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A CHARACTERIZATION OF NON-LINEAR FUNCTIONALS ON W_1^p POSSESSING AUTONOMOUS KERNELS. I

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Let Ω be a domain in R^n and let a nonlinear functional N be given on the first order Sobolev space $W_1^p(\Omega)$, $1 \leqq p \leqq \infty$. We are concerned with obtaining a characterization of those functionals N of the form

$$(1.1) \qquad N(u) = \int g(u, D_1u, \cdots, D_nu)dm\ , \quad u \in W_1^p(\Omega)\ ,$$

where $g: R^{n+1} \to R$ is a continuous function, $D_iu (i = 1, \cdots, n)$ denotes the distribution derivative of u relative to its ith coordinate variable and m denotes Lebesgue measure. In the present paper we confine ourselves to the case $n = 1$. The general case will be considered in the second part of this work.

In recent years, characterizations have been obtained for nonlinear functionals defined on Banach lattices of functions such as the L^p spaces and Orlicz spaces [1], [3], [4], [6], but the methods utilized in those works all depend crucially on the normality of these lattices and are unavailable in the present context. On the other hand, very recently a characterization was obtained for nonlinear functionals of the form

$$N(u) = \int_J g(t, D^ku(t))dt \qquad u \in \mathring{W}_k^p(J)\ , \quad k \geqq 1\ ,$$

where J is an interval on the line, g satisfies Caratheodory conditions and $g(t, 0) \equiv 0$ [2]. Such a functional possesses the property of D^k-disjoint additivity,

$$N(u + v) = N(u) + N(v) \quad \text{provided} \quad D^ku \cdot D^kv = 0\ ,$$

which permits a reduction of the problem to that for a disjointly additive functional on a closed subspace of $L^p(J)$. However even when $n = 1$ functionals of the form (1.1) are generally *not* D-disjointly additive and hence the methods of [2] do not apply in this case.

If g in (1.1) satisfies a suitable growth condition then the integrand Gu, where

$$(1.2) \qquad (Gu)(x) = g(u(x), D_1u(x), \cdots, D_nu(x))\ ,$$

belongs to $L^1(\Omega)$ for all $u \in W_1^p(\Omega)$, so that N is real-valued. In such cases it can be seen that N possesses the following properties:

(A) N is *additively invariant under swapping*: if H is a hyperplane in R^n which partitions Ω into sets $\{\Omega_1, \Omega_2\}$ and u, v are swappable across H in the sense that

$$u\chi_{\Omega_1} + v\chi_{\Omega_2},\ v\chi_{\Omega_1} + u\chi_{\Omega_2} \quad \text{are in} \quad W_1^p(\Omega)$$

(either of these conditions implies the other), then

$$N(u) + N(v) = N(u\chi_{\Omega_1} + v\chi_{\Omega_2}) + N(v\chi_{\Omega_1} + u\chi_{\Omega_2})\ ;$$

(B) N is *invariant under* 1-*equimeasurability*:

$$N(u) = N(v)$$

whenever $u, v \in W_1^p(\Omega)$ are such that the $(n+1)$-tuples

$$\boldsymbol{u} = (u, D_1u, \cdots, D_nu)\ , \quad \boldsymbol{v} = (v, D_1v, \cdots, D_nv)$$

are stochastically equivalent as mappings from Ω to R^{n+1}:

$$m(\boldsymbol{u}^{-1}(B)) = m(\boldsymbol{v}^{-1}(B))$$

for every Borel set $B \subset R^{n+1}$.

(C) N *is continuous*:

$$N(u_k) \longrightarrow N(u_0) \quad \text{whenever} \quad \|u_k - u_0\|_{W_1^p(\Omega)} \longrightarrow 0\ .$$

We are concerned with the extent to which properties (A), (B), (C) characterize functionals of the form (1.1). It will be shown that in conjunction with an additional hypothesis of "locally uniform continuity in variation," the conditions (A)-(C) *do* characterize functionals of the form (1.1).

A similar characterization is given for nonlinear operators G: $W_1^p(\Omega) \longrightarrow L^q(\Omega)$ having the form (1.2).

The present paper is devoted to an exposition of these results when R^n is the real line and Ω is a bounded open interval. The situation for $n > 1$ is as follows. The case $p > n$ can be treated by essentially the same method that is used in this paper, but the details are more subtle and intricate. On the other hand, in the case $p \leqq n$ the characterization involves several new problems. This work will appear in the second part.

2. Representation for functionals. We develop here the desired representation result for a class of nonlinear functionals and operators on the spaces $W_1^p(J)$, $1 \leqq p \leqq \infty$, where J is a bounded interval on the line.

