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LEVEL CROSSING PROBABILITIES FOR A MULTI-PARAMETER BROWNIAN PROCESS

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Let $\{\xi_{s_1,s_2}: -\infty < s_1, <\infty, -\infty < s_2 < \infty\}$ be a Gaussian process with $\xi_{s_1,s_2} = 0$ if $s_1 = 0$ or $s_2 = 0$, mean values $E(\xi_{s_1,s_2}) = 0$, and covariances $E(\xi_{s_1,s_2}\xi_{s_1',s_1'}) = 1/2 \min(s_1,s_1') \min(s_2,s_2')$. This is the two parameter Brownian process studied by J. D. Kuelbs, W. J. Park, P. T. Strait, and J. Yeh. In this paper, upper and lower bounds for level crossing probabilities of this process are derived.

More specifically, let (t_1, t_2) and (τ_1, τ_2) be two pairs of constants chosen so that $0 < t_1 < \infty$, $0 < t_2 < \infty$, $0 < \tau_1 < \infty$, and $0 < \tau_2 < \infty$. Let $\delta_1 = \tau_1/m$, $\delta_2 = \tau_2/n$ where m and n are integers, and define random variables $X_{h,k}$ for $h = 0, 1, 2, \cdots, m$ and $k = 0, 1, 2, \cdots, n$ as follows.

$$(1)\qquad X_{h,k} = \begin{cases} 0 & \text{for}\quad h = 0 \quad\text{or}\quad k = 0 \\ \xi_{t_1+(h-1)\delta_1,\, t_2+(k-1)\delta_2} & \text{for}\quad h = 1, 2, \cdots, m;\ k = 1, 2, \cdots, n\,. \end{cases}$$

For any given number a, define

$$(2)\qquad \begin{aligned} &P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \\ &\quad = P(X_{i,j} > a \text{ for } i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)\,. \end{aligned}$$

In this paper, upper and lower bounds (Theorems 1 and 2) for $\lim_{m\to\infty\, n\to\infty} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2)$ are derived.

2. Preliminary lemmas.

LEMMA 1. *Let $\zeta_{h,k} = X_{h,k} - X_{h,k-1} - X_{h-1,k} + X_{h-1,k-1}$ for $h = 1, \cdots, m$ and $k = 1, \cdots, n$. Then,*

(i) *$\zeta_{h,k}$, $h = 1, \cdots, m$, $k = 1, \cdots, n$ are independent Gaussian random variables with means 0 and variances $\sigma^2_{h,k}$ given by*

$$(3)\qquad \begin{aligned} \sigma^2_{1,1} &= \frac{1}{2} t_1 t_2 \\ \sigma^2_{1,k} &= \frac{1}{2}\delta_1 t_2 \quad \textit{for}\quad k = 2, \cdots, n \\ \sigma^2_{h,1} &= \frac{1}{2}\delta_2 t_1 \quad \textit{for}\quad h = 2, \cdots, m \\ \sigma^2_{h,k} &= \frac{1}{2}\delta_1\delta_2 \quad \textit{for}\quad h = 2, \cdots, m;\ k = 2, \cdots, m \end{aligned}$$

and

$$(4)\quad \text{(ii)}\quad X_{i,j} = \sum_{h=1}^{i} \sum_{k=1}^{j} \zeta_{h,k} \quad \textit{for} \quad i = 1, \cdots, m; j = 1, \cdots, n .$$

Proof of Lemma 1. To prove part (i) of the lemma, observe that

$$(5)\qquad E(\zeta_{h,k}) = E(X_{h,k} - X_{h,k-1} - X_{h-1,k} + X_{h-1,k-1}) = 0 \quad \text{for} \quad h = 1, \cdots, m; k = 1, \cdots, n .$$

