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SPAN AND STABLY TRIVIAL BUNDLES

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E. Thomas [19] introduced the notion of span of a differentiable manifold (or of a vector bundle). The notion of span can be extended in an obvious way to PL-microbundles, topological microbundles and spherical fibrations. In the case of a vector bundle or a microbundle the dimension of the fibre will be referred to as its rank. A spherical fibration with fibre homotopically equivalent to S^{k-1} will be said to be of rank k. In this paper we study stably trivial objects of rank k over a CW-complex of dimension $\leqq k$ from each of the above collections. Then we determine the span of such stably trivial objects over CW-complexes of a "special type" yielding generalizations of the Bredon-Kosinski, Thomas theorem on the span of a closed differentiable π-manifold [3], [19]. Though originally PL-microbundles were defined only over simplicial complexes, in this paper by a PL-microbundle of rank k over a CW-complex X we mean an element of the set $[X, BPL(k)]$ of homotopy classes of maps of X into $BPL(k)$.

Throughout this paper X will denote a CW-complex and X^k will denote the k-skeleton of X. We write $\xi \in \mathrm{Vect}(X)$ $\{PL\,\mathrm{mic}(X)$, Topmic(X) or Sph$(X)\}$ to denote that ξ is a vector bundle a PL-microbundle, a topological microbundle or a spherical fibration over X. We write ξ^k to denote that ξ is of rank k. We write $R(X)$ for any one of Vect(X), $PL\,\mathrm{mic}(X)$, Topmic(X) or Sph(X). The trivial object of rank k in $R(X)$ will be denoted by $\epsilon^k_{R,X}$. We write $\xi \in R_+(X)$ to denote that ξ is orientable. We write O^k_X, θ^k_X, ϵ^k_X and k_X respectively for the *trivial* vector bundle, PL-microbundle, topological microbundle and spherical fibration of rank k over X.

Section 2 is concerned with stably trivial elements $\xi^k \in R(X)$ when $\dim X \leqq k$. In Section 3 we introduce the notion of a Gauss map for a $\xi \in R(X)$. If $\xi^k \in R(X)$ is stably trivial, $\dim X \leqq k$ and $R \neq$ Topmic we prove the existence of a Gauss map for ξ. If $R =$ Topmic the same result is true whenever $k \neq 4$. In Section 4 we prove the main result of this paper (Theorem 4.3). An an immediate consequence of this theorem the analogue of Bredon-Kosinski, Thomas theorem could be derived in all the categories Diff, PL, Top or Poincare Complexes with "obvious" exceptions.

1. The kernel of $\pi_k(B_k) \to \pi_k(B_{k+1})$. We write B_k for any one of $BSO(k)$, $BPL^+(k)$, $B\mathrm{Top}^+(k)$ or $BSH(k)$. For our later results we need information about the kernel of $\pi_k(B_k) \rightarrow {}_k(B_{k+1})$. When $B_k \neq B\mathrm{Top}^+(k)$ the kernel of $\pi_k(B_k) \rightarrow \pi_k(B_{k+1})$ is well-known. Using the results of Kirby-Siebenmann **[13]** and Lashof-Rothenberg **[16]** we get information about the kernel when $B_k = B\mathrm{Top}^+(k)$, for $k \neq 4$. Let $T_{S^k}, t_{S^k}, \tau_{S^k}$ and λ_{S^k} denote the tangent vector bundle, tangent PL-microbundle, tangent microbundle and the tangent spherical fibration of S^k. Let

$$K_k = ker\, \pi_k(BSP(k)) \rightarrow \pi_k(BSO(k)),$$
$$C_k = ker\, \pi_k(BPL^+(k)) \rightarrow \pi_k(BPL^+(k+1)),$$
$$K_k = ker\, \pi_k(B\mathrm{Top}^+(k)) \rightarrow \pi_k(B\mathrm{Top}^+(k+1))$$

and

$$K''_k = ker\, \pi_k(BSH(k)) \rightarrow \pi_k(BSH(k+1)).$$

It is well-known that the obvious map $\pi_k(BSO(k) \rightarrow \pi_k(BSH(k))$ carries K_k isomorphically onto K''_k and that

$$(1) \qquad K_k \simeq K''_k \simeq \begin{cases} Z \text{ if } k \text{ is even} \\ O \text{ if } k = 1, 3 \text{ or } 7 \\ Z_2 \text{ if } k \text{ is odd and } \neq 1, 3, 7. \end{cases}$$

with T_{S^k} (respy λ_{S^k}) as generator.