Recall that $W_1^p(J)$ denotes the Sobolev space

$$W_1^p(J) = \{u \in L^p(J): Du \in L^p(J)\}$$

where Du denotes the distribution derivative of u. (All functions are taken to be real.) It is known that a function u belongs to $W_1^p(J)$ if and only if it is equivalent to a function in $C(J)$ which is absolutely continuous and whose first derivative belongs to $L^p(J)$. Hereafter any function in $W_1^p(J)$ will be assumed to be the continuous representative of its equivalence class. The space $W_1^p(J)$ is a Banach space under the norm

$$||u||_{W_1^p(J)} = ||u||_{L^p(J)} + ||Du||_{L^p(J)} .$$

Its structure under the metric

$$\rho(u, v) = \sigma(u - v) + \sigma(Du - Dv) \tag{2.1}$$

where

$$\sigma(f) = \inf_{\varepsilon>0} \{\varepsilon + m(\{t: |f(t)| > \varepsilon\})\}$$

will also play a role in what follows.

The class of functionals to be characterized consists of those functionals representable in the form

$$N(u) = \int_J g(u(t), u'(t))dt \quad u \in W_1^p(J) ,$$

for an appropriate function $g: R^2 \to R$. It will be necessary for our purposes to analyze the behavior of such functionals in some detail.

We adopt the following notations and conventions. Given a function f we set $K(f) = \{t: f(t) \neq 0\}$. The functions $u, v \in W_1^p(J)$ are 1-*equimeasurable*, denoted $w \approx_1 v$, provided that the pairs

$$\boldsymbol{u} = (u, Du) , \quad \boldsymbol{v} = (v, Dv)$$

are stochastically equivalent: for each Borel set $B \subset R^2$

$$m(\boldsymbol{u}^{-1}(B)) = m(\boldsymbol{v}^{-1}(B)) . \tag{2.2}$$

The functions $u, v \in W_1^p(J)$ are 1-*disjoint* provided that $K(Du) \cap K(Dv)$ is a null set. A 1-disjoint pair u, v is said to be *envelope compatible* provided there exists a partition of J into two subintervals J', J'' such that

$$\text{(2.3)} \quad \begin{array}{ll} \text{(i)} & K(Du) \subset J',\ K(Dv) \subset J'' \\ \text{(ii)} & u\chi_{J'} + v\chi_{J''} =: u \oplus v \quad \text{is in} \quad W_1^p(J)^1 . \end{array}$$

The function $z = u \oplus v$, which is independent of the choice of partition $\{J', J''\}$, is called the 1-*envelope* of u and v. Note that by

[1] "=:" means "is, by definition,".

(2.3), for any finite partition of J into subintervals $\{J_i\}_{i=1}^{l}$, numbered from left to right, say, it is possible to decompose each $u \in W_1^p(J)$ into functions $u^{J_i} \in W_1^p(J)$, $1 \leqq i \leqq l$, such that

$$(2.4) \qquad \begin{aligned} &\text{(a)} \quad K(D(u^{J_i})) \subset J_i\,, \quad i \geqq 1, \cdots, l \\ &\text{(b)} \quad u = ((u^{J_1} \oplus u^{J_2}) \oplus \cdots) \oplus u^{J_l}\,. \end{aligned}$$

Indeed, (2.4) holds if, for any subinterval $J' \subset J$, $u^{J'}$ denotes the element of $C(J)$ which coincides with u in J' and is constant on the left and right of J'.

Now let $g: R^2 \rightarrow R$ be a continuous function. For each $u \in W_1^p(J)$ the function Gu defined by

$$(2.5) \qquad Gu(t) = g(u(t), u'(t)) \qquad t \in J\,,$$

is measurable. Hence if Gu is in $L^1(J)$ for all $u \in W_1^p(J)$, in particular if g satisfies a growth condition of the form

$$(2.6) \qquad |g(x_0, x_1)| \leqq K_M(1 + |x_1|^p) \quad \text{whenever} \quad |x_0| \leqq M\,,$$

then one can form the nonlinear functional

$$(2.7) \qquad N(u) = \int_J Gu \qquad u \in W_1^p(J)\,.$$

As mentioned in §1 the functional N has the following properties:

(A) *N is additively invariant under swapping.*

Note that in the case $n = 1$ two functions u, v are swappable across a point α in J if $u(\alpha) = v(\alpha)$.

(B) *N is invariant under 1-equimeasurability*:

$$N(u) = N(v) \quad \text{whenever} \quad u \underset{1}{\approx} v\,.$$

(C) *N is continuous*:

$$N(u_n) \longrightarrow N(u_0) \quad \text{whenever} \quad \|u_n - u_0\|_{W_{1,p}(J)} \rightarrow 0\,.$$

For the continuity of g implies that the sequence $\{Gu_n\}$ converges to Gu_0 in measure. Moreover, the L^p-convergence of Du_n to Du_0 implies that every subsequence $\{u_{n'}\}$ possesses a subsequence $\{u_{n''}\}$ for which the sequence $\{Du_{n''}\}$ is dominated by an $L^p(J)$ function. Hence it follows from (2.6) that $\{Gu_{n''}\}$ converges to Gu_0 in measure, dominatedly in $L^1(J)$. Thus $\{N(u_{n''})\}$ converges to $N(u_0)$ and the continuity of N follows.

