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SPECTRAL PROPERTIES OF LOCALLY HOLOMORPHIC VECTOR-VALUED FUNCTIONS

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This paper deals with spectral properties of commutative locally holomorphic Banach algebra valued functions. One of the main concepts is that of a spectral set of such a function. This concept, which is due to L. Mittenthal, extends that of a spectral set of a single Banach algebra element. It will be shown that the spectral idempotent associated with a nonvoid spectral set is nonzero. This result is a generalization of a well-known theorem in ordinary spectral theory. It will be used to prove a correctly stated but incorrectly proven theorem of L. Mittenthal.

We investigate spectral properties of a commutative locally holomorphic function F defined on an open subset of the complex plane and with values in a complex Banach algebra B. In particular we will be dealing with two concepts which were introduced by L. Mittenthal in his dissertation [4] (see also [5]).

The first concept is that of a spectral set (i.e., a separating singular subset in terms of [4] and [5]) of F. We will show (Theorem 4) that the spectral idempotent associated with F and a (nonvoid) spectral set of F is nonzero. This result, which extends a well-known theorem in ordinary spectral theory (see [3], §5.6), seems to be new.

The second concept is that of the spectral resultant (i.e., the root operator in terms of [4] and [5]) of F and a spectral set S of F. This resultant r is an element of the Banach algebra pBp. Here p denotes the spectral idempotent associated with F and S. Our second main result (Theorem 7) shows that S is precisely the spectrum of r relative to pBp. This also extends a well-known result in ordinary spectral theory (see [3], §5.6). Further, we will prove (Theorem 9) a generalization of the spectral mapping theorem (see [3], §5.3).

For the case when B is the Banach algebra of all bounded linear operators on a complex Banach space, Mittenthal has results similar to those mentioned in the preceding paragraph (see [4], Theorems 2-4 and 2-6, and [5], Theorem 9 and Corollary 10). However, his proofs do not seem to be quite correct. In our argument, Theorem 4, cited above, plays a crucial role.

1. Preliminaries. In this section we present some definitions and notations. The symbol $\boldsymbol{C}$ denotes the complex plane. The closure of a subset V of $\boldsymbol{C}$ is denoted by $\bar{V}$. We shall often use the concept of a Cauchy domain. For the definition of this notion, we refer to [6], § 5.6. The (positively oriented) boundary of a Cauchy domain D is denoted by ∂D.

The domain of a function f will be denoted by $\Delta(f)$. A Banach algebra valued function g is said to be *commutative* if

$$g(\lambda)g(\mu) = g(\mu)g(\lambda) \qquad (\lambda, \mu \in \Delta(g)) .$$

We shall freely use the standard notions concerning locally holomorphic vector-valued functions. For a fairly complete survey of these notions we refer to [2], § III.14.

Let F be a locally holomorphic function defined on an open subset Δ of $\boldsymbol{C}$ and with values in a complex Banach algebra B with unit element e. We do not require the norm of e to be one (cf. [3], § 1.15).

The set $R(F)$ of all $\lambda \in \Delta$ such that $F(\lambda)$ is regular in B is called the *resolvent set* of F. It is an open subset of $\boldsymbol{C}$. The function F^{-1} defined by

$$F^{-1}(\lambda) = F(\lambda)^{-1} \qquad (\lambda \in R(F))$$

is called the *resolvent* of F. It is a locally holomorphic function with values in B. The set $S(F)$ of all $\lambda \in \Delta$ such that $F(\lambda)$ is singular in B is called the *spectrum* of F. Observe that

$$S(F) = \Delta \backslash R(F) ,$$

and that $R(F)$ is closed in the relative topology of Δ.