According to a result of W. M. Hirsch the map $\pi_k(BSO(k)) \rightarrow \pi_k(BPL^+(k))$ carries K_k onto C_k. A reference for this is **[7]**. Since the composite map $K_k \rightarrow C_k \rightarrow K''_k$ is an isomorphism, it follows that

$$(2) \qquad K_k \simeq C_k \text{ and that } t_{S^k} \text{ generates } C_k.$$

PROPOSITION 1.1. *For $k \neq 4$, K'_k is cyclic and is generated by τ_{S^k}.*

$$(3) \qquad \textit{Moreover} \quad K'_k \simeq \begin{cases} Z \textit{ if } k \textit{ is even and } \neq 4 \\ O \textit{ if } k = 1, 3 \text{ or } 7 \\ Z_2 \textit{ if } k \textit{ is odd and } \neq 1, 3, 7. \end{cases}$$

Proof. Since the composite map $K_k \rightarrow K'_k \rightarrow K''_k$ is an isomorphism it follows that $K_k \rightarrow K'_k$ is an injection for all k.

Let $k \geqq 5$. In the following commutative diagram where the horizontal rows are exact and the vertical maps are the obvious ones,

$$\begin{array}{ccccc}
O \to & K_k & \to \pi_k(BSO(k)) & \to & \pi_k(BSO(k+1)) \\
& \text{onto} \downarrow & \downarrow & & \downarrow \\
O \to & C_k & \to \pi_k(BPL^+(k)) & \to & \pi_k(BPL^+(k+1)) \\
& \downarrow & \downarrow \text{onto} & & \downarrow \\
O \to & K_k & \to \pi_k(B\mathrm{Top}^+(k)) & \to & \pi_k(B\mathrm{Top}^+(k+1))
\end{array}$$

DIAGRAM 1

the map $\pi_k(BPL^+(k)) \to \pi_k(B\mathrm{Top}^+(k))$ is onto and $\pi_k(BPL^+(k+1)) \to \pi_k(B\mathrm{Top}^+(k+1))$ for $k \geqq 5$ by **[13]** or **[16]**. As already observed $K_k \to C_k$ is onto according to a result of M. W. Hirsch **[7]**. Standard diagram chasing using Diagram 1 yields $K_k \to K'_k$ is onto for $k \geqq 5$.

For $k \leqq 3$ it is known that $SO(k) \to \mathrm{Top}^+(k)$ is a homotopy equivalence **[15]**. Hence for $k \leqq 2$ we have $K_k \simeq K'_k$. When $k = 3$ we have $O = \pi_2(SO(3)) \simeq \pi_3(BSO(3)) \simeq \pi_3(B\mathrm{Top}^+(3))$. Hence $K_3 = O = K'_3$. This completes the proof of 1.1.

2. Stably trivial elements $\xi \in R(X)$. Suppose $\dim X \leqq k$ and $\xi^{k+1} \in R(X)$ is stably trivial. Then for $R \neq$ Topmic it is known that $\xi^{k+1} \simeq \epsilon^{k+1}_{R,X}$. This is actually a consequence of

$$(4) \qquad \pi_i(B_{k+1}, B_k) = 0 \quad \text{for} \quad i \leqq k$$

whenever $B_k = BSO(k)$, $BPL^+(k)$ or $BSH(k)$. For $B_k = BSH(k)$, 4 is due to I. M. James **[10]**. When $B_k = BPL^+(k)$ it is due to Haefliger and Wall **[7]**. We write B_∞ to denote one of BSO, BPL^+, $B\mathrm{Top}^+$ or BSH.

LEMMA 2.1. *Let* $\dim X \leqq k$ *and* $\xi^{k+1} \in \mathrm{Topmic}(X)$ *be stably trivial. Then* $\xi^{k+1} \simeq \epsilon^{k+1}_X$ *whenever* $k \neq 3$.