Now suppose in addition that $g: R^2 \rightarrow R$ satisfies

$$(2.8) \qquad g(x_0, 0) = 0 \quad \text{for all} \quad x_0 \in R\,.$$

In this case condition (A) is readily seen to imply

(A′) N is 1-*envelope additive*:

$$N(u \oplus v) = N(u) + N(v) \quad \text{whenever} \quad u,\, v \in W_1^p(J)$$

are envelope compatible,

and condition (B) is seen to imply

(B′) N is invariant under *generalized* 1-*equimeasurability*:

$$N(u) = N(v) \quad \text{whenever} \quad u \underset{1}{\sim} v\ ,$$

where $u \underset{1}{\sim} v$ means that $\boldsymbol{u} = (u, Du)$, $\boldsymbol{v} = (v, Dv)$ are stochastically equivalent on $K(Du)$ and $K(Dv)$, respectively: for each Borel set $B \subset R^2$

$$m(\boldsymbol{u}^{-1}(B) \cap K(Du)) = m(\boldsymbol{v}^{-1}(B) \cap K(Dv))\ .$$

Note that it follows from (2.4) and (A′) that N determines for each $u \in W_1^p(J)$ an additive set function ν_u defined on the subintervals of J by

$$\nu_u(J') = \int_{J'} Gu = N(u^{J'}) \quad u \in W_1^p(J)\ .$$

This enables us to deduce the following additional property of N.

(D) *N is locally uniformly continuous in* (interval) *variation*:

$$\lim_{\delta \to 0} V_M(\delta; N) = 0 \quad \text{for each} \quad M > 0\ ,$$

where the quantity $V_M(\delta; N)$ is given by

$$V_M(\delta; N) = \sup \sum_{i=1}^{l} |N(u_i^{J_i}) - N(v_i^{J_i})|\ ,$$

with the supremum being taken over all finite partitions of J into intervals $\{J_i\}_{i=1}^{l}$ and all sets of pairs $u_i,\, v_i \in W_1^\infty(J)$ satisfying

$$\|u_i\|_{W_1^\infty(J_i)}\ , \quad \|v_i\|_{W_1^\infty(J_i)} \leqq M\ , \quad \rho(U, V) \leqq \delta\ ,$$

where $U := \Sigma u_i \chi_{J_i}$, $DU := \Sigma Du_i \chi_{J_i}$ and similarly for V.

This follows from the uniform continuity and boundedness of g on sets of the form

$$\{\boldsymbol{x} = (x_0, x_1) \colon |x_i| \leqq M\} \subset R^2\ .$$

In fact, putting $|\boldsymbol{x}| = \max\{|x_0|, |x_1|\}$ we have

$$\sum_{i=1}^{l} |N(u_i^{J_i}) - N(v_i^{J_i})| = \sum_{i=1}^{l} |\nu_{u_i}(J_i) - \nu_{v_i}(J_i)|$$
$$\leqq \sum_{i=1}^{l} \int_{J_i} |g(u_i(t), u_i'(t)) - g(v_i(t), v_i'(t))| dt$$
$$\leqq \Big(\sup_{\substack{|x-y|\leqq\delta \\ |x|,|y|\leqq M}} |g(x) - g(y)| \Big) m(J) + 2\Big(\max_{|x|\leqq M} |g(x)|\Big)\delta .$$

We are concerned here with the extent to which these properties characterize functionals of the form (2.7). The principal part of our result is the following theorem.

THEOREM 2.1. *Let J be a bounded interval and let N be a real functional on $W_1^p(J)$, $1 \leqq p \leqq \infty$, which possesses the properties:*

(A′) $N(u \oplus v) = N(u) + N(v)$ *whenever* u, v *are envelope compatible,*

(B′) $N(u) = N(v)$ *whenever* $u \underset{1}{\sim} v$,

(C) $N(u_m) \rightarrow N(u_0)$ *whenever* $\|u_m - u_0\|_{W_1^p(J)} \rightarrow 0$, $1 \leqq p < \infty$,

(D) $\lim_{\delta\rightarrow 0} V_M(\delta; N) = 0$ *for each* $M > 0$.