$$(6)\qquad E(\zeta^2_{h,k}) = E[(X_{h,k} - X_{h-1,k} - X_{h,k-1} + X_{h-1,k-1})^2] = \begin{cases} \frac{1}{2}t_1t_2 & \text{for} \quad h = 1, k = 1 \\ \frac{1}{2}\delta_1 t_2 & \text{for} \quad h = 1; k = 2, \cdots, n \\ \frac{1}{2}\delta_2 t_1 & \text{for} \quad h = 2, \cdots, m; k = 1 \\ \frac{1}{2}\delta_1\delta_2 & \text{for} \quad h = 2, \cdots, m; k = 2, \cdots, n . \end{cases}$$

$$(7)\quad E(\zeta_{h,k}\zeta_{p,q}) = E[X_{h,k} - X_{h-1,k} - X_{hk-1} + X_{h-1,k-1})(X_{p,q} - X_{p-1,q} - X_{p,q-1} + X_{p-1,q-1})] = 0 \quad \text{when } (h, k) \neq (p, q) .$$

(In each of the equations (5), (6), and (7), the last term is derived by direct computation of the expression of the preceding term.) Thus, the random variables $\zeta_{h,k}$, $h = 1, \cdots, m$, $k = 1, \cdots, n$ are independent, Gaussian random variables with mean values 0 and variances given by equation (3).

To prove part (ii) of the lemma, observe that

$$(8)\qquad \sum_{h=1}^{i} \sum_{k=1}^{j} \zeta_{h,k} = \sum_{h=1}^{i} \sum_{k=1}^{j} (X_{h,k} - X_{h-1,k} - X_{h,k-1} + X_{h-1,k-1}) = X_{i,j}$$

For the remaining lemmas and theorems we add the following notations and definitions. Let $C = \{(i, j): i = 1, \cdots, m; j = 1, \cdots, n\}$, $C^* = C - \{(1, 1)\}$. Furthermore, let $P^-_{m,n}$ and $P^*_{m,n}$ be probabilities defined as follows.

$$(9)\quad P^-_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = P(X_{i,j} > a \text{ for all } (i, j) \in C^* \text{ and } X_{1,1} \leqq a)$$

$$(10)\qquad P^*_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = P(X_{i,j} > a \text{ for all } (i, j) \in C^*) .$$

Then clearly,

$$(11)\qquad P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = P^*_{m,n}(a, t_1, t_2, \tau_1, \tau_2) - P^-_{m,n}(a, t_1, t_2, \tau_1, \tau_2) .$$

LEMMA 2. *Let η_1, η_2, η_3 be normal random variables with $E(\eta_i) = 0$ and $\mathrm{Var}(\eta_i) = (1/6)t_1t_2$ for $i = 1, 2, 3$. Assume also that $\eta_1, \eta_2, \eta_3, \zeta_{1,1}, \zeta_{1,2}, \cdots, \zeta_{m,n}$ form a set of independent random variables. Then*

$$
\begin{aligned}
&P^*_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \\
&\quad \geqq P\Big(\eta_1 + \sum_{h=2}^{i} \sum_{k=2}^{j} \zeta_{h,k} > \frac{a}{3} \quad \text{for all} \quad (i, j) \in C^*\Big) \\
&\quad \cdot P\Big(\eta_2 + \sum_{k=2}^{j} \zeta_{1,k} > \frac{a}{3} \quad \text{for all} \quad j = 1, 2, \cdots, n\Big) \qquad (12) \\
&\quad \cdot P\Big(\eta_3 + \sum_{h=2}^{i} \zeta_{h,1} > \frac{a}{3} \quad \text{for all} \quad i = 1, 2, \cdots, m\Big) .
\end{aligned}
$$

Proof of Lemma 2. For $i < 2$ or $j < 2$ define

$$
\sum_{h=2}^{i} \sum_{k=2}^{j} \zeta_{h,k} = 0 \,, \quad \sum_{k=2}^{j} \zeta_{1,k} = 0 \,, \quad \text{and} \quad \sum_{h=2}^{i} \zeta_{h,1} = 0 \,.
$$

The proof of Lemma 2 follows.