Proof. From Kirby-Siebenmann **[13]** or Lashof-Rothenberg **[16]** we have $\pi_i(B\mathrm{Top}^+(l+1), B\mathrm{Top}^+(l)) = 0$ for $i \leqq l$ and $l \geqq 5$. As an immediate consequence of this and obstruction theory one gets $[X, B\mathrm{Top}^+(k+1)] \to [X, B\mathrm{Top}^+]$ to be an isomorphism for $k \geqq 4$.

Now let $k \leqq 2$. Since $\pi_i(B\mathrm{Top}^+, BPL^+) \simeq \pi_{i-1}(\mathrm{Top}^+, PL^+) = 0$ for $i \neq 4$, we see that $[X, BPL^+] \to [X, B\mathrm{Top}^+]$ is an isomorphism. Also $SO(k+1) \to PL^+(k+1)$ and $PL^+(k+1) \to \mathrm{Top}^+(k+1)$ are homotopy equivalences for $k \leqq 2$. Hence each of the maps $[X, BSO(k+1)] \to [X, BPL^+(k+1)]$, $[X, BPL^+(k+1)] \to [X, B\mathrm{Top}^+(k+1)]$ is an isomorphism. From 4 we see that $[X, BPL^+(k+1)] \to [X, BPL^+]$ is an isomorphism. Now Diagram 2 below immediately gives $[X, B\mathrm{Top}^+(k+1)] \to [X, B\mathrm{Top}^+]$ an isomorphism.

$$\begin{array}{ccc} [X, BPL^+(k+1)] & \xrightarrow{\simeq} & [X, BPL^+] \\ \simeq \downarrow & & \simeq \downarrow \\ [X, B\mathrm{Top}^+(k+1)] & \longrightarrow & [X, B\mathrm{Top}^+] \end{array}$$

DIAGRAM 2

This completes the proof of Lemma 2.1.

PROPOSITION 2.2. *Let X be a CW-complex of dimension $\leqq k$ where $k = 3$ or 7. Let $\xi^k \in R_+(x)$ be such that $\xi^k \mid X^{k-1} \simeq \epsilon^k_{R,X^{k-1}}$. Then $\xi \simeq \epsilon^k_{R,X}$ whenever $R \neq$* Sph.

Proof. We have

$$O = \pi_3(BSO(3)) \simeq \pi_3(BPL^+(3)) \simeq \pi_3(B\mathrm{Top}^+(3)) \tag{5}$$

From results in Section 1 we see that *ker* $\pi_7(B_7) \to \pi_7(B_8)$ is zero. From $\pi_i(B_{k+1}, B_k) = 0$ for $i \leqq k$ and $k \geqq 5$ it now follows that $\pi_7(B_7) \to \pi_7(B_8)$ and $\pi_7(B_8) \to \pi_7(B_\infty)$ are isomorphisms. From Bott **[2]** $\pi_6(SO) = 0$. From Hirsch and Mazur **[8]**, **[9]** $\pi_7(BPL^+, BSO) \simeq \Gamma_6$ the group of concordance classes of smooth structures on S^6. It is known **[12]** that $\Gamma_6 = 0$. Combining these with the result $\pi_7(B\mathrm{Top}^+, BPL^+) = 0$ of Kirby-Siebenmann we get

$$O = \pi_7(BSO(7)) \simeq \pi_7(BPL^+(7)) \simeq \pi_7(B\mathrm{Top}^+(7)) \tag{6}$$

Let $\mu : X^{k-1} \to X$ denote the inclusion. If $X = X^{k-1} \cup_{i \in J} e^k_i$ we have a cofibration $\mu : X^{k-1} \to X$ with cofibre $\vee_{i \in J} S^k_i$. Let $c : X \to \vee_{i \in J} S^k_i$ be got by collapsing X^{k-1} to a point. In the Puppe exact sequence

$$\left[\bigvee_{i \in J} S^k_i, B_k\right] \xrightarrow{c^*} [X, B_k] \xrightarrow{\mu^*} [X^{k-1}, B_k]$$

we have $\mu^*(\xi^k) = 0$, since $\xi^k \mid X^{k-1}$ is trivial. Hence $\exists$ an $x [\vee_{i \in J} S^k_i, B_k]$ such that $c^*(x) = \xi^k$. By 5 and 6, $\pi_k(B_k) = 0$ for $k = 3$ and 7, whenever $B_k \neq BSH(k)$. Hence $x = 0$, which in turn yields $\xi^k = 0$ in $[X, B_k]$.

