PACIFIC JOURNAL OF MATHEMATICS
Vol. 52, No. 2, 1974

PROJECTIONS IN THE SPACES OF BOUNDED LINEAR OPERATORS

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For Banach spaces X, Z, let $B(X, Z)$ denote the space of bounded linear operators from X into Z and $K(X, Z)$ (resp. $W(X, Z)$) the subspace of compact (resp. weakly compact) operators. It is shown that (a) if X contains an isomorph of c_0, then $K(X, l^\infty)$ is not complemented in $B(X, l^\infty)$, (b) if S is a compact Hausdorff space which is not scattered, then $K(C(S), Z)$ is not complemented in $W(C(S), Z)$ for $Z = c_0$ or l^∞. In particular, $K(l^\infty, c_0)$ is not complemented in $B(l^\infty, c_0)$, which gives a negative answer to a question proposed by Arterburn and Whitley.

A subspace Y of a Banach space X is complemented if there is a projection $P: X \to X$ with range Y, i.e., a bounded linear operator of X such that $P^2 = P$ and $P(X) = Y$. There is a general conjecture afoot that if $K(X, Z)$ is a proper subspace of $B(X, Z)$ (resp. $W(X, Z)$) then it is not complemented in $B(X, Z)$ (resp. $W(X, Z)$). This conjecture was first studied by Thorp in [8], where he proved that $K(X, Z)$ is not complemented in $B(X, Z)$ when X, Z are certain Banach spaces of sequences. Later, various types of pairs X, Z for which the conjecture is known to be true were exhibited in [1] and [9]. We only recall that if weak and norm sequential convergence are not the same in the dual of a separable Banach space X, then $K(X, Z)$ is not complemented in $W(X, Z)$ for $Z = c_0$ or l^∞.

Let S be a compact Hausdorff space. S is called *scattered* if it contains no nonempty perfect subset. From the known results, we shall first establish some basic tools to determine certain situation where $K(X, Z)$ or $W(X, Z)$ is uncomplemented, then restrict ourselves to the projections in $B(X, Z)$ when X contains an $\mathscr{L}^\infty$-space in the sense of [4] and especially when $X = C(S)$. To avoid lengthy statements, we only discuss below the projections of $B(X, Z)$ onto $K(X, Z)$ and remark here that the statements in Proposition 1 through Theorem 6 remain true if we replace $B(\cdot, \cdot)$ by $W(\cdot, \cdot)$ everywhere; and also if, instead, we replace $K(\cdot, \cdot)$ by $W(\cdot, \cdot)$ everywhere. Our results are consistent with the conjecture. Furthermore, no counter-examples to the conjecture are known at present. In the sequel, let X^* denote the dual space of a Banach space X and let X be embedded into X^{**} under the canonical isometry.

PROPOSITION 1. *Let Z be a Banach space such that Z is comple-*

mented in Z^{**}. *Suppose* $K(X, Z)$ *is not complemented in* $B(X, Z)$, *then* $K(Z^*, X^*)$ *is not complemented in* $B(Z^*, X^*)$.

Proof. The map $T \to T^*$ is an isometrical isomorphism of $B(X, Z)$ into $B(Z^*, X^*)$ such that T^* is compact if and only if T is. Also T^{**} is a linear extension of T. Suppose now Q is a projection of Z^{**} onto Z and R is a projection of $B(Z^*, X^*)$ onto $K(Z^*, X^*)$; define $P: B(X, Z) \to B(X, Z)$ by

$$(PT)(x) = Q((RT^*)^*(x)) .$$

P is then a projection of $B(X, Z)$ onto $K(X, Z)$, a contradiction.

As an application, since $K(l^1, l^1)$ is not complemented in $B(l^1, l^1)$ [8], it follows that $K(l^\infty, l^\infty)$ is not complemented in $B(l^\infty, l^\infty)$, a simple result which is not contained in previous work.

PROPOSITION 2. *There exists an isometrical isomorphism of* $B(X, Z^*)$ *onto* $B(Z, X^*)$ *such that* $K(X, Z^*)$ *corresponds to* $K(Z, X^*)$. *Thus if* $K(X, Z^*)$ *is not complemented in* $B(X, Z^*)$, *neither is* $K(Z, X^*)$ *in* $B(Z, X^*)$.

Proof. Consider $T \in B(X, Z^*)$. Since Z is weak* dense in Z^{**}, the map $\tau: T \to T^*|_Z$, the restriction of T^* to Z, is an isometrical isomorphism. τ is also surjective, for given any $U \in B(Z, X^*)$, we have $\tau(U^*|_X) = U$. The correspondence of the subspaces of compact operators is trivial.

