PACIFIC JOURNAL OF MATHEMATICS
Vol. 60, No. 2, 1975

BROWNIAN MOTION AND SETS OF MULTIPLICITY

ROBERT KAUFMAN

$X(t)$ is Brownian motion on the axis $-\infty < t < \infty$, with paths in R^n, $n \geqq 2$. $X(t)$ leads to composed mappings $f \circ X$, where f is a real-valued function of class $\Lambda^\alpha(R^n)$, whose gradient never vanishes. To define the class $\Lambda^\alpha(R^n)$, when $\alpha > 1$, take the integer p in the interval $\alpha - 1 \leqq p < \alpha$ and require that f have continuous partial derivatives of orders $1, \cdots, p$ and these fulfill a Lipschitz condition in exponent $\alpha - p$ on each compact set; to specify further that grad $f \neq 0$ throughout R^n, write Λ_+^α. Then a closed set T is a set of "Λ^α-multiplicity" if every transform $f(T) \subseteq R^1 (f \in \Lambda_+^\alpha)$ is a set of strict multiplicity—an M_0-set (see below). Henceforth we define $b = \alpha^{-1}$ and take S to be a closed linear set.

THEOREM 1. *In order that $X(S)$ be almost surely a set of Λ^α-multiplicity, it is sufficient that the Hausdorff dimension of S exceed b. It is not sufficient that $\dim S = b$.*

An M_0-set in R is one carrying a measure $\mu \neq 0$ whose Fourier-Stieltjes transform vanishes at infinity; the theory of M_0-sets is propounded in [**1**, p. 57] and [**8**, pp. 344, 348, 383] and Hausdorff dimension is treated in [**1**, II—III]. Theorem 1 reveals a difference between multi-dimensional Brownian motion and the linear process; for linear paths the critical point is dim $S = \frac{1}{2} b$ [**5**]. Theorem 2 below contains a sharper form of the sufficiency condition.

THEOREM 2. *Let S be a compact set, carrying a probability measure μ for which*

$$h(u) \equiv \sup \mu(x, x+u) = o(u^b) \cdot |\log u|^{-1}.$$

Then $X(S)$ is almost surely a set of Λ^α-multiplicity.

1. (Proof of Theorem 2) We can assume that S is mapped by X entirely within some fixed ball B in R^n and that all elements f appearing below are bounded in Λ^α-norm over B (defined in analogy with the norms in Banach spaces of Lipschitz functions). Moreover we can assume that all gradients fulfill an inequality $\|\nabla\| \geqq \delta > 0$ on all of B, and even on all of R^n.

(a) There is a function $\xi(u) > 0$ of u so that $\lim u^{-1}\xi(u) = +\infty$ and $h(\xi(u)) = o(u^b)\ |\log u|^{-1}$ as $u \to 0+$. In proving that all sets $f \circ X(S)$ are M_0-sets, we study integrals $\int \exp - 2\pi i y f \circ X(s) \cdot \mu(ds)$, since these are the Fourier-Stieltjes transforms of probability measures carried by $f \circ X(S)$. Our plan is to estimate the probability of an event $|\int| > \eta$ for an individual f and y, and then combine a large enough number of these inequalities to obtain a bound for *all* functions f in question. The individual estimations are obtained as in **[5**, pp. 60–61], using the independence of increments of X. To obtain a uniform estimate on the expected values, similar to that in **[5]**, we divide S into intervals of length rather larger than y^{-2}. The expected values are then integral involving the normal density in R^n, and these are handled by integration first along straight lines approximately parallel to ∇f. For each $\eta > 0$ we find

$$P\{|\int \exp - 2\pi i y f \circ X(s)\, \mu(ds)| > \eta\} < \exp - A(\eta)\psi(y) \log y \cdot y^{2b}$$

where $A(\eta) > 0$ and $\psi(y) \to +\infty$ with y.

(b) To each large y and $\eta > 0$ we shall find a determinate set $L(y)$ in Λ_+^α, with this property: there is a random number y_0, almost surely finite, and a random set S^* of μ-measure $1 - \eta$; to each function f in Λ_+^α there is a function f_1 in $L(y)$, such that $|f - f_1| \leqq \eta y^{-1}$ on $X(S^*)$—all this for $y > y_0$. Moreover $L(y)$ contains at most $\exp A'(\eta) y^{2b} \log y$ elements f_1. When $L(y)$ has been secured, we let y tend to $+\infty$ along the sequence $1, \sqrt{2}, \cdots, k^{1/2}, \cdots$ for example, and use the Borel-Cantelli Lemma to estimate the integrals involving $f_1 \in L(y)$. The properties of $L(y)$ allow us to extend our almost-sure inequalities to all of Λ_+^α.