REMARKS.

2.3. If $F(k)$ denotes the subspace of $SH(k+1)$ consisting of base point preserving maps it is known **[10]** that

$$\pi_3(BSH(3)) \simeq \pi_2(SH(3)) \simeq \pi_2(F(3)) \simeq \pi_5(S^3) \simeq Z_2$$

and that

$$\pi_7(BSH(7)) \simeq \pi_6(SH(7)) \simeq \pi_6(F(7)) \simeq \pi_{13}(S^7) \simeq Z_2.$$

Let $k = 3$ or 7. We have a CW structure X on S^k such that $X^{k-1} = *$ (base point). If $\xi^k \in \mathrm{Sph}(X)$ is represented by the nonzero element of $[X, BSH(k)] \simeq \pi_{k-1}(SH(k)) \simeq Z_2$ then clearly $\xi^k \,|\, X^{k-1}$ is trivial, but ξ^k itself is not trivial.

2.4. Any $\xi^1 \in R_+(X)$ is trivial whatever be the dimension of X.

PROPOSITION 2.5. *Let $\eta^k \in R(X)$ be stably trivial and* $\dim X \leqq k$. *Then*

$$\eta^k \oplus \epsilon^1_{R,X} \simeq \epsilon^{k+1}_{R,X}.$$

Proof. As commented already, this is well-known when $R \neq$ Topmic. For $R =$ Topmic and $k \neq 3$ this is an immediate consequence of Lemma 2.1. Let now $k = 3$. Then $\eta^3 | X^2$ is stably trivial. From Lemma 2.1 applied to $\eta^3 | X^2$ we get $\eta^3 | X^3 \simeq \epsilon^3_{R,X^2}$. Now proposition 2.2 yields $\eta^3 \simeq \epsilon^3_{R,X}$. Hence $\eta \oplus \epsilon^1_{R,X} \simeq \epsilon^3_{R,X}$.

3. Gauss maps.

DEFINITION 3.1. Let $\xi^k \in R(X)$. A map $f\colon X \to S^k$ will be called a Gauss map for ξ if $\xi \simeq f^*(\tau_{R,S^k})$ in $R(X)$, where $\tau_{R,S^k} = T_{S^k}, t_{S^k}, \tau_{S^k}$ or λ_{S^k} according as $R =$ Vect, PL mic, Topmic or Sph.

When $\xi \in R(X)$ admits of a Gauss map then necessarily ξ is stably trivial. The main result of this section is the following:

THEOREM 3.2. *Let* $\dim X \leqq k$ *and* $\xi^k \in R(X)$ *stably trivial. There exists a Gauss map for ξ whatever be k if $R \neq$* Topmic *and for $k \neq 4$ if $R =$* Topmic.

In the proof of this theorem we will be making use of the following lemma.

LEMMA 3.3. *Let Y be a CW complex of dimension $\leqq k-1$. Then $[\Sigma Y, B_k] \to [\Sigma Y, B_{k+1}]$ is onto whatever be k if $B_k \neq B\mathrm{Top}^+(k)$, and for $k \neq 3,4$ if $B_k = B\mathrm{Top}^+(k)$.*

Proof. Let $Y = Y^{k-2} \cup_{\nu \in J} e^{k-1}_\nu$, $i\colon Y^{k-2} \to Y$, $j\colon B_k \to B_{k+1}$ the inclusion maps and $h\colon Y \to \vee_{\nu \in J} S^{k-1}$ got by collapsing Y^{k-2} to a point. Lemma 3.3 follows immediately by diagram chasing using the following commutative diagram coming from Puppe exact sequences where $(\Sigma h)^*$, $(\Sigma i)^*$ and all the j_* are group homomorphisms.

$$\begin{array}{ccccccc}
\left[\Sigma \bigvee_{\nu\in J} S^{k-1}, B_k\right] & \xrightarrow{(\Sigma h)^*} & [\Sigma Y, B_k] & \xrightarrow{(\Sigma i)^*} & [\Sigma(Y^{k-2}), B_k] & \xrightarrow{\partial} & \left[\bigvee_{\nu\in J} S^{k-1}, B_k\right] \\
\textit{onto } a \downarrow j_* & & b \downarrow j_* & & c \downarrow j_* & & d \downarrow j_* \\
\left[\Sigma \bigvee_{\nu\in J} S^{k-1}, B_{k+1}\right] & \xrightarrow{(\Sigma h)^*} & [\Sigma Y, B_{k+1}] & \longrightarrow & [\Sigma(Y^{k-2}), B_{k+1}] & F\!\!\longrightarrow & \left[\bigvee_{\nu\in J} S^{k-1}, B_{k+1}\right]
\end{array}$$

DIAGRAM 3

Here the maps j_* marked by c and d are isomorphisms under the conditions in Lemma 3.3 and the j_* marked by a is onto.