Then there exists a unique continuous function $g: R^2 \rightarrow R$ *satisfying*:

$$g(x_0, 0) = 0 \quad \textit{for all} \quad x_0 \in R , \tag{2.9}$$

such that

$$N(u) = \int_J g(u(t), u'(t))dt \quad \textit{for all} \quad u \in W_1^p(J) . \tag{2.10}$$

Moreover, if $1 \leqq p < \infty$, *then* g *satisfies a growth condition of the form*

$$|g(x_0, x_1)| \leqq K_M(1 + |x_1|)^p \quad \textit{whenever} \ |x_0| \leqq M . \tag{2.11}$$

The proof of this result utilizes the Lebesgue differentiability of the interval function ν_u, for each $u \in W_1^\infty(J)$. The Lebesgue derivative f_u of ν_u is shown to belong to $L^1(J)$, and ν_u is shown to be the indefinite integral of f_u. Then it is shown that there exists a unique continuous function $g: R^2 \rightarrow R$ such that

$$f_u(t) = g(u(t), u'(t)) \text{ a.e., for each } u \in W_1^\infty(J) .$$

The representation (2.10) follows immediately for piecewise linear functions in $W_1^\infty(J)$. It is then extended to arbitrary $u \in W_1^\infty(J)$ and eventually to all of $W_1^p(J)$ by a limiting process.

We continue with the detailed proof of the theorem.

LEMMA 2.1. *The interval function ν_u is finitely additive on the semi-algebra S_J of subintervals of J, for every function $u \in W_1^p(J)$.*

If u is in $W_1^\infty(J)$ then ν_u is absolutely continuous and countably additive.

Proof. By (2.4), whenever $\{J_i\}_{i=1}^l$ are disjoint intervals, indexed from left to right, whose union is an interval J_0, then the relation

$$u^{J_0} = (\cdots (u^{J_1} \oplus u^{J_2}) \oplus \cdots) \oplus u^{J_l}$$

holds. Thus the finite additivity of ν_u on S_J follows from (A′).

The absolute continuity of ν_u for $u \in W_1^\infty(J)$ is proved as follows. Let $\{J_i'\}_{i=1}^l$ denote any family of disjoint intervals in J and let $\{J_i\}_{i=1}^L$ denote the minimal partition of J (into subintervals) generated by the $\{J_i'\}$, indexed so that $J_i = J_i'$, $1 \leqq i \leqq l$. Then we may write:

$$\sum_{i=1}^{l} |\nu_u(J_i')| = \sum_{i=1}^{L} |\nu_{u_i}(J_i) - \nu_{v_i}(J_i)| \leqq V_M(\delta; N)\ , \tag{2.12}$$

where

$$u_i = \begin{cases} u & i = 1, \cdots, l \\ 0 & \text{otherwise}\ , \end{cases} \qquad v_i = 0,\ i = 1 \cdots, L\ ,$$

and $M = \|u\|_{W_1^\infty(J)}$, $\delta = \sum_{i=1}^l m(J_i')$. The countable additivity and absolute continuity of ν_u is, by (D), an immediate consequence of (2.12).

By a result of Lebesgue's on differentiation of interval functions [5, pp. 115, 119], Lemma 1 implies that whenever u is in $W_1^\infty(J)$ then the Lebesgue derivative f_u of ν_u is defined almost everywhere, belongs to $L^1(J)$ and satisfies:

$$\nu_u(I) = \int_I f_u\ , \quad \text{for all} \quad I \in S_J\ . \tag{2.13}$$

We have need of a somewhat more precise result.

LEMMA 2.2. *Given a point $\boldsymbol{x} = (x_0, x_1) \in R^2$, let $\mathscr{F}_x$ denote the family of all affine functions u on J with the property that for some $t_0 = t_0(u)$ interior to J,*

$$(u(t_0),\ u'(t_0)) = \boldsymbol{x}\ .$$

Then for each $u \in \mathscr{F}_x$, the Lebesgue derivative of ν_u at t_0 exists and has a value which is independent of the choice of $u \in \mathscr{F}_x$:

$$(D\nu_u)(t_0) = g(\boldsymbol{x}) \quad \textit{for all} \quad u \in \mathscr{F}_x\ . \tag{*}$$

Moreover the function $g\colon R^2 \to R$ defined in this way is continuous and satisfies (2.9).

Proof. For simplicity of notation assume that J is an open interval. Given a function $u \in \mathscr{F}_x$ and a *closed* subinterval $I_0 \subset J$ containing the point $t_0(u)$, put $\eta = m(I_0)$ and denote

$$I_j = I_0 + j\eta,\ u_j(t) = u(t - j\eta),\ j = \pm 1, \pm 2, \cdots \tag{2.14}$$

Let $\mathscr{M}(I_0) = \{I_j\}_{j=-k}^{l}$ denote the maximal family of these intervals containing points of J. Let $u_{\mathscr{M}(I_0)}$ denote the (single-valued) piecewise continuous function on J given by

$$u_{\mathscr{M}(I_0)} = \sum_{-k}^{l} u_j \chi_{I'_j}\,,$$

where $I'_0 = I_0$, and I'_j, $j \neq 0$, is $I_j \cap J$ with either its right endpoint excised ($j < 0$) or its left endpoint excised ($j > 0$).