REMARK. In particular, $K(c_0, l^\infty)$ is thus uncomplemented in $B(c_0, l^\infty)$ because $K(l^1, l^1)$ is uncomplemented in $B(l^1, l^1)$. This proof avoids direct expressions for the norms of operators in terms of matrix coefficients as in the original proof of [8].

Let Y be a subspace of X. A bounded linear operator $E: B(Y, Z) \to B(X, Z)$ is called a *simultaneous extension* if $R_Y E(T) = T$ for every $T \in B(Y, Z)$, where R_Y denotes the restriction to Y. Suppose in addition that $E(K(Y, Z)) \subset K(X, Z)$ and that P is a projection of $B(X, Z)$ onto $K(X, Z)$; then $R_Y PE$ is a projection of $B(Y, Z)$ onto $K(Y, Z)$. Hence we have:

LEMMA 3. *Suppose* $K(Y, Z)$ *is not complemented in* $B(Y, Z)$ *and that there exists a simultaneous extension* $E: B(Y, Z) \to B(X, Z)$ *such that* $E(K(Y, Z)) \subset K(X, Z)$; *then* $K(X, Z)$ *is not complemented in* $B(X, Z)$.

LEMMA 4. *If* Y *is complemented in* X, *then there exists a simultaneous extension* E *such that* $E(K(Y, Z)) \subset K(X, Z)$.

LEMMA 5. *If Z is complemented in Z^{**} and $Y \subset Y_1 \subset Y^{**}$, then there exists a simultaneous extension E from $B(Y, Z)$ to $B(Y_1, Z)$ with $E(K(Y, Z)) \subset K(Y_1, Z)$.*

Proof. The map $T \to T^{**}$ is an isometrical isomorphism from $B(Y, Z)$ into $B(Y^{**}, Z^{**})$ such that T^{**} is an extension of T and T^{**} is compact if and only if T is. Let P be a projection of Z^{**} onto Z. Define $E: B(Y, Z) \to B(Y_1, Z)$ by $(ET)(y) = P(T^{**}(y))$, $y \in Y_1$; then E is the desired simultaneous extension.

THEOREM 6. *If Z is complemented in Z^{**} and Y is an $\mathscr{L}^\infty$-space such that $K(Y, Z)$ is not complemented in $B(Y, Z)$ then $K(X, Z)$ is not complemented in $B(X, Z)$ for any X containing a subspace isomorphic to Y.*

Proof. We can assume without loss of generality that $Y \subset X$, because if Y is isomorphic to $\tilde{Y}$ then $K(Y, Z)$ is complemented in $B(Y, Z)$ if and only if $K(\tilde{Y}, Z)$ is complemented in $B(\tilde{Y}, Z)$. Then Y^{**} can be regarded as a subspace of X^{**}. Since Y^{**} is an injective space [4, p. 291], there exists a projection Q from X^{**} onto Y^{**}. Let P be the projection from Z^{**} onto Z. On account of Lemma 4 and Lemma 5, we define $E: B(Y, Z) \to B(X, Z)$ by $(ET)(x) = P(T^{**}(Q(x))$, $x \in X$. Then E is a simultaneous extension such that $E(K(Y, Z)) \subset K(X, Z)$, which in turn proves that $K(X, Z)$ is not complemented in $B(X, Z)$.

REMARKS. (a) Z is complemented in Z^{**} if and only if Z is isomorphic to a complemented subspace of a dual space. (b) A bounded linear operator $T \in B(Y, Z)$ is weakly compact if and only if T^{**} maps Y^{**} into Z, i.e., $T \in W(Y, Z) \Leftrightarrow T^{**} \in W(Y^{**}, Z)$. Hence if $B(Y, Z) = W(Y, Z)$, or if we are merely looking for a projection of $W(X, Z)$ onto $K(X, Z)$, the assumption that Z is complemented in Z^{**} is redundant.

Observe that c_0 is an $\mathscr{L}^\infty$-space [4, p. 283]. Therefore, since there exists no projection of $B(c_0, l^\infty)$ onto $K(c_0, l^\infty)$ and since every infinite-dimensional Banach space whose dual is an L^1 space contains a subspace isomorphic to c_0 [10], we have

COROLLARY 7. *If X contains a subspace isomorphic to c_0, which is in particular the case when X is isomorphic to a $C(S)$ space or X is an infinite-dimensional Banach space whose dual is an L^1 space, then $K(X, l^\infty)$ is not complemented in $B(X, l^\infty)$.*

REMARK. An infinite-dimensional Banach space whose dual is an

L^1 space need not be isomorphic to a $C(S)$ space. As an example, given by Benyamini and Lindenstrauss, there exists a predual of l^1 which is not isomorphic to any $C(S)$ space [2].

In connection with the linear extension of operators, we have the following corollary, which will serve as a lemma for the next theorem.