By Q_F we denote the function given by

$$Q_F(\lambda, \mu) = \begin{cases} \dfrac{F(\lambda) - F(\mu)}{\lambda - \mu} & (\lambda, \mu \in \Delta; \lambda \neq \mu) , \\ F'(\lambda) & (\lambda = \mu \in \Delta) . \end{cases}$$

Here F' denotes, as usually, the derivative of F. A subset S of $S(F)$ is called a *spectral set* of F if the following three conditions are satisfied:

(i) S is both open and closed in the relative topology of $S(F)$;

(ii) S is a nonvoid compact subset of $\boldsymbol{C}$;

(iii) $Q_F(\lambda, \mu)$ is regular for all $\lambda, \mu \in S$.

This notion corresponds with Mittenthal's concept of a separating singular subset.

By way of illustration, we consider the special case when

$$(*) \qquad F(\lambda) = \lambda e - t \qquad (\lambda \in \boldsymbol{C}) ,$$

where t is some element of B. Then

$$S(F) = \sigma(t)\ , \quad R(F) = \rho(t)\ , \quad F^{-1} = R(\cdot\,; t)\ .$$

Here $\sigma(t)$, $\rho(t)$, and $R(\cdot\,; t)$ denote, as usually, the spectrum, resolvent set and resolvent of t (cf. [3], Definition 4.7.1). Further, the spectral sets of F are precisely the spectral sets of t (cf. [3], Definition 5.6.1). This justifies our terminology.

2. Spectral idempotents. In the following S denotes a spectral set of a commutative locally holomorphic function F defined on an open subset $\varDelta$ of $\boldsymbol{C}$ and with values in a complex Banach algebra B with unit element e. Using methods of Mittenthal, we shall introduce an "operational calculus". Further we shall define the spectral idempotent p associated with F and S. The main result of this section is that p is nonzero.

Let $\mathscr{F}$ be the set of all complex-valued functions f such that

(i) $\varDelta(f)$ is an open neighborhood of S;

(ii) f is locally holomorphic.

Let $\mathscr{G}$ be the set of all functions g with values in B such that

(i) $\varDelta(g)$ is an open neighborhood of S;

(ii) g is locally holomorphic;

(iii) $g(\lambda)F(\mu) = F(\mu)g(\lambda)$ for all $\lambda \in \varDelta(g)$ and $\mu \in \varDelta$.

In $\mathscr{F}$ and $\mathscr{G}$ we define algebraic operations—scalar multiplication, addition and multiplication—in an obvious way. We shall now define for each function h, which belongs either to $\mathscr{F}$ or to $\mathscr{G}$, an element $F_h \in B$ in such a way that the mappings $h \to F_h$ $(h \in \mathscr{F})$ and $h \to F_h$ $(h \in \mathscr{G})$ preserve the algebraic operations. The definition is

$$F_h = \frac{1}{2\pi i}\int_{\partial D} h(\lambda)F'(\lambda)F^{-1}(\lambda)d\lambda\ ,$$

where D is any bounded Cauchy domain such that

$$S \subset D \subset \bar{D} \subset \varDelta(h) \cap [\varDelta \backslash \{S(F) \backslash S\}]\ .$$

Since $\varDelta(h) \cap [\varDelta \backslash \{S(F) \backslash S\}]$ is an open neighborhood of the compact set S, there do exist bounded Cauchy domains of the required sort. It follows from Cauchy's theorem (see [2], § III.14) that the value of the above integral is independent of the choice of D. Thus, F_h is well-defined (cf. [6], § 5.6). The following theorem is essentially due to Mittenthal. The proof, which is similar to that of [4], Theorem 1–3 (also [5], Theorems 1 and 2), will be omitted.

THEOREM 1. *Let* $\alpha \in \boldsymbol{C}$ *and either* $f, g \in \mathscr{F}$, *or* $f, g \in \mathscr{G}$. *Then*

(i) $F_{\alpha f} = \alpha F_f$;

(ii) $F_{f+g} = F_f + F_g$;

(iii) $F_{fg} = F_f F_g$.