$$
\begin{aligned}
&P^*_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = P(\mathrm{X}_{i,j} > a \quad \text{for} \quad (i, j) \in C^*) \\
&\quad = P\Big(\sum_{h=1}^{i} \sum_{k=1}^{j} \zeta_{h,k} > a \quad \text{for} \quad (i, j) \in C^*\Big) \\
&\quad = P\Big(\sum_{h=2}^{i} \sum_{k=2}^{j} \zeta_{h,k} + \sum_{k=2}^{j} \zeta_{1,k} + \sum_{h=2}^{i} \zeta_{h,1} + \zeta_{1,1} > a \text{ for } (i, j) \in C^*\Big) \\
&\quad = P\Big(\sum_{h=2}^{i} \sum_{k=2}^{j} \zeta_{h,k} + \sum_{k=2}^{i} \zeta_{1,k} + \sum_{h=2}^{i} \zeta_{h,1} + \eta_1 + \eta_2 + \eta_3 > a \\
&\qquad \text{for } (i, j) \in C^*\Big) \qquad (13) \\
&\quad \geqq P\Big[\Big(\eta_1 + \sum_{h=2}^{i} \sum_{k=2}^{j} \zeta_{h,k} > \frac{a}{3}\Big) \quad \text{and} \quad \Big(\eta_2 + \sum_{k=2}^{j} \zeta_{1,k} > \frac{a}{3}\Big) \\
&\qquad \text{and} \quad \Big(\eta_3 + \sum_{h=2}^{i} \zeta_{h,1} > \frac{a}{3}\Big) \quad \text{for} \quad (i, j) \in C^* \Big] \\
&\quad = P\Big(\eta_1 + \sum_{h=2}^{i} \sum_{k=2}^{j} \zeta_{h,k} > \frac{a}{3} \quad \text{for} \quad (i, j) \in C^* \Big) \\
&\qquad \cdot P\Big(\eta_2 + \sum_{k=2}^{j} \zeta_{1,k} > \frac{a}{3} \quad \text{for} \quad j = 1, 2, \cdots, n \Big) \\
&\qquad \cdot P\Big(\eta_3 + \sum_{h=2}^{i} \zeta_{h,1} < \frac{a}{3} \quad \text{for} \quad i = 1, 2, \cdots, m\Big) .
\end{aligned}
$$

LEMMA 3.

$$
\begin{aligned}
&\lim_{\substack{m \to \infty \\ n \to \infty}} P\Big(\eta_1 + \sum_{h=2}^{i} \sum_{k=2}^{j} \zeta_{h,k} > a \quad \text{for} \quad (i, j) \in C^*\Big) \qquad (14) \\
&\quad \geqq \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}} \,, \quad \alpha = \frac{a}{\sqrt{3t_1t_2}} \,.
\end{aligned}
$$

Proof of Lemma 3. Let $Y_{h,k}$, $h = 1, \cdots, m$, $k = 1, \cdots, n$ be independent standard normal random variables. Let

$$
\begin{aligned}
L_{m,n} &= \min_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left(\sum_{h=1}^{i} \sum_{k=1}^{j} Y_{h,k} \right) \\
L_{m,n}^{-} &= \min (0, L_{m,n}) \qquad (15)\\
U_{m,n} &= \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left(\sum_{h=1}^{i} \sum_{k=1}^{j} Y_{h,k} \right), \quad U_{m,n}^{+} = \max (0, U_{m,n})
\end{aligned}
$$

It is shown in [6] that

$$E(U_{m,n}^{+}) < 4\sqrt{mn}\,. \tag{16}$$

To prove Lemma 3, it is first shown that

$$
\begin{aligned}
&P\Big(\eta_1 + \sum_{h=2}^{i} \sum_{k=2}^{j} \zeta_{h,k} > a \quad \text{for} \quad (i, j) \in C^*\Big) \\
&= \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - E^{-}\Big(\int_0^{-\sqrt{3\delta_1\delta_2/t_1t_2}\,L_{m-1,n-1}} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\Big)
\end{aligned} \tag{17}
$$

where E^{-} denotes that portion of the expectation obtained by integration over the negative range of $L_{m-1,n-1}$. (Later, E^{+} shall also be used to denote expectation obtained by integration over the positive range of values.)