We examine the quantity $\hat{N}(u_{\mathscr{M}(I_0)})$ which is defined as follows:

$$\hat{N}(u_{\mathscr{M}(I_0)}) = \sum_{-k}^{l} \nu_{u_j}(I'_j) = \sum_{-k}^{l} N(u_j^{I'_j})\,.$$

Since

$$u_j^{I'_j} \underset{1}{\sim} u^{I_0}\,, \quad -k + 1 \leqq j \leqq l - 1\,,$$

we deduce from (B′) the relation

$$\hat{N}(u_{m(I_0)}) = (k + l - 1)N(u^{I_0}) + N(u_{-k}^{I'_{-k}}) + N(u_l^{I'_l})\,. \tag{2.15}$$

Moreover, by our construction

$$(k + l - 1)\eta < m(J) \leqq (k + l + 1)\eta\,.$$

Thus (2.15) implies:

$$\begin{aligned}\frac{1}{\eta}\nu_u(I_0) &= \frac{1}{(k+l-1)\eta}\hat{N}(u_{\mathscr{M}(I_0)}) - \frac{1}{(k+l-1)\eta}[N(u_{-k}^{I'_{-k}}) + N(u_l^{I'_l})] \\ &= \frac{1}{m(J) - \theta\cdot 2\eta}\hat{N}(u_{\mathscr{M}(I_0)}) - \frac{1}{m(J) - \theta\cdot 2\eta}[N(u_{-k}^{I'_{-k}}) + N(u_l^{I'_l})]\,,\end{aligned} \tag{2.16}$$

for some $\theta \in (0, 1)$.

In view of properties (D) and (A′), (2.16) implies that

$$\frac{1}{m(I_0)}\nu_u(I_0) - \frac{1}{m(J)}\hat{N}(u_{\mathscr{M}(I_0)}) \longrightarrow 0 \quad \text{when} \quad m(I_0) \longrightarrow 0\,. \tag{2.17}$$

Moreover, given a positive ε, property (D) implies that

$$|\hat{N}(u_{\mathscr{M}(I_0)}) - \hat{N}(u_{\mathscr{M}(I_0^*)})| < \varepsilon \tag{2.18}$$

whenever I_0, I_0^* are closed intervals of sufficiently small measure containing the point t_0. Hence ν_u possesses a Lebesgue derivative at $t_0 = t_0(u)$.

Next we observe that the Lebesgue derivative

$$(D\nu_u)(t_0(u)) = \lim_{m(I_0)\to 0} \frac{\nu_u(I_0)}{m(I_0)}$$

is the same for all $u \in \mathscr{F}_x$. For if u, $\check{u}$ are in $\mathscr{F}_x$ then whenever the interval $I_0 \ni t_0(u)$ is sufficiently small, there is a corresponding interval $\check{I}_0 \ni t_0(\check{u})$, of the same length, situated so that

$$\check{u}^{\check{I}_0} \underset{1}{\sim} u^{I_0} .$$

Property (B′) then implies that

$$\frac{\nu_u(I_0)}{m(I_0)} = \frac{\nu_{\check{u}}(\check{I}_0)}{m(\check{I}_0)} ,$$

from which we deduce that

$$(D\nu_u)(t_0(u)) = (D\nu_{\check{u}}(t_0(\check{u})) .$$

We denote the common value of all these Lebesgue derivatives by $g(\boldsymbol{x})$:

$$g(\boldsymbol{x}) := (D\nu_u)(t_0(u)) \quad \text{for all} \quad u \in \mathscr{F}_x .$$

It is evident from property (A′) that this definition implies, when $\boldsymbol{x} = (x_0, 0)$, that $\nu_u = 0$ for all $u \in \mathscr{F}_x$, so that (2.9) holds.

Finally, let $\boldsymbol{x}$ and $\boldsymbol{x}^*$ be two points in R^2 and let u be a function in $\mathscr{F}_x$, with $I_0 \subset J$ a closed interval containing $t_0(u)$. Select $v \in \mathscr{F}_{x^*}$ such that $t_0(v) = t_0(u)$. Then given a positive ε, property (D) implies that if $\boldsymbol{x}$, $\boldsymbol{x}^*$ are sufficiently close and if $m(I_0)$ is sufficiently small then

$$|\hat{N}(u_{\mathscr{M}(I_0)}) - \hat{N}(v_{\mathscr{M}(I_0)})| < \varepsilon . \tag{2.19}$$

Thus by (2.17), (2.18) and (2.19), g is a continuous function on R^2. This completes the proof.

COROLLARY 2.1. *The representation*

$$N(u) = \int_J g(u(t), u'(t))dt$$

is valid whenever $u \in W_1^\infty(J)$ is piecewise linear.