COROLLARY 2. *Let $p \in B$ be given by*

$$p = \frac{1}{2\pi i} \int_{\partial D} F'(\lambda)F^{-1}(\lambda)d\lambda ,$$

where D is any bounded Cauchy domain such that

$$S \subset D \subset \bar{D} \subset \Delta \backslash [S(F) \backslash S] .$$

Then p is an idempotent.

The element $p \in B$ defined in Corollary 2 plays a crucial role in this paper. It is called the *spectral idempotent associated with F and S.* Suppose that F is as in formula $(*)$ of § 1. Then p is the spectral idempotent associated with t and the spectral set S of t (cf. [3], Theorem 5.6.1). This justifies our terminology. Furthermore, we note that, in this case, $F_h = ph(t) = h(t)p$ for all $h \in \mathscr{F}$. For the definition of $h(t)$ we refer to [3], Theorem 5.2.4 (see also [6], § 5.6).

For the proof of the next theorem, containing the main result of this section, we need a lemma.

LEMMA 3. *Let D be a bounded Cauchy domain such that $\bar{D} \subset \Delta$ and $\partial D \subset R(F)$. Suppose that*

$$\int_{\partial D} F'(\lambda)F^{-1}(\lambda)d\lambda = 0 .$$

Then D is a subset of $R(F)$.

Proof. Since F is commutative, the set $\{F(\lambda) \mid \lambda \in \Delta\}$ is contained in a maximal commutative subset A of B. Observe that A is a closed commutative subalgebra of B with unit element e. An element of A is regular in A with inverse y if and only if it is regular in B with inverse y. Hence, without loss of generality, we may assume B to be commutative.

From the Gelfand representation theory (see [3], §§ 4.13 and 4.14) we know that an element $b \in B$ is regular in B if and only if $\beta(b) \neq 0$ for each (nonzero) multiplicative linear functional β on B. Let β be such a functional and put $f = \beta \circ F$. Then f is a locally holomorphic complex-valued function and $f' = \beta \circ F'$. For $\lambda \in R(F)$ we have $f(\lambda) \neq 0$ and $f(\lambda)^{-1} = \beta(F^{-1}(\lambda))$. It is easy to verify that

$$\int_{\partial D} \frac{f'(\lambda)}{f(\lambda)} d\lambda = \beta\Big(\int_{\partial D} F'(\lambda)F^{-1}(\lambda)d\lambda\Big) = \beta(0) = 0 .$$

By a well-known result from complex analysis (see [1], Ch. III, § 4, Satz 16), this implies that $\beta(F(\lambda)) = f(\lambda) \neq 0$ for all $\lambda \in D$, and the proof is complete.

THEOREM 4. *The spectral idempotent p associated with F and S is nonzero.*

Proof. Suppose that $p = 0$. Then

$$\int_{\partial D} F'(\lambda)F^{-1}(\lambda)d\lambda = 2\pi i p = 0 ,$$

where D is as in Corollary 2. By Lemma 3, this implies that $D \subset R(F)$. Consequently $S \subset R(F)$. But $S \subset S(F)$ too. It follows that $S \subset R(F) \cap S(F) = \emptyset$. This contradicts the fact that, by definition, a spectral set is nonvoid.

3. The spectral resultant. In this section we shall define the spectral resultant r of F and S. Our main result is that r is an element of the complex Banach algebra pBp whose spectrum (relative to pBp) is precisely S. Further, we shall prove a generalization of the spectral mapping theorem.