To prove equation (17), observe that

$$
\begin{aligned}
&P\Big(\eta_1 + \sum_{h=2}^{i} \sum_{k=2}^{j} \zeta_{h,k} > a \quad \text{for} \quad (i, j) \in C^*\Big) \\
&= P\Big(\eta_1 - a > - \sum_{h=2}^{i} \sum_{k=2}^{j} \zeta_{h,k} \quad \text{for} \quad (i, j) \in C^*\Big) \\
&= P\Big(\eta_1 - a > \sqrt{\frac{\delta_1\delta_2}{2}} \sum_{h=2}^{i} \sum_{k=2}^{j} Y_{h,k} \quad \text{for} \quad (i, j) \in C^*\Big) \\
&= P\Big(\eta_1 - a > - \sqrt{\frac{\delta_1\delta_2}{2}} L_{m-1,n-1}^{-}\Big) \\
&= E\Big(\int_{-\sqrt{\delta_1\delta_2/2}\,L_{m-1,n-1}^{-}}^{\infty} \frac{1}{\sqrt{\pi \dfrac{t_1t_2}{3}}} e^{-(w+a)^2/(1/3)t_1t_2} dw\Big) \\
&= E\Big(\int_{-\sqrt{3\delta_1\delta_2/t_1t_2}\,L_{m-1,n-1}^{-}}^{\infty} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\Big) \\
&= E^{+}\Big(\int_{-\sqrt{3\delta_1\delta_2/t_1t_2}\,L_{m-1,n-1}^{-}}^{\infty} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\Big) \qquad (18) \\
&\quad + E^{-}\Big(\int_{-\sqrt{3\delta_1\delta_2/t_1t_2}\,L_{m-1,n-1}^{-}}^{\infty} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= E^{+}\left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
&\quad + E^{-}\left(\int_{-\sqrt{3\delta_1\delta_2/t_1t_2}L_{m-1,n-1}}^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
&= E^{+}\left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) + E^{-}\left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
&\quad - E^{-}\left(\int_0^{-\sqrt{3\delta_1\delta_2/t_1t_2}L_{m-1,n-1}} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) \\
&= \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - E^{-}\left(\int_0^{-\sqrt{3\delta_1\delta_2/t_1t_2}L_{m-1,n-1}} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right).
\end{aligned}
$$

Next, apply the inequality

$$
\int_0^x e^{-1/2(u+\alpha)^2} du < x \quad \text{for} \quad x > 0 \tag{19}
$$

to obtain

$$
E^{-}\left(\int_0^{-\sqrt{3\delta_1\delta_2/t_1t_2}L_{m-1,n-1}} \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du\right) < -\sqrt{\frac{2}{\pi}}\sqrt{\frac{3\delta_1\delta_2}{t_1t_2}} E^{-}(L_{m-1,n-1}). \tag{20}
$$

Now, observe that

$$
E^{-}(L_{m-1,n-1}) = E(L^{-}_{m-1,n-1}) = -E(U^{+}_{m-1,n-1}) \tag{21}
$$

then combine this with equation (16), to obtain

$$
E^{-}(L_{m-1,n-1}) > -4\sqrt{(m-1)(n-1)}. \tag{22}
$$

Finally, apply equations (17), (20), and (22) to obtain

$$
\begin{aligned}
&\lim_{\substack{m\to\infty \\ n\to\infty}} P\left(\eta_1 + \sum_{h=2}^{i}\sum_{k=2}^{j} \zeta_{h,k} > a \quad \text{for} \quad (i,j) \in C^*\right) \\
&\geqq \lim_{\substack{m\to\infty \\ n\to\infty}} \left\{\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du + \sqrt{\frac{2}{\pi}}\sqrt{\frac{3\delta_1\delta_2}{t_1t_2}} E^{-}(L_{m-1,n-1})\right\} \\
&\geqq \lim_{\substack{m\to\infty \\ n\to\infty}} \left\{\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du + \sqrt{\frac{2}{\pi}}\sqrt{\frac{3\delta_1\delta_2}{t_1t_2}} (-4\sqrt{(m-1)(n-1)})\right\} \\
&= \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - 4\sqrt{\frac{6\delta_1\delta_2}{\pi t_1 t_2}}.
\end{aligned} \tag{23}
$$

LEMMA 4.