Proof of Theorem 3.2. Let $X = X^{k-1} \cup_{\gamma\in J} e^k_\gamma$, $\mu: X^{k-1} \to X$ the inclusion and $c: X \to \bigvee_{\gamma\in J} S^k$ the map collapsing X^{k-1} to a point. Consider the following diagram where the horizontal rows are part of Puppe exact sequences of the confibration μ.

$$\begin{array}{ccccccc}
[\Sigma(X^{k-1}), B_k] & \xrightarrow{\partial} & \left[\bigvee_{\gamma\in J} S^k, B_k\right] & \xrightarrow{c^*} & [X, B_k] & \xrightarrow{\mu^*} & [X^{k-1}, B_k] \\
\downarrow j_* & & \downarrow j_* & & \downarrow j_* & & \downarrow j_* \\
[\Sigma(X^{k-1}, B_{k+1}] & \xrightarrow{\partial} & \left[\bigvee_{\gamma\in J} S^k, B_{k+1}\right] & \xrightarrow{c^*} & [X, B_{k+1}] & \xrightarrow{\mu^*} & [X^{k-1}, B_{k+1}]
\end{array}$$

DIAGRAM 4

By Lemma 2.1 we have $\mu^*(\xi^k) = 0$ in $[X^{k-1}, B_k]$ whenever $R \neq$ Topmic and $k - 1 \neq 3$. By proposition 2.5, $j_*(\xi^k) = 0$ in $[X, B_{k+1}]$. From $\mu^*(\xi) = 0$ we get an element $u \in [\vee_{\gamma\in J} S^k, B_k]$ such that $c^*(\mu) = \xi$. Then $j_*(\mu) = x \in [\vee_{\gamma\in J} S^k, B_{k+1}]$ satisfies $c^*(x) = j_*(\xi) = 0$. Hence $\exists\, b \in [\Sigma(X^{k-1}), B_{k+1}]$ such that $x^b = 0$ where x^b is got from x by the action of $[\Sigma(X^{k-1}), B_{k+1}]$ on $[\vee_{\gamma\in J} S^k, B_{k+1}]$.

By Lemma 3.3, $\exists\, a \in [\Sigma(X^{k-1}), B_k]$ such that $j_*(a) = b$ except when $R =$ Topmic and $k = 3$ or 4. Then the element $\mu' = \mu^a \in [\vee_{\gamma\in J} S^k, B_k]$ satisfies $j_*(\mu') = 0$ and $c^*(\mu') = \xi$. Identifying $[\vee_{\gamma\in J} S^k, B_k]$ with the direct product $\Pi_{\gamma\in J}\, [S^k, B_k]$, μ' corresponds to an element $(\mu')_{\gamma\in J}$ where $\mu'_\gamma \in \ker j_*: \Pi_k(B_k) \to \Pi_k(B_{k+1})$. Using 1, 2, 3 of §1 we see that $\mu'_\gamma = d_\gamma \tau_{R,S}k$ {for some $d_\gamma \in Z$ if k is even, $d_\gamma \in Z_2$ if k is odd}. Let $g_\gamma: S^k \to S^k$ be a map of degree d_γ and $\varphi: S^k \to B_k$ a classifying map for τ_{R,S^k}. Then clearly the composite map

$$\bigvee_{\gamma\in J} S^k \xrightarrow{\vee g_\gamma} \bigvee_{\gamma\in J} S^k \xrightarrow{\nabla} S^k \xrightarrow{\varphi} B_k \quad \text{represents} \quad \mu' = (\mu'_\gamma)_{\gamma\in J}.$$