At the corresponding stage in the treatment of linear Brownian motion, Kolmogorov's estimates of entropy in the space $\Lambda^\alpha[-1, 1]$ are exploited; an interesting aspect of the argument below is the minor role of the dimension n. Compare **[6**, Ch. 9–10].

(c) In carrying out the program of (b) we let y increase through the sequence $2^{k\alpha}$ $(k = 1, 2, 3, \cdots)$ and observe that the sets $L(2^{k\alpha})$ will serve for $2^{(k-1)\alpha} \leqq y \leqq 2^{k\alpha}$. To each $\eta > 0$ we can find a constant C_1 so large that the inequality $\|X(t)\| \leqq C_1$, $0 \leqq t \leqq 1$, is valid with $P > 1 - \frac{1}{2}\eta$. We divide the t-axis into adjacent intervals I of length 4^{-k} and write μ_k^* for the total μ-measure of those t-intervals on which $X(t)$ oscillates more than $2C_1 \cdot 2^{-k}$. By the scaling of X, and by independence of increments, we find upper bounds for the mean and variance of μ_k^*, namely $E(\mu_k^*) < \frac{1}{2}\eta$ and $\sigma^2(\mu_k^*) \leqq 0(1)h(4^{-k})$. By Chebyshev's inequality, $P\{\mu_k^* > \eta\} \leqq 0(1)h(4^{-k})$, and from $\Sigma\, h(4^{-k}) < +\infty$ we conclude that $\mu_k^* < \eta$ for large k, almost surely. The complementary intervals now form S^*, so that $X(S^*)$ is contained in $0(4^k)$ subsets of R^n, of diameter $C_1 \cdot 2^{1-k}$. (By our standing assumptions, $\|X(S^*)\| \leqq B$). Let η_1 be a small constant, depending on η and the Lipschitz constants of the functions f, and let us cover the ball $\| X \| \leqq B$ with a grid of rectangles of side $\eta_1 2^{-k}$; for large n the grid contains $< 2^{(n+1)k}$ cells. Moreover $X(S^*)$ is contained in $C_2 4^k$ of these cells, and these cells can be chosen in at most $\exp C_3 k 4^k$ different ways. For each set T_0, composed of $C_2 4^k$ cells, we construct a "matching set" $L(y, T_0) \subseteq \Lambda_+^\alpha$ of the proper cardinality. As the sets T_0 are not too numerous, the join of all sets $L(y, T_0)$ in Λ_+^α will be our set $L(y)$.

On each cell we replace each f by its Taylor expansion about the center, up to derivatives of order p; if η_1 is sufficiently small, the Taylor expansion deviates from f by at most $1/8\ \eta \cdot 2^{-k\alpha}$, and the totality of functions so constructed has dimension $\leqq (p+1)^n \cdot C_2 4^k$. At points common to two or more cells in T, we replace the Taylor expansion by 0. Now we have a finite dimensional subspace of the Banach space of bounded functions on T—and by the inequality between "widths and entropy" [**6**, p. 164] the totality of approximating functions is contained in $\exp C_4 k 4^k$ sets of diameter $1/8\ \eta 2^{-k\alpha}$. From elementary inequalities in metric spaces, we can cover all the functions f by the same number of balls, of radius $\frac{1}{2}\eta \cdot 2^{-k\alpha}$ in the uniform norm on T, centered at functions f. Now $k 4^k = 0(1) y^{2b} \log y$ so the set $L(y)$ is small enough to complete the proof of Theorem 2.

2. (Proof of Theorem 1). First we find a set S of Hausdorff dimension b_1, arbitrarily close to b, such that $X(S)$ is not a set of Λ^α-multiplicity.