(a) $$\lim_{n\to\infty} P\left(\eta_2 + \sum_{k=2}^{j} \zeta_{1,k} > 0 \quad \text{for} \quad j = 1, 2, \cdots, n\right) = \frac{1}{\pi}\sin^{-1}\left[\left(1 + \frac{3\tau_1}{t_1}\right)^{-1/2}\right]$$

(24)

(b) $\lim_{m\to\infty} P\Big(\eta_3 + \sum_{h=2}^{i} \zeta_{h,1} > 0 \quad \text{for} \quad i = 1, 2, \cdots, m\Big)$

$$= \frac{1}{\pi} \sin^{-1}\Big[\Big(1 + \frac{3\tau_2}{t_2}\Big)^{-1/2}\Big]$$

(c) *For small* $\frac{\tau_1}{t_1}$,

$$\lim_{n\to\infty} P\Big(\eta_2 + \sum_{k=2}^{j} \zeta_{1,k} > a \quad \text{for} \quad j = 1, 2, \cdots, n\Big)$$

$$= \Phi\Big(-\frac{6a}{t_1t_2}\Big) - \frac{1}{\pi}\Big(\frac{3\tau_1}{t_1}\Big)^{1/2} e^{-3a^2/t_1t_2} + O\Big(\frac{3\tau_1}{t_1}\Big)$$

(d) *For small* $\frac{\tau_2}{t_2}$,

$$\lim_{m\to\infty} P\Big(\eta_3 + \sum_{h=2}^{i} \zeta_{h,1}a \quad \text{for} \quad i = 1, 2, \cdots, m\Big)$$

$$= \Phi\Big(-\frac{6a}{t_1t_2}\Big) - \frac{1}{\pi}\Big(\frac{3\tau_2}{t_2}\Big)^{1/2} e^{-}\frac{3a^2}{t_1t_2} + O\Big(\frac{3\tau_2}{t_2}\Big)$$

where $\Phi(\ )$ *is the standardized normal distribution function*

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \ .$$

Proof of Lemma 4. Let

$$Y_k = \sqrt{\frac{2}{\delta_1 t_2}}\zeta_{1,k} \quad \text{for} \quad k = 2, \cdots, n \tag{25}$$

so that $Y_2, Y_3, \cdots, Y_n$ are independent standard normal random variables. Let

$$\delta = \frac{\delta_1 t_2}{2} \tag{26}$$

(27) $X_1 = \eta_2$

$$X_2 = \eta_2 + \zeta_{1,2} = \eta_2 + \sqrt{\frac{\delta_1 t_2}{2}} Y_2 = X_1 + \delta^{1/2} Y_2$$

$$\vdots$$

$$X_n = \eta_2 + \sum_{k=2}^{n} \zeta_{1,k} = \eta_2 + \sqrt{\frac{\delta_1 t_2}{2}} \sum_{k=2}^{n} Y_k = X_1 + \delta^{1/2} \sum_{k=1}^{n} Y_k$$

where X_1 is normal, $E(X_1) = 0$, $\mathrm{Var}\,(X_1) = (1/6)t_1t_2$. Let

$$t = \frac{1}{6} t_1 t_2 \ , \quad \tau = \frac{n\delta_1 t_2}{2} = \frac{\tau_1 t_2}{2} \ . \tag{28}$$

For random variables $X_1, X_2, \cdots, X_n$ satisfying the conditions given above, J. A. McFadden and J. L. Lewis proved in [3] (equation (34) on page 310) that

$$\lim_{n\to\infty} P(X_i > 0 \text{ for all } i = 1, \cdots, n) = \lim_{n\to\infty} P_{n+1}(0, t) = \frac{1}{\pi}\sin^{-1}\left[\left(1+\frac{\tau}{t}\right)^{-1/2}\right]. \tag{29}$$