From $c^*(\mu') = \xi$ it follows that $f^*(\tau_{R,S^k}) \simeq \xi$ where

$$f = \nabla \circ \left(\bigvee_{\gamma\in J} g_\gamma\right) \circ c : X \to S^k.$$

To complete the proof of Theorem 3.2 we have still to consider the case R = Topmic, $k = 3$. In this case $\xi \mid X^2$ is stably trivial of rank 3 over a 2-dimensional complex. By Lemma 2.1, $\xi \mid X^2 = \epsilon^3_{X^2}$. By Proposition 2.2, $\xi \simeq \epsilon^3_X$. Since $\tau_{S^3} \simeq \tau_{S^3}$ we have $f^*(\tau_{S^3}) \simeq \xi$. This completes the proof of Theorem 3.2.

4. Span of any $\xi \in R(X)$. We now recall the definition of span originally due to E. Thomas **[19]**.

DEFINITION 4.1. Let $\xi \in R(X)$. The span of ξ is defined to be the largest integer l with the property $\xi \simeq \epsilon^l_{R,X} \oplus \eta$ for some $\eta \in R(X)$.

In this section we will be interested in complexes of the form $X = L \cup e^k$ where $\dim L \leqq k - 1$. It is easy to see using the exact homology sequence of the pair (X, L) and the fact that $H_{k-1}(L)$ is free abelian that either $H_k(X) = 0$ or $H_k(X) \simeq Z$. If we further assume that $Ext(H_{k-1}(X), Z) = 0$ it follows from the universal co-efficient theorem that either $H^k(X) = 0$ or $H^k(X) \simeq Z$. By Hopf's classification theorem $[X, S^k] \simeq H^k(X)$. When $H_k(X) = 0$ every map $X \to S^k$ is homotopically trivial, when $H_k(X) \simeq Z$ the map $[f] \to \deg f$ provides an isomorphism of $[X, S^k]$ with l. Let $l \leqq k$ and $\pi: V_{k+1,l+1} \to S^k$ denote the map which carries any orthonormal $(l + 1)$ frame $(\vec{\nu}_1, \ldots, \vec{\nu}_{l+1})$ in $\boldsymbol{R}^{k+1}$ to the vector $\vec{\nu}_{l+1}$. We will be considering mainly complexes $X = L \cup e^k$ with $\dim L \leqq k - 1$ and satisfying the following condition:

(**) Suppose $\theta: X \to S^k$ is a map admitting of a lift $\varphi: X \to V_{k+1,l+1}$ (i.e. $\pi \circ \varphi = \theta$) and suppose $\deg \theta = 1$. Then $l \leqq \sigma_k$, where $\sigma_k = 2^{c(k)} + 8d(k) - 1$ with $k + 1 = 2^{c(k)} 16^{d(k)} b_k$, $0 \leqq c(k) \leqq 3$, $d(k) \geqq 0$ and b_k odd.

DEFINITION 4.2. Let k be an integer $\geqq 4$. A CW-complex X will be referred to as a "special complex" of dimension k

(i) $X = L \cup e^k$ with $\dim L \leqq k - 1$

(ii) $\mathrm{Ext}(H_{k-1}(X), Z) = 0$ and

(iii) condition (**) is valid whenever k is odd.

Observe that when $H_k(X) = 0$ condition (**) is emptily valid, since there are no maps $\theta: X \to S^k$ of degree 1 then.

THEOREM 4.3.

(A) *Let $\xi^2 \in R_+(X)$ with X an arbitrary CW-complex. Then span $\xi = 0$ or 2.*

(B) *Let $k = 1, 3$ or 7 and $\xi^k \in R(X)$ stably trivial with* $\dim X \leqq k$. *Then span $\xi = k$.*

(C) *Let $k \geqq 4$ and $\neq 7$, X a special complex of dimension k and $\xi^k \in R(X)$ stably trivial. Then*

(i) span $\xi = \sigma_k$ *or k whenever $R =$* Vect

(ii) *if $R = PL$*mic *or* Sph, *span $\xi = \sigma_k$ or k whenever $k \neq 15$*

(iii) *if $R =$* Topmic, *span $\xi = \sigma_k$ or k whenever $k \neq 4$ and 15.*

LEMMA 4.4. *Let X be a CW-complex of dimension $\leqq k, \xi^k$ a vector bundle, $\alpha \in R(X)$ the object in $R(X)$ underlying ξ. Let l be any integer $\leqq (k-1)/2$. Then $\alpha \simeq \beta \oplus \epsilon^l_{R,X}$ in $R(X)$ if and only if $\xi \simeq \eta \oplus O^l_X$ in* Vect(X).