Proof. By (A′)

$$N(u) = \sum_{j=1}^{q} N(u^{I_j'})$$

where $\{I_j'\}_{j=1}^q$ is a partition of J into subintervals on each of which

u is linear. Hence by (2.9) it suffices to prove that for any fixed u which is *linear* on J the Lebesgue derivative (equivalently by (2.13), the Radon-Nikodym derivative) of ν_u satisfies:

$$f_u(t) = (D\nu_u)(t) = g(u(t), u'(t)) \quad \text{a.e. } t \in J . \tag{2.20}$$

However the linear function u satisfies

$$u \in \mathscr{F}_{x(\tau)} \quad \text{for all} \quad \tau \in \mathring{J} ,$$

where

$$\boldsymbol{x}(\tau) = (u(\tau), u'(\tau)), \quad \tau \in \mathring{J} .$$

Consequently (2.20) follows from Lemma 2.2.

We now extend the representation to all $u \in W_1^\infty(J)$.

LEMMA 2.3. *The representation*

$$N(u) = \int_J g(u(t), u'(t))dt$$

is valid whenever u is in $W_1^\infty(J)$.

Proof. Given $u \in W_1^\infty(J)$ we proceed to construct a sequence $\{u_n\} \in W_1^\infty(J)$ of piecewise linear functions satisfying

$$\|u_n\|_{W_1^\infty(J)} \leqq 2\|u\|_{W_1^\infty(J)}, \; u_n \to u \quad \text{and} \quad u_n' \longrightarrow u' \text{ a.e.} \tag{2.21}$$

(It then follows by (D) that $N(u_n) \to N(u)$.)

Consider the function $Du \in L^\infty(J)$. Now there exists a sequence $\{z_n\}$ of step functions based on *subintervals* of J such that

$$\|z_n\|_{L^\infty(J)} \leqq \|Du\|_{L^\infty(J)} , \quad z_n \longrightarrow Du \text{ a.e.} \tag{2.22}$$

Select a point $t_0 \in \mathring{J}$ and define the sequence $\{u_n\}$ as follows:

$$u_n(t) = u(t_0) + \int_{t_0}^t z_n(\tau)d\tau, \quad n \geqq 1 .$$

Clearly u_n is piecewise linear and by (2.22) satisfies the relation

$\|u_n\|_{W_1^\infty(J)} \leqq 2\|u\|_{W_1^\infty(J)} =: M$, for sufficiently large n; $u_n \to u$ and $u_n' \to u'$ a.e.

Hence by property (D)

$$N(u) = \lim_{n\to\infty} N(u_n) = \lim_{n\to\infty} \int_J g(u_n(t), u_n'(t))dt ,$$

while by the continuity of g

$$|g(u_n(t), u_n'(t))| \leqq L_M = \sup_{|x| \leqq M} |g(x)| ,$$

$g(u_n(t), u_n'(t)) \to g(u(t), u'(t))$ a.e. The use of Lebesgue's dominated convergence theorem now yields the desired result.

In order to complete the proof of the theorem in the case $1 \leqq p < \infty$ it will be necessary to establish the growth condition (2.11).

LEMMA 2.4. *The function g defined by* (*) *in Lemma* 2.2 *satisfies, if* $1 \leqq p < \infty$, (**) *a growth condition of the form*:

$$(**) \qquad |g(x_0, x_1)| \leqq K_M(1 + |x_1|)^p \quad \textit{whenever} \quad |x_0| \leqq M, \quad \textit{for all } M > 0 .$$

Proof. The given growth condition is equivalent to the assertion that for each $M > 0$

$$\sup_{|x_0| \leqq M} k_{x_0} < \infty \tag{2.23}$$

where

$$k_{x_0} : = \sup_{x_1 \in R} \frac{|g(x_0, x_1)|}{(1 + |x_1|)^p} .$$

Suppose that for some M the assertion in (2.23) is false. Select a sequence $\{\theta_j\}$ in $(0, 1)$ and an increasing sequence $\{D_j\}$ in R^+ such that

$$\Sigma \theta_j = \frac{1}{2} , \quad \Sigma D_j \theta_j = \infty . \tag{2.24}$$

We construct a sequence $\{(c_n, d_n)\} \in [-M, M] \times R$ as follows. Let $c_n \in [-M, M]$ satisfy $k_{c_n} > 2D_n$; then select $d_n \in R$ so that

$$|g(c_n, d_n)| > 2D_n(1 + |d_n|)^p , \quad n \geqq 1 . \tag{2.25}$$

Clearly we can suppose without loss of generality that all $\{d_n\}$ are of the same sign and that all $\{g(c_n, d_n)\}$ are of the same sign, say positive in both cases. Moreover, since any infinite family $\{c_n\}$ satisfying $k_{c_n} > D_n$, $|c_n| \leqq M$, has a cluster point we may suppose, by going to a subsequence if necessary, that

$$\Sigma |c_{n+1} - c_n| \leqq \frac{1}{2} m(J) . \tag{2.26}$$

Now, starting from the left endpoint of J construct a family of subintervals $\{J_j'\}$ of J satisfying

$$\text{(2.27)}\qquad \begin{aligned}&\text{(a)}\quad m(J_j') = \frac{\theta_j}{(1+|d_j|)^{p-1}} m(J)\\ &\text{(b)}\quad \operatorname{dist}\{J_j', J_{j+1}'\} = |c_{j+1} - c_j|\ .\end{aligned}$$