Hence in this case

$$\begin{aligned} &\lim_{n\to\infty} P\left(\eta_2 + \sum_{k=2}^{j}\zeta_{1,k} > 0 \quad \text{for} \quad j = 1, 2, \cdots, n\right) \\ &\quad = \lim_{n\to\infty} P(X_i > 0 \quad \text{for all} \quad i = 1, \cdots, n) \\ &\quad = \frac{1}{\pi}\sin^{-1}\left[\left(1+\frac{\tau}{t}\right)^{-1/2}\right] = \frac{1}{\pi}\sin^{-1}\left[\left(1+\frac{3\tau_1}{t_1}\right)^{-1/2}\right]. \end{aligned} \tag{30}$$

Similarly,

$$\begin{aligned} &\lim_{m\to\infty} P\left(\eta_3 + \sum_{h=2}^{i}\zeta_{h,1} > 0 \text{ for } i = 1, \cdots, m\right) \\ &\quad = \frac{1}{\pi}\sin^{-1}\left[\left(1+\frac{3\tau_2}{t_2}\right)^{-1/2}\right]. \end{aligned} \tag{31}$$

For the case $a \neq 0$ and τ_1/t_1, τ_2/t_2 small, use equation (37) of McFadden and Lewis [3] which states that

$$\begin{aligned} &\lim_{n\to\infty} P(X_i > a \quad \text{for all} \quad i = 1, \cdots, n) \\ &\quad = \lim_{n\to\infty} P_{n+1}(a, t) = \Phi\left(\frac{-a}{t^{1/2}}\right) - \frac{1}{\pi}\left(\frac{\tau}{t}\right)^{1/2} e^{-a^2/2t} + O(\tau/t) \end{aligned} \tag{32}$$

to obtain

$$\begin{aligned} &\lim_{n\to\infty} P\left(\eta_2 + \sum_{k=2}^{j}\zeta_{i,k} > a \quad \text{for} \quad i = 1, 2, \cdots, n\right) \\ &\quad = \Phi\left(-\frac{a}{\sqrt{\frac{1}{6}t_1 t_2}}\right) - \frac{1}{\pi}\left(\frac{3\tau_1}{t_1}\right)^{1/2} e^{-3a^2/t_1 t_2} + O\left(\frac{3\tau_1}{t_1}\right). \end{aligned} \tag{33}$$

Similarly,

$$\begin{aligned} &\lim_{m\to\infty} P\left(\eta_3 + \sum_{h=2}^{i}\zeta_{h,1} > 0 \quad \text{for} \quad i = 1, 2, \cdots, m\right) \\ &\quad = \Phi\left(-\frac{\sqrt{6}a}{\sqrt{t_1 t_2}}\right) - \frac{1}{\pi}\left(\frac{3t_2}{t_2}\right)^{1/2} e^{-3a^2/t_1 t_2} + O\left(\frac{3\tau_2}{t_1}\right). \end{aligned} \tag{34}$$

LEMMA 5.

$$\lim_{\substack{m\to\infty\\ n\to\infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = \lim_{\substack{m\to\infty\\ n\to\infty}} P^*_{m,n}(a, t_1, t_2, \tau_1, \tau_2)\ . \tag{35}$$

Proof of Lemma 5. Consider the term $P^-_{m,n}(a, t_1, t_2, \tau_1, \tau_2)$ of equation (11).

$$\begin{aligned} P^-_{m,n}(a, t_1, t_2, \tau_1, \tau_2) &= P(X_{i,j} > a \text{ for } (i, j) \in C^* \text{ and } X_{1,1} \leqq a) \\ &= P(X_{i,j} > a \text{ for } (i, j) \in C^* \text{ and } X_{1,1} < a)\ . \end{aligned} \tag{36}$$

Therefore, by continuity of the sample paths (see [4] for proof of this fact)

$$\lim_{\substack{m\to\infty\\ n\to\infty}} P^-_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = 0 \quad \text{for} \quad a \neq 0\ . \tag{37}$$

But this implies that equation (11) is equivalent to

$$P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) = P^*_{m,n}(a, t_1, t_2, \tau_1, \tau_2) - P^-_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \tag{38}$$

Upon taking limits on both sides of (38), equation (35) of Lemma 5 is obtained.

