt'd\lambda\ . \tag{36}$$

Let $t > R$, to show $f_1(k_1a_t) = 0$, it suffices to show that $\sum_{p=-s}^{s} f_{p,j}(t) = 0$ for all j. Let

$$h_p(t') = \int_{-\infty}^{\infty}(p^2+\lambda^2)g(p,\lambda)e^{-i\lambda t'}d\lambda\ . \tag{37}$$

By the classical Paley-Wiener theorem, $h_p(t') = 0$ if $t' > R$. Thus

$$f_{p,j}(t) = \int_{-\infty}^{\infty}h_p(t')D^s_{-p,j}(v_\theta^{-1})D^s_{j,-p}(v_{\theta'})dt'\ . \tag{38}$$

Put $x_1 = e^{t'}$, $x_2 = e^{-t'}$, then by (5), (28) we get

$$\begin{aligned}D^s_{-p,j}(v_\theta^{-1})D^s_{j,-p}(v_{\theta'}) = &\frac{(-1)^{s+j}e^{-jt}}{(s+j)!\,(s-j)!\,(2\sinh t)^{2s}}e^{-pt'}\frac{\partial^{2s}}{\partial x_1^{s-p}\partial x_2^{s+p}}\\ &\times[(x_1x_2 - e^{-t}(x_1+x_2)+e^{-2t})^{s-j}\\ &\times(x_1x_2 - e^{t}(x_1+x_2)+e^{2t})^{s+j}]\ .\end{aligned} \tag{39}$$

The above expression is just the linear combination of terms

$$e^{-pt'}\frac{\partial^{2s}}{\partial x_1^{s-p}\partial x_2^{s+p}}[(x_1x_2)^{r_1}(x_1^{r_2}+x_2^{r_i})]$$

with coefficients as functions of t, and $r_1, r_2 \geqq 0$, $r_1 + r_2 \leqq 2s$. Pick one of these terms and consider the two integrals

$$\begin{aligned}&\sum_{p=-s}^{s}\int_{-\infty}^{\infty}h_p(t')e^{-pt'}\frac{\partial^{2s}}{\partial x_1^{s-p}\partial x_2^{s+p}}[x_1^{r_1+r_2}x_2^{r_1}]dt'\\ &= \sum_{p=\max\{-s,s-r_1-r_2\}}^{\min\{s,r_1-s\}}\int_{-\infty}^{\infty}h_p(t')\frac{(r_1+r_2)!\,r_1!}{(r_1+r_2-s+p)!\,(r_1-s-p)!}e^{(r_2+p)t'}dt'\\ &= 2\pi\sum_{p=s-r_1-r_2}^{r_1-s}\frac{(r_1+r_2)!\,r_1!}{(r_1+r_2-s+p)!\,(r_1-s-p)!}\\ &\quad\times[p^2+(-i(r_2+p))^2]g(p,-i(r_2+p))\\ &= -2\pi\sum_{p=s-r_1-r_2}^{r_1-s}\frac{(r_1+r_2)!\,r_1!}{(r_1+r_2-s+p)!\,(r_1-s-p)!}\\ &\quad\times r_2(r_2+2p)g(p,-i(r_2+p))\end{aligned} \tag{40}$$

$$\begin{aligned}&\sum_{p=-s}^{s}\int_{-\infty}^{\infty}h_p(t')e^{-pt'}\frac{\partial^{2s}}{\partial x_1^{s-p}\partial x_2^{s+p}}[x_1^{r_1}x_2^{r_1+r_2}]dt'\\ &= 2\pi\sum_{p=\max\{-s,s-r_1\}}^{\min\{s,r_1+r_2-s\}}\frac{(r_1+r_2)!\,r_1!}{(r_1-s+p)!\,(r_1+r_2-s-p)!}\int_{-\infty}^{\infty}h_p(t')e^{(p-r_2)t}dt'\\ &= 2\pi\sum_{p=s-r_1}^{r_1+r_2-s}\frac{r_1!\,(r_1+r_2)!}{(r_1-s+p)!\,(r_1+r_2-s-p)!}r_2(2p-r_2)g(p,i(r_2-p))\,.\end{aligned} \tag{41}$$

By changing the index and the fact that

$$g(p, i(r_2 - p)) = g(p - r_2, -ip) ,$$

we get the sum of (40) and (41) is zero. Now the lemma is clear.