Since p is a nonzero idempotent (see Theorem 4), pBp is a closed subalgebra of B with unit element p. The resolvent set, spectrum and resolvent of an element $x \in pBp$ relative to pBp will be denoted by $\rho_p(x)$, $\sigma_p(x)$, and $R_p(\cdot; x)$ respectively. An element $x \in B$ belongs to pBp if and only if $x = px = xp(= pxp)$. As an easy consequence of Theorem 1 we have that $F_h \in pBp$ for each h which belongs either to $\mathscr{F}$ or to $\mathscr{G}$. In particular, the element $r \in B$, given by

$$r = \frac{1}{2\pi i} \int_{\partial D} \lambda F'(\lambda)F^{-1}(\lambda)d\lambda ,$$

where D is any bounded Cauchy domain such that

$$S \subset D \subset \bar{D} \subset \Delta \backslash [S(F) \backslash S] ,$$

belongs to pBp. It is called the *spectral resultant of F and S.* This notion corresponds with Mittenthal's concept of the root operator. If F is as in formula $(*)$ of § 1, then p is the spectral idempotent associated with t and the spectral set S of t, $r = tp = pt$ and $\sigma_p(r) = S$ (see the proof of [3], Theorem 5.6.1). We shall prove that the last equality holds in general.

LEMMA 5. *Let $\mu \in \boldsymbol{C} \backslash S$. Then $\mu \in \rho_p(r)$ and*

$$R_p(\mu; r) = \frac{1}{2\pi i} \int_{\partial D} \frac{F'(\lambda)F^{-1}(\lambda)}{\mu - \lambda} d\lambda ,$$

where D is any bounded Cauchy domain such that

$$S \subset D \subset \bar{D} \subset [\boldsymbol{C} \backslash \{\mu\}] \cap [\Delta \backslash \{S(F) \backslash S\}] .$$

Proof. The proof is similar to that of [4], Theorem 2-2 (cf. also the first part of the proof of [5], Theorem 9). Define $g: \boldsymbol{C} \rightarrow \boldsymbol{C}$ and $h: \boldsymbol{C}\backslash\{\mu\} \rightarrow \boldsymbol{C}$ by $g(\lambda) = \mu - \lambda$ and $h(\lambda) = (\mu - \lambda)^{-1}$. Clearly, both g and h belong to $\mathscr{F}$. By Theorem 1, we have $F_g F_h = F_h F_g = p$ and $F_g = \mu p - r$. Thus $\mu p - r$ is regular in pBp with inverse F_h. This proves the lemma.

LEMMA 6. $\sigma_p(r) = \{\lambda \in S \mid pF(\lambda)$ *singular in* $pBp\}$.

Proof. The proof is similar to that of [4], Theorem 2-4 (cf. also the second part of the proof of [5], Theorem 9). From Lemma 5 we know that $S \subset \sigma_p(r)$. Therefore, it suffices to show that an element $\mu \in S$ belongs to $\sigma_p(r)$ if and only if $pF(\mu)$ is singular in pBp.

Let $\mu \in S$. Using the function Q_F, which was introduced in § 1, we define the function $Q: \varDelta \rightarrow B$ by $Q(\lambda) = Q_F(\lambda, \mu)$. It is not difficult to prove that Q belongs to the set $\mathscr{G}$. Since S is a spectral set of F, we have $S \subset R(Q)$. It follows that the resolvent $P(= Q^{-1})$ of the function Q belongs to $\mathscr{G}$ too. Applying Theorem 1, we obtain $F_P F_Q = F_Q F_P = p$. Hence F_Q is regular in pBp.

Clearly, $F \in \mathscr{G}$ and $F(\mu) = F(\lambda) + (\mu - \lambda)Q(\lambda)$ for all $\lambda \in \varDelta$. Using Theorem 1, we find $pF(\mu) = F_F + (\mu p - r)F_Q$. It follows from Cauchy's theorem that $F_F = 0$. So $pF(\mu) = (\mu p - r)F_Q = F_Q(\mu p - r)$. Since F_Q is regular in pBp, it follows that $\mu p - r$ is singular in pBp if and only if $pF(\mu)$ is singular in pBp. This proves the lemma.

The next theorem contains the main result of this section. Mittenthal has a similar result (cf. [4], Theorem 2-4 and [5], Theorem 9). His proof, however, is not quite correct. In fact, Mittenthal only proved what we have called Lemma 6. Our argument is based on Theorem 4.

THEOREM 7. $\sigma_p(r) = S$.