3. The main theorems.

THEOREM 1. *A lower bound.*

$$\begin{aligned} &\lim_{\substack{m\to\infty\\ n\to\infty}} P_{m,n}(0, t_1, t_2, \tau_1, \tau_2) \\ &\quad \geqq \left(1 - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}\right)\cdot\left(\frac{1}{\pi}\sin^{-1}\left[\left(1 + \frac{3\tau_1}{t_1}\right)^{-1/2}\right]\right) \\ &\quad \cdot\left(\frac{1}{\pi}\sin^{-1}\left[\left(1 + \frac{3\tau_2}{t_2}\right)^{1/2}\right]\right) \end{aligned} \tag{39}$$

and for $a \neq 0$, τ_1/t_1 *small*, τ_2/t_2 *small*,

$$\begin{aligned} &\lim_{\substack{m\to\infty\\ n\to\infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \geqq \left(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du \ - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}\right) \\ &\quad \cdot\left[\Phi\left(-\frac{6a}{t_1t_2}\right) - \frac{1}{\pi}\left(\frac{3\tau_1}{t_1}\right)^{1/2} e^{-a^2/3t_1t_2} O\left(\frac{3\tau_1}{t_1}\right)\right] \\ &\quad \cdot\left[\Phi\left(-\frac{6a}{t_1t_2}\right) - \frac{1}{\pi}\left(\frac{3\tau_2}{t_2}\right)^{1/2} e^{-a^2/3t_1t_2} + O\left(\frac{3\tau_2}{t_2}\right)\right], \end{aligned} \tag{40}$$

where $\alpha = \dfrac{a}{\sqrt{3t_1t_2}}$

Proof of Theorem 1. Apply Lemma 2 to Lemma 5 to obtain

$$\lim_{\substack{m\to\infty \\ n\to\infty}} P_{mn}(a, t_1, t_2, \tau_1, \tau_2)$$

$$(41) \qquad \begin{aligned} &\geqq P\Big(\eta_1 + \sum_{h=2}^{i}\sum_{k=2}^{j}\zeta_{h,k} > \frac{a}{3} \text{ for } (i, j) \in C^*\Big) \\ &\cdot P\Big(\eta_2 + \sum_{k=2}^{j}\zeta_{1,k} > \frac{a}{3} \text{ for } j = 1, 2, \cdots, n\Big) \\ &\cdot P\Big(\eta_3 + \sum_{h=2}^{i}\zeta_{h,1} > \frac{a}{3} \text{ for } i = 1, 2, \cdots, m\Big) . \end{aligned}$$

Then use Lemmas 3 and 4 to obtain

$$(42) \qquad \begin{aligned} &\lim_{\substack{m\to\infty \\ n\to\infty}} P_{m,n}(0, t_1, t_2, \tau_1, \tau_2) \\ &\geqq \Big(1 - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}\Big)\Big(\frac{1}{\pi}\sin^{-1}\Big[\Big(1+\frac{3\tau_1}{t_1}\Big)^{1/2}\Big]\Big)\Big(\frac{1}{\pi}\sin^{-1}\Big[\Big(1+\frac{3\tau_2}{t_2}\Big)^{-1/2}\Big]\Big) \end{aligned}$$

and for $a \neq 0$,

$$(43) \qquad \begin{aligned} &\lim_{\substack{m\to\infty \\ n\to\infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \\ &\geqq \Big(\int_0^\infty \sqrt{\frac{2}{\pi}} e^{-1/2(u+\alpha)^2} du - 4\sqrt{\frac{6\tau_1\tau_2}{\pi t_1 t_2}}\Big) \\ &\cdot \Big[\Phi\Big(-\frac{6a}{t_1 t_2}\Big) - \frac{1}{\pi}\Big(\frac{3\tau_1}{t_1}\Big)^{1/2} e^{-a^2/3t_1t_2} + O\Big(\frac{3\tau_1}{t_1}\Big)\Big] \\ &\cdot \Big[\Phi\Big(-\frac{6a}{t_1 t_2}\Big) - \frac{1}{\pi}\Big(\frac{3\tau_2}{t_2}\Big)^{1/2} e^{-a^2/3t_1t_2} + O\Big(\frac{3\tau_2}{t_2}\Big)\Big] \end{aligned}$$

where $\alpha = \dfrac{a}{\sqrt{3t_1t_2}}$.