COROLLARY 8. *If Y is an $\mathscr{L}^\infty$-space and X contains Y, then there exists a simultaneous extension E from $W(Y, Z)$ to $W(X, Z)$ such that $E(K(Y, Z)) \subset K(X, Z)$. If in addition Z is complemented in Z^{**}, then there exists a simultaneous extension from $B(Y, Z)$ to $B(X, Z)$ with $K(Y, Z)$ and $W(Y, Z)$ corresponding to subspaces of $K(X, Z)$ and $W(X, Z)$ respectively.*

THEOREM 9. *Let S be a compact Hausdorff space which is not scattered, then $K(C(S), Z)$ is not complemented in $W(C(S), Z)$ for $Z = c_0$ or $Z = l^\infty$.*

Proof. Consider the space $C([0, 1])$. Since weak and norm sequential convergence are not the same in $C([0, 1])^*$, it is known by the aforementioned result in [1] that $K(C([0, 1]), Z)$ is not complemented in $W(C([0, 1]), Z)$ when Z is c_0 or l^∞. Now if S is not scattered, the interval $[0, 1]$ is a continuous image of S [7], hence $C(S)$ contains an isometric copy of $C([0, 1])$. Therefore by Corollary 8, there exists a simultaneous extension from $W(C([0, 1]), Z)$ to $W(C(S), Z)$ such that $K(C([0, 1]), Z)$ corresponds to a subspace of $K(C(S), Z)$. It follows then from Lemma 3 that $K(C(S), Z)$ is not complemented in $W(C(S), Z)$.

In answer to a question raised by Arterburn and Whitley in [1], where they asked whether $K(l^\infty, c_0)$ is complemented in $B(l^\infty, c_0)$, we have the following corollary, though an independent proof has been given in [9].

COROLLARY 10. *$K(l^\infty, c_0)$ is not complemented in $B(l^\infty, c_0)$.*

Proof. Since l^∞ can be identified with $C(\beta N)$ and βN is not scattered, the desired result follows immediately from Theorem 9.

Finally, to complete the examples studied by Tong and Wilken in [9], we consider the space of bounded linear operators $B(C(S), Z)$, $Z = c_0$ or $Z = l^p$, $1 \leqq p \leqq \infty$. Suppose S is scattered; then, since $C(S)^*$ is isometric to $l^1(S)$, $K(C(S), Z) = W(C(S), Z)$. (Recall that weak convergent sequences in $l^1(S)$ are norm convergent [3, p. 33] and a bounded linear operator T is compact if and only if T^* has the same property.) But it is well known that $W(C(S), Z) = B(C(S), Z)$ for an arbitrary Banach space Z containing no subspace isomorphic to c_0 [6], hence $K(C(S), Z) = B(C(S), Z)$ for $Z = l^p, 1 \leqq p < \infty$. When $Z = c_0$ or $Z = l^\infty$, then since $C(S)$ contains a complemented copy of c_0, $K(C(S), Z)$ is not complemented in $B(C(S), Z)$. If S is not scattered and $Z = c_0$ or $Z = l^\infty$, it is clear that $K(C(S), Z)$ is not complemented in $W(C(S), Z)$ (and hence not complemented in $B(C(S), Z)$) by Theorem 9. For $Z = l^p, 2 \leqq p \leqq \infty$, $K(C(S), Z)$ is not complemented in $B(C(S), Z)$ by the main theorem in [9] and the fact that there exists a noncompact operator from $C(S)$ into Z as indicated there. When $Z = l^p, 1 \leqq p < 2$, the question of the existence of a noncompact operator was left open in the same reference; the answer is no, as follows from a factorization theorem:

THEOREM 11. *Every bounded linear operator from an $\mathscr{L}^\infty$-space into $l^p, 1 \leqq p < 2$ is compact.*

Proof. By Theorem 5.2 in [4], every bounded linear operator from an $\mathscr{L}^\infty$-space into $l^p, 1 \leqq p < 2$ can be factorized through a Hilbert space. Indeed, since l^p is separable, the Hilbert space H can further be chosen to be l^2. For if $T: H \rightarrow l^p$, then T can be factorized as $H \xrightarrow{\Phi} H/N \xrightarrow{\hat{T}} l^p$, where N is the null space of T, Φ is the quotient map and $\hat{T}$ is the induced injective map. Now since $\hat{T}^*: l^q \rightarrow H/N$ has a weak* dense image (hence weakly dense, since H/N is reflexive), H/N must be separable, which implies that H/N is isomorphic to l^2. The desired result then follows from the fact that every bounded linear operator from l^2 into $l^p, 1 \leqq p < 2$ is compact.

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Received August 15, 1973 and in revised form January 21, 1974. The author is grateful to Professor J. J. Schäffer for his help during the preparation of this paper and to the referee for a suggestion simplifying the proof of Theorem 6.

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