Proof. Immediate consequence of a classical result of I. M. James [Proposition 1.2 in **[10]**] and obstruction theory.

LEMMA 4.5. *The span of $\tau_{R,S^k} = \sigma_k$.*

For $R =$ Vect this is a classical result of J. F. Adams **[1]**. For $R =$ Topmic this is Theorem 1.1 in **[20]**. For $R = PL$mic or Sph the proof is exactly similar to that of Theorem 1.1 in **[20]**.

LEMMA 4.6. *Let l be any integer $\leqq (k-1)/2$, $f: X \to S^k$ a Gauss map for $\alpha^k \in R(X)$ and* $\dim X \leqq k$. *Suppose $\alpha \simeq \beta \oplus \epsilon^l_{R,X}$. Then $\exists$ a map $\varphi: X \to V_{k+1,l+1}$ such that $f = \pi \circ \varphi$.*

Proof. This is an immediate consequence of Lemma 4.4 applied to the vector bundle $\xi^k = f^*(T_{S^k})$.

LEMMA 4.7. *Let X be a CW-complex of dimension k satisfying conditions* (i) *and* (ii) *of Definition* 4.2. *Suppose k is odd, $H_k(X) \neq 0$ and a Gauss map $f: X \to S^k$ for $\xi^k \in R(X)$ has odd degree. Then any map $g: X \to S^k$ of degree* 1 *is a Gauss map for ξ.*

Proof. This is an immediate consequence of the fact that $2\tau_{R,S^k} = 0$ in $\pi_k(B_k)$ whenever k is odd.

LEMMA 4.8. *Let X be a CW-complex of dimension $k \geqq 4$ and satisfying* (i) *and* (ii) *of Definition* 4.2. *Suppose k is even, a Gauss map $f: X \to S^k$ for $\xi^k \in R(X)$ has* $\deg f \neq 0$. *Then span $\xi = 0 = \sigma_k$.*

Proof. Denote the span of ξ by $\sigma(\xi)$. If $\sigma(\xi) \neq 0$ we can find a $\eta^{k-1} \in R(X)$ such that $\xi \simeq \eta \oplus \epsilon^1_{R,X}$. Since $1 \leqq (k-1)/2$, by Lemma 4.6 $\exists$ a map $\varphi: X \to V_{k+1,2}$ satisfying $\pi \circ \varphi = f$. Since $H_k(V_{k+1,2}) \simeq Z_2$ it follows that $\deg f = 0$, contradicting the assumption $\deg f \neq 0$.

LEMMA 4.9. *Let X be a CW-complex of dimension k, satisfying conditions* (i) *and* (ii) *of Definition* 4.2. *Suppose $f: X \to S^k$ is a Gauss map for $\xi^k \in R(X)$. Then $\xi^k \simeq \epsilon^k_{R,X}$ whenever one of the following holds good.*

(a) $H_k(X) = 0$

(b) $H_k(X) \neq 0$ *(hence $H_k(X) \simeq Z$) and* $\deg f = 0$

(c) $H_k(X) \neq 0, k$ *odd and* $\deg f$ *is even.*

Proof. (a) and (b) are immediate consequences of Hopf's classification theorem. (c) is immediate from $2\tau_{R,S^k} = 0$ in $\pi_k(B_k)$ whenever k is odd.

Proof of Theorem 4.3. We write $\sigma(\xi)$ for the span of ξ.

(A) If $\sigma(\xi^2) \neq 0, \xi^2 \simeq \eta \oplus \epsilon^1_{R,X}$ for some $\eta^1 \in R_+(X)$. By Remark 2.4, $\eta^1 = \epsilon^1_{R,X}$. Hence $\xi^2 \simeq \epsilon^2_{R,X}$. Thus $\sigma(\xi^2) = 2$.

(B) Immediate consequence of Theorem 3.2 and the fact $\tau_{R,S^k} \simeq \epsilon^k_{R,S^k}$ for $k = 1, 3, 7$.