THEOREM 2. *An upper bound.*

$$(44) \qquad \begin{aligned} &\lim_{\substack{m\to\infty \\ n\to\infty}} P_{m,n}(0, t_1, t_2, \tau_1, \tau_2) \\ &\leqq \min \Big\{\Big(\frac{1}{\pi}\sin^{-1}\Big[\Big(1+\frac{3\tau_1}{t_1}\Big)^{-1/2}\Big]\Big), \\ &\qquad \Big(\frac{1}{\pi}\sin^{-1}\Big[\Big(1+\frac{3\tau_2}{t_2}\Big)^{-1/2}\Big]\Big)\Big\} \end{aligned}$$

and for $a \neq 0$, τ_1/t_1 *small, and* τ^2/t_2 *small,*

$$(45) \qquad \begin{aligned} &\lim_{\substack{m\to\infty \\ n\to\infty}} P_{m,n}(a, t_1, t_2, \tau_1, \tau_2) \\ &\leqq \min\Big\{\Big[\Phi\Big(-a\sqrt{\frac{6}{t_1t_2}}\Big) - \frac{1}{\pi}\Big(\frac{3\tau_1}{t_1}\Big)^{1/2} e^{-a^2/3t_1t_2} + O\Big(\frac{3\tau_1}{t_1}\Big)\Big], \\ &\Big[\Phi\Big(-a\sqrt{\frac{6}{t_1t_2}}\Big) - \frac{1}{\pi}\Big(\frac{3\tau_2}{t_2}\Big)^{1/2} e^{-a^2/3t_1t_2} + O\Big(\frac{3\tau_2}{t_2}\Big)\Big]\Big\} . \end{aligned}$$

Proof of Theorem 2. Observe that

$$P(X_{i,j} > a \text{ for } (i, j) \in C) \leqq \min\{P(X_{1,j} > a, j = 1, \cdots, n), P(X_{i,1} > a, i = 1, \cdots, m)\} . \tag{46}$$

For the case $a = 0$, we apply equation (29) to obtain

$$\lim_{\substack{m\to\infty \\ n\to\infty}} P(X_{i,j} > 0 \text{ for } (i, j) \in C) \leqq \min\left\{\left(\frac{1}{\pi}\sin^{-1}\left[\left(1 + \frac{3\tau_1}{t_1}\right)^{-1/2}\right]\right), \left(\frac{1}{\pi}\sin^{-1}\left[\left(1 + \frac{3\tau_2}{t_2}\right)^{-1/2}\right]\right)\right\} . \tag{47}$$

For the case $a \neq 0$, apply equation (32) to obtain

$$\begin{aligned} &\lim_{\substack{m\to\infty \\ n\to\infty}} P(X_{i,j} > a \text{ for } (i, j) \in C) \\ &\leqq \min\left[\left\{\Phi\left(-a\sqrt{\frac{6}{t_1t_2}}\right) - \frac{1}{\pi}\left(\frac{3\tau_1}{t_1}\right)^{1/2} e^{-3a^2/t_1t_2} + O\left(\frac{3\tau_1}{t_1}\right)\right\} ,\right. \\ &\left.\left\{\Phi\left(-a\sqrt{\frac{6}{t_1t_2}}\right) - \frac{1}{\pi}\left(\frac{3\tau_2}{t_2}\right)^{1/2} e^{-3a^2/t_1t_2} + O\left(\frac{3\tau_2}{t_2}\right)\right\}\right] \end{aligned} \tag{48}$$

for small $\dfrac{\tau_1}{t_1}, \dfrac{\tau_2}{t_2}$.

4. **Remark.** The method used here may be generalized to apply to the N-parameter Brownian process.

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