Combine the above discussion, we get the following analogue of Paley-Wiener theorem.

PROPOSITION 4.5. *The Fourier transform f to $\hat{f}$ defined in* (30) *is a one-to-one algebra homomorphism of $I_{c,s}(G)$ onto H_s. A function f in $I_{c,s}(G)$ has support in the ball of radius R if and only if $\hat{f}$ is in $H_s(R)$.*

Let $L_s^1(G)$ be the closure of $I_{c,s}(G)$ in $L'(G)$. Given $f \in L_s^1(G)$, by Proposition 3.4, the integral

$$\hat{f}(p, \lambda) = \int_G f(x)\Phi_s^{p,\lambda}(x)dx \tag{42}$$

is defined for $(p, \lambda) \in B_s$ with $\lambda = \xi + i\eta$, $|\eta| \leqq 1$. Then we have the following analogue of Riemann Lebesgue lemma.

PROPOSITION 4.6. *Let $f \in L_s^1(G)$ and define $\hat{f}$ as in* (42), *then $\lim_{\xi\to\pm\infty} \hat{f}(p, \xi + i\eta) = 0$ uniformly for $|\eta| \leqq 1$.*

Proof. Given $\varepsilon > 0$, choose g in $I_{c,s}(G)$ such that $\|f - g\|_1 < \varepsilon/2$. But then we have

$$|\hat{f}(p, \lambda) - \hat{g}(p, \lambda)| \leqq \int_G |f(x) - g(x)|\, dx < \varepsilon/2 . \tag{43}$$

Choose R, C such that

$$|\hat{g}(p, \lambda)| \leqq C(1 + |\lambda|)^{-1} \exp R|\eta| \leqq C(1 + |\lambda|)^{-1} \exp R \tag{44}$$

since $|\eta| \leqq 1$. Combine (43), (44) we get $|\hat{f}(p, \lambda)| < \varepsilon$ when $|\xi|$ is large enough.

Let $B = \{(s, p, \lambda)$: s is a nonnegative integer or half integer, $(p, \lambda) \in B_s\}$. Given $f \in I_c(G)$ and $(s, p, \lambda) \in B$, define

$$\hat{f}(s, p, \lambda) = \int_G f(x)\Phi_s^{p,\lambda}(x)dx . \tag{45}$$

It is clear that $\hat{f}(s, p, \lambda) = \hat{f}_s(p, \lambda)$.

LEMMA 4.7. *Let $f \in I_c(G)$. Then f has support in the ball of radius R if and only if f_s has support in the ball of radius R for all s.*

Proof. By definition, $f_s(x) = \int_K f(k^{-1}x)\chi_s(k)dk$. Thus if f has support in the ball of radius R and $f_s(k_1 a_t) \neq 0$ with $k_1 \in K$, $a_t \in A_p$, we have $f(k^{-1}k_1 a_t) \neq 0$ for some $k \in K$ and therefore $|t| \leqq R$. The converse follows from the fact that $\sum_s f_s$ converges to f absolutely, [8, vol. I, p. 264].

PROPOSITION 4.8. *The map $f \to \hat{f}$ defined in* (45) *is a one-to-one algebra homomorphism of $I_c(G)$ into the algebra of all functions g on B satisfying* (i) *$g(s, p, \lambda)$ is entire holomorphic in λ,* (ii) $g(s, p, \lambda) = g(s, -p, -\lambda)$, $(s, p, \lambda) \in B$, (iii) *$g(s, p, \lambda) = g(s, i\lambda, -ip)$ if both (s, p, λ) and $(s, i\lambda, -ip)$ are in B,* (iv) *there exists $R > 0$, for each given positive integer m, there exists $C_{m,s}$ such that*

$$|g(s, p, \lambda)| \leqq C_{m,s}(1 + |\lambda|)^{-m} \exp R|\eta|, \ \xi + i\eta \in \boldsymbol{R} + i\boldsymbol{R} .$$

Proof. This is clear by Proposition 4.6 and Lemma 4.7.

COROLLARY. *Let $f \in L^1(G)$. Then $\hat{f}(s, p, \lambda)$ is defined for $\lambda = \xi + i\eta$, $|\eta| \leqq 1$ and $\lim_{\xi \to \pm\infty} \hat{f}(s, p, \xi + i\eta) = 0$ for $|\eta| \leqq 1$.*

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