Proof. In view of Lemma 6 it is sufficient to prove that $pF(\mu)$ is singular in pBp for all $\mu \in S$. The case $p = e$ is trivial. Therefore, we may assume $p \neq e$.

Put $q = e - p$. Then q is a nonzero idempotent and qBq is a closed subalgebra of B with unit element q. From the definition of p it is clear that $F(\lambda)$ commutes with p and q for all $\lambda \in \varDelta$. Hence $pF(\lambda) \in pBp$ and $qF(\lambda) \in qBq$ for all $\lambda \in \varDelta$. By F_p and F_q we denote the functions, with domain $\varDelta$, given by $F_p(\lambda) = pF(\lambda)$ and $F_q(\lambda) = qF(\lambda)$. Observe that F_p is a commutative locally holomorphic function with values in the complex Banach algebra pBp. Similarly, F_q is a commutative locally holomorphic function with values in qBq.

Let $S_p(F_p)$ denote the spectrum of F_p (relative to pBp), and let $S_q(F_q)$ denote the spectrum of F_q (relative to qBq). We have to prove that $S \subset S_p(F_p)$. Since $S \subset S(F) = S_p(F_p) \cup S_q(F_q)$, it suffices to show that $S \cap S_q(F_q) = \varnothing$.

Put $S_q = S \cap S_q(F_q)$, and suppose that S_q is nonvoid. Then S_q is a spectral set of F_q. The spectral idempotent associated with F_q and S_q is equal to $qp = 0$. This contradicts Theorem 4, and the proof is complete.

Let $h \in \mathscr{F}$. The preceding theorem shows that $\sigma_p(r) = S$. Hence $\sigma_p(r) \subset \Delta(h)$. We use the symbol $h(r)_p$ to denote the element of pBp given by

$$h(r)_p = \frac{1}{2\pi i} {}_{\partial D}\!\int h(\mu) R_p(\mu; r) d\mu ,$$

where D is any bounded Cauchy domain such that

$$\sigma_p(r) = S \subset D \subset \bar{D} \subset \Delta(h) .$$

From ordinary operational calculus we know that $h(r)_p$ is well-defined (cf. [3], Theorem 5.2.4 and [6], § 5.6). It will be shown that $h(r)_p = F_h$. A similar result appears in the work of Mittenthal (see [4], pp. 42, 43, 49 and [5], pp. 126–129), but again his arguments are not quite satisfactory. We shall give a new proof.

LEMMA 8. $h(r)_p = F_h$ $(h \in \mathscr{F})$.

Proof. Let $h \in \mathscr{F}$. Choose two bounded Cauchy domains U and V such that

$$S \subset U \subset \bar{U} \subset V \subset \bar{V} \subset \Delta(h) \cap [\Delta \backslash \{S(F) \backslash S\}] .$$

Then

$$h(r)_p = \frac{1}{2\pi i} {}_{\partial V}\!\int h(\mu) R_p(\mu; r) d\mu .$$

By Lemma 5

$$R_p(\mu; r) = \frac{1}{2\pi i} {}_{\partial U}\!\int \frac{F'(\lambda)F^{-1}(\lambda)}{\mu - \lambda} d\lambda$$

for all $\mu \in \partial V$. Hence

$$h(r)_p = \left(\frac{1}{2\pi i}\right)^2 {}_{\partial V}\!\int \left[{}_{\partial U}\!\int \frac{h(\mu)}{\mu - \lambda} F'(\lambda) F^{-1}(\lambda) d\lambda \right] d\mu .$$

By changing the order of integration, we find

$$h(r)_p = \left(\frac{1}{2\pi i}\right)^2 \int_{\partial U}\left[\int_{\partial V} \frac{h(\mu)}{\mu - \lambda} d\mu\right] F'(\lambda)F^{-1}(\lambda)d\lambda .$$

Cauchy's integral formula yields that

$$h(\lambda) = \frac{1}{2\pi i} \int_{\partial V} \frac{h(\mu)}{\mu - \lambda} d\mu$$

for all $\lambda \in \partial U$. Thus

$$h(r)_p = \frac{1}{2\pi i} \int_{\partial U} h(\lambda)F'(\lambda)F^{-1}(\lambda)d\lambda .$$

By definition, the right hand side of this equation is equal to F_h, and so the proof is complete.