Let α_1 and c be chosen so that $b_1^{-1} > \alpha_1 > \alpha$ and $1 < c < \alpha^{-1}\alpha_1$. Then let M be a sequence of positive integers m such that each set $\{m \in M, m \leqq k\}$ has at least $b_1 k$ elements; then the set $S = S_M$ of all sums $\Sigma \pm 2^{-m}$ has Hausdorff dimension at least b_1. In addition, we assume that M contains infinitely many pairs of consecutive elements q, q_1 such that $q_1 > \alpha_1 q$. Sequences M exist because $\alpha_1 b_1 < 1$. Each number q of this type determines a division of S into at most 2^q subsets S_p, based on the coordinates for $m \leqq q$: each S_p has diameter $< 4 \cdot 2^{-q_1}$, and the sets S_p have mutual distances $\geqq 2^{-q-1}$.

For large enough q, the sets $X(S_p)$ are dispersed in a sense to be made precise in a moment. Taking an integer $s > 1 + (c-1)^{-1}$ we investigate the event that s distinct sets S_p are mapped within $d = 2^{-qc/2}$ of each other. By a famous inequality of Paul Lévy, the sets $X(S_p)$ have diameters $o(q_1 2^{-q_1/2}) = o(d)$ for large q, so we can simplify the calculation by taking $t_p \in S_p$ and bounding the probability that s numbers t_p are mapped within $2d$ of each other. We use the scaling property and independence of increments, with the observation that $n = 2$ is the least favorable case. An s-tuple leads to an event of probability $0(1) \cdot \Pi d^2 | u_{j+1} - u_j |^{-1}$. We sum this for all s-tuples chosen from the numbers t_p and recall that u_1 takes at most 2^q values. Each factor $d^2 | u_{j+1} - u_j |^{-1}$ adds a factor $2^q q \cdot d^2$ to the sum. From the formula

$d = 2^{-qc/2}$ and the inequality $(s-1)c-(s-1) > 1$, we find that the sum has magnitude $2^{-\delta q}$ for some $\delta > 0$. The Borel–Cantelli Lemma then shows that the dispersion property holds for large q, with probability 1.

Now $X(S)$ is a union of sets of diameter $< d_1 = q_1 2^{-q_1/2}$ and at most $s-1$ sets $X(S_p)$ have mutual distances $< d$. Moreover $d > d_1^\beta$ for some $\beta < \alpha^{-1}$ because $c < \alpha^{-1}\alpha_1$. It is proved in [2, 5, p. 66] that $f \circ X(S)$ is not an M_0-set (nor even an M-set) for all f in Λ^α except a set of first category. Of course Λ_+^α is an open subset of Λ^α so the same is true of Λ_+^α.

To finish the proof of the negative statement in Theorem 1, we let b_1 increase to b along a sequence and choose a union of sets S_M, wherein M depends on b_1. As the union is countable, the union of the meager sets obtained for each S_M is again meager, and it is classical that, for measures μ such that $\hat{\mu}(\infty) = 0$, the entire space $L^1(\mu)$ inherits this property. This completes the proof of the second assertion in Theorem 1.

The positive assertion is a consequence of Theorem 2: by a theorem of Frostman [1, II–III] any closed set of Hausdorff dimension $> b$ carries a measure μ fulfilling the inequalities of Theorem 2.

A problem that appears much more difficult is the behavior of sets S with "strong dimension" b: S is not the union of a sequence US_m, $\dim S_m < b$. These sets can be characterized in the theory of Hausdorff measures [7]. Some of the analysis is done in [3,4].

References

1. J.-P. Kahane and R. Salem, *Ensembles parfaits et séries trigonométriques*, Hermann, Paris, 1963.
2. R. Kaufman, *A functional method for linear sets*, Israel J. Math. **5** (1967), 785–787.
3. R. Kaufman, *Une propriété métrique du mouvement brownien*, C.R. Acad. Sci. Paris **268 A** (1969), 727–728.
4. R. Kaufman, *Brownian motion and dimension of perfect sets*, Canad. J. Math. **22** (1970), 674–680.
5. R. Kaufman, *Brownian motion, approximation of functions, and Fourier analysis*, Studia Math. (to appear).
6. G. G. Lorentz, *Approximation of Functions*, Holt, New York, 1966.
7. C. A. Rogers, *Sets non-σ-finite for Hausdorff measures*, Mathematika **9** (1962), 95–103.
8. A. Zygmund, *Trigonometric Series I.* Cambridge, 1959 and 1966.

Received December 11, 1973 and in revised form March 15, 1974. Alfred P. Sloan Fellow.

UNIVERSITY OF ILLINOIS