(C) By Theorem 3.2, $\exists$ a Gauss map $f: X \to S^k$ for ξ. If $H_k(X) = 0$, by Lemma 4.9 (a) we get $\sigma(\xi) = k$. If $H_k(X) \neq 0$ and $\deg f = 0$, by Lemma 4.9 (b) we get $\sigma(\xi) = k$. If k is odd and deg f is even by Lemma 4.9 (c) we get $\sigma(\xi) = k$. If $k \geqq 4$ is even and $\deg f \neq 0$, by Lemma 4.8 we get $\sigma(\xi) = 0 = \sigma_k$.

Hence to complete the proof of (C) we have only to consider the case $k \geqq 5$ odd and $\neq 7$ and deg f odd. The existence of a Gauss map implies that $\sigma(\xi) \geqq \sigma_k$. By Lemma 4.7, any map $g: X \to S^k$ of deg 1 is a Gauss map for ξ. If possible let $\sigma(\xi) > \sigma_k$. For R = Vect this means that $\exists$ a map $\varphi: X \to V_{k+1,l+1}$ satisfying $\pi \circ \varphi = g$ for some $l > \sigma_k$, contradicting the validity of condition (**). Now suppose $R \neq$ Vect. For $k \geqq 5$ odd, $k \neq 7$ and 15 direct checking shows $\sigma_k + 1 \leqq (k-1)/2$. If $\sigma(\xi) > \sigma_k$ then $\xi \simeq \eta \oplus \epsilon^l_{R,X}$ with $l = \sigma_k + 1$. From Lemma 4.6 we see that $\exists \circ \varphi: X \to V_{k+1,l+1}$ such that $\pi \circ \varphi = g$, again contradicting (**).

5. Poincare complexes with $\nu_X = 0$. For any Poincare complex X let $\nu_X \in J(X)$ denote the spivak normal fibration of X. From the results of C.T.C. Wall **[21]**, it follows that any Poincare complex X of formal dimension $k \neq 2$ is of the homotopy type of a CW-complex of dimension k and that if $k \neq 3, X$ is homotopically equivalent to $L \cup e^k$ with $\dim L \leqq k-1$. The methods employed in **[5]**, **[6]** allow one to define unstable tangent spherical fibration for Poincare complexes of formal dimension $\neq 2$.

LEMMA 5.1. *Any connected Poincare complex X of formal dimension $k \geqq 4$ with $\nu_X = 0$ is of the homotopy type of a "special complex" of dimension k* (as given in Definition 4.2).

Proof. From $H_{k-1}(X) \simeq H^1(X) \simeq \mathrm{Hom}(H_1(X), Z)$ and finite generation of $H_1(X)$ we see that $H_{k-1}(X)$ is free abelian. Hence $\mathrm{Ext}(H_{k-1}(X), Z) = 0$. As already commented X is of the homotopy type of $L \cup e^k$ where $\dim L \leqq k-1$. The Thom space of the normal fibration ν_k is reducible. Since $\nu_X = 0$ it follows that the Thom space of the trivial vector bundle σ_X^{k+1} is reducible. Suppose $k \geqq 5$ is odd. By the Browder-Novikov theorem **[4]**, **[11]** it now follows that $\exists$ a closed C^∞ manifold M^k of dimension k and a homotopy equivalence $f: M^k \to X$ such that $f^*(O_X^{k+1}) = O_M^{k+1}$ is the stable normal bundle of M. This means M is a closed differentiable π-manifold. Lemma 5.1 is now an immediate consequence of Lemma 3.2 in **[3]**.

For any PL (respy topological) manifold M the PL (respy topological) span of M is defined to be the span of the PL (respy topological) tangent microbundle of M. For a Poincare complex X the spherical span of X is defined to the span of the unstable tangent spherical fibration of X. As an immediate consequence of Theorem 4.3 we get all the following results at one stroke.

THEOREM 5.2. (1) *Let M^k be a closed* Diff, *PL-or Top π-manifold of dimension k, with $k \neq 15$ in the case of a PL-manifold and $k \neq 4$ and* 15 *in the case of a topological manifold. Then the span* (respy PL-span or Top span) *of M is either σ_k or k.*

(2) *If X is a Poincare complex of formal dimension $k \neq 2$ and* 15 *with $\nu_X = 0$ in $J(X)$, then the spherical span of $X = \sigma_k$ or k.*

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