Combining Theorem 7, Lemma 8 and the spectral mapping theorem, we obtain the following result (cf. [4], Theorem 2-6 and [5], Corollary 10).

THEOREM 9. $\sigma_p(F_h) = h[S]$ $(h \in \mathscr{F})$.

Proof. From Lemma 8 we know that $F_h = h(r)_p$. The spectral mapping theorem (see [3], Theorem 5.3.1) yields that $\sigma_p(h(r)_p) = h[\sigma_p(r)]$. Now the desired result is immediate from Theorem 7, which says that $\sigma_p(r) = S$.

The preceding result may be viewed as a generalization of the spectral mapping theorem. To see this, take F as in formula $(*)$ of §1 and $S = \sigma(t)$.

Let L be the logarithmic derivative of F. Thus L is the function defined on $R(F)$ by

$$L(\lambda) = F'(\lambda)F^{-1}(\lambda) .$$

In view of the preceding results (in particular Theorem 1), the question arises whether L satisfies the resolvent equation. The following example shows that, in general, the answer is negative.

EXAMPLE 10. Let t be a nilpotent element of B of order of nilpotence 2. Define F on $\boldsymbol{C}$ by

$$F(\lambda) = \lambda e + \lambda^2 t .$$

Then F is holomorphic and commutative. Using the fact that $t^2 = 0$, one easily shows that $S(F) = \{0\}$ and

$$F^{-1}(\lambda) = \frac{1}{\lambda} e - t \qquad (\lambda \neq 0) .$$

Since $F'(0) = e$ is regular, we have that $\{0\}$ is a spectral set of F.

Now assume that the logarithmic derivative L of F satisfies the resolvent equation on a deleted neighborhood U of 0. Thus

$$L(\lambda) - L(\mu) = (\mu - \lambda)L(\lambda)L(\mu) \qquad (\lambda, \mu \in U) .$$

Using the expression for $F^{-1}(\lambda)$ obtained above, it is easily seen that

$$L(\lambda) = \frac{1}{\lambda}e + t \qquad (\lambda \neq 0) .$$

Substituting this in the resolvent equation, we get

$$\left(\frac{1}{\lambda}e + t\right) - \left(\frac{1}{\mu}e + t\right) = (\mu - \lambda)\left(\frac{1}{\lambda}e + t\right)\left(\frac{1}{\mu}e + t\right) \quad (\lambda, \mu \in U) .$$

It follows by a straightforward computation that

$$(\lambda^2 - \mu^2)t = 0 \qquad (\lambda, \mu \in U) .$$

But this implies that $t = 0$, which contradicts the hypothesis that the order of nilpotence of t is 2. The conclusion is that L does not satisfy the resolvent equation.

REFERENCES

1. H. C. H. Behnke and F. Sommer, *Theorie der analytischen Funktionen einer komplexen Veränderlichen*, Springer-Verlag, Berlin, 1962.
2. N. Dunford and J. T. Schwartz, *Linear Operators Part I*, Interscience, New York, 1967.
3. E. Hille and R. S. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Coll. Publ. XXXI, Providence, 1957.
4. L. Mittenthal, *Operator-valued polynomials in a complex variable, and generalizations of spectral theory*, Ph. D. Dissertation, University of California, Los Angeles, 1966.
5. ———, *Operator valued analytic functions and generalizations of spectral theory*, Pacific J. Math., **24** (1968), 119–132.
6. A. E. Taylor, *Introduction to Functional Analysis*, Wiley and Sons, New York, 1967.

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