The existence of such a family follows from (2.24) and (2.26). Denote by $\{J_j''\}$ the sequence of intervals forming the gaps between the $\{J_j'\}$:

$$\text{(2.28)}\qquad J_j'' \text{ is contiguous to } J_j' \text{ and } J_{j+1}',\ j \geqq 1$$

(Note that J_j'' is empty whenever $c_{j+1} = c_j$.)

We construct on each interval J_j' a continuous piecewise linear function v_j as follows:

(i) each linear piece of v_j has slope d_j or -1,

(ii) v_j attains the value c_j on each (maximal) subinterval of J_j' where v_j is linear, as well as at the endpoints of J_j',

$$\text{(2.29)}\qquad \begin{aligned}&\text{(iii)}\quad |g(v_j(t), v_j'(t)) - g(c_j, d_j)| < \frac{1}{2}|g(c_j, d_j)|\\ &\qquad\qquad \text{wherever } v_j'(t) = d_j\ .\end{aligned}$$

It can be seen from (i), (ii) that the sets

$$A_j = \{t \in J_j' \colon v_j'(t) = d_j\},\ B_j = \{t \in J_j' \colon v_j'(t) = -1\}$$

satisfy:

$$\text{(2.30)}\qquad \begin{aligned} m(A_j) &= \frac{1}{1+d_j} m(J_j') = \frac{\theta_j}{(1+d_j)^p} m(J)\ ,\\ m(B_j) &= \frac{d_j}{1+d_j} m(J_j') = \frac{d_j\theta_j}{(1+d_j)^p} m(J),\ j \geqq 1\ .\end{aligned}$$

By (2.27), we can construct on each nonempty interval J_j'' a linear function w_j, of slope either 1 or -1, taking the value c_j at the left endpoint of J_j'' and the value c_{j+1} at the right endpoint.

Note that by (2.29)(i), (ii) and (2.30), $|v_j(t) - c_j| \leqq m(J_j')$ for $j = 1, 2, \cdots$. Thus,

$$\sup_{t \in J_j'} |v_j(t)| \leqq M + m(J) =: M_1\ , \qquad j = 1, 2, \cdots\ .$$

By the definition of w_j we have also $\sup_{t \in J_j''} |w_j(t)| \leqq M,\ j = 1, 2, \cdots$.

Now put $c_0 = \lim_{j\to\infty} c_j$ (see (2.26)) and examine the continuous function defined by

$$u = \sum_{j=1}^{\infty} v_j \chi_{J_j'} + \sum_{j=1}^{\infty} w_j \chi_{J_j''} + c_0 \chi_{\tilde{J}}$$

where $\tilde{J} = J \backslash \bigcup_{j=1}^{\infty} (J_j' \cup J_j'')$. Obviously u is locally absolutely continuous on the interior of the intervals $\tilde{J}$ and $J\backslash\tilde{J}$. We now note that u is in $W_1^p(J)$ since by (2.29), (2.30), (2.24):

$$\begin{aligned} ||Du||_{L^p(J)} &= \left[\sum_{j=1}^{\infty} \int_{J_j'} |v_j'(t)|^p dt + \sum_{j=1}^{\infty} \int_{J_j''} |w_j'(t)|^p dt\right]^{1/p} \\ &\leqq \left[\sum_{j=1}^{\infty} (d_j^p m(A_j) + m(B_j)) + m(J)\right]^{1/p} \qquad (2.31) \\ &\leqq \left(\frac{5}{2}\right)^{1/p} m(J)^{1/p} < \infty \,. \end{aligned}$$

Moreover, the sequence $\{u^n\} \subset W_1^p(J)$ which is defined by

$$u^n = \sum_{j=1}^{n} v_j \chi_{J_j'} w_j \chi_{J_j''} + \sum_{j=1}^{n} w_j \chi_{J_j''} + c_{n+1} \chi_{\tilde{J}_n}$$

where $\tilde{J}_n = J\backslash\bigcup_{j=1}^{n} (J_j' \cup J_j'')$, is easily seen to converge to u:

$$||u - u^n||_{W_1^p(J)} \longrightarrow 0 \,. \tag{2.32}$$

Consider the sequence $\{N(u^n)\}$. By use of Lemma 2.3, 2.29(iii), (2.24), (2.25) and (2.30) we have:

$$\begin{aligned} N(u^n) &= \sum_{j=1}^{n} \int_{J_j'} g(v_j(t), v_j'(t))dt + \int_{J_j''} g(w_j(t), w_j'(t))dt \\ &\leqq \sum_{j=1}^{n} \left[\frac{1}{2} g(c_j, d_j) m(A_j) - \left(\sup_{|x_0| \leqq M_1} |g(x_0, -1)|\right) m(B_j)\right] \\ &\quad - \sum_{j=1}^{n} \left(\sup_{|x_0| \leqq M} |g(x_0, \pm 1)\right) m(J_j'') \\ &\geqq \left(\sum_{j=1}^{n} D_j \theta_j - \sup_{|x_0| \leqq M_1} |g(x_0, \pm 1)|\right) m(J) \longrightarrow \infty \,. \end{aligned}$$

However, by (2.32) this contradicts property (C). The lemma is proved.

We are now in a position to complete the proof of the theorem in the case $1 \leqq p < \infty$.

Proof of Theorem 2.1. Given a function $u \in W_1^p(J)$, let $\{u_n\}$ denote a sequence in $W_1^\infty(J)$ converging to u:

$$||u_n - u||_{W_1^p(J)} \longrightarrow 0 \,. \tag{2.33}$$

Then it follows that the (continuous) functions $\{u_n\}$ converge uniformly to u; in particular,

$$|u_n(t)| \leqq M < \infty \,, \quad t \in J \,, \quad n \geqq 1 \,. \tag{2.34}$$

Moreover, (2.33) implies that, by selecting a subsequence $\{u_{n_i}\}$, we can require:

$$\mathrm{D}u_{n_i} \longrightarrow Du \text{ a.e. and } \sum_{i=1}^{\infty} \|Du_{n_i} - Du\|_{L^p(J)} < \infty . \tag{2.35}$$

It follows from (2.35) that the functions $\{Du_{n_i}\}$ satisfy:

$$|Du_{n_i}(t)| \leqq z(t) \text{ a.e., for some } z \in L^p(J) . \tag{2.36}$$

Hence, by (2.34) and Lemma 2.4 we deduce that the functions $\{Gu_{n_i}\}$ are dominated by an integrable function:

$$\begin{aligned} |Gu_{n_i}(t)| = |g(u_{n_i}(t), u'_{n_i}(t))| &\leqq K_M(1 + |u'_{n_i}(t)|)^p \\ &\leqq K_M(1 + |z(t)|)^p \text{ a.e., } i \geqq 1 . \end{aligned} \tag{2.37}$$

Now by Lemma 2.3

$$N(u_{n_i}) = \int Gu_{n_i} , \quad n_i \geqq 1 .$$

Combining (2.36) and (2.37) we deduce by means of property (C) the relation:

$$N(u) = \lim_{n_i \to \infty} N(u_{n_i}) = \lim_{n_i \to \infty} \int g(u_{n_i}(t), u'_{n_i}(t))dt = \int g(u(t), u'(t))dt ,$$

where the last equality utilizes the Lebesgue dominated convergence theorem. This completes the argument.

REMARK 2.1. We point out that the proof of Theorem 2.1 actually utilized only the following weaker form of condition (B′):

(B″) N is invariant under *inessential translation*:

$$N(T_h u) = N(u) \quad \text{whenever} \quad T_h u \underset{1}{\sim} u ,$$

where $T_h u$, $|h| < \operatorname{mes} J$, is defined by:

$$(T_h u)(t) = \begin{cases} u(\inf J) , & t < h + \inf J \\ u(t - h) , & t - h \in J \\ u(\sup J) , & t > h + \sup J , \end{cases} \qquad t \in J .$$

Given any functional $N: W_1^p(J) \to R$ which satisfies conditions (A), (B), (C) there is a canonical decomposition of N

$$N = \tilde{N} + M_f , \tag{2.38}$$

where $\tilde{N}$ is envelope additive and thus satisfies (A′), (B), (C) while M_f is a "lower order" functional of the form:

$$M_f(u) = \int_J f(u(t))dt \ .$$

To see this, define the function $f: R \to R$ in terms of the values that N assumes on the one-dimensional subspace of constant functions:

$$(2.39) \qquad f(c) := N(c)/m(J) \ , \quad c \in R \ .$$

Then f is continuous by property (C) of N, from which it readily follows that M_f satisfies (A), (B), (C).

Consequently, the function $\tilde{N}$ satisfies (A), (B), (C) and in addition annihilates the constant functions:

$$(2.40) \qquad \tilde{N}(c) = 0 \ , \quad c \in R \ .$$

These facts imply that $\tilde{N}$ also satisfies the envelope additivity condition (A′); for whenever $u, v \in W_1^p(J)$ are envelope compatible we have from (A)

$$N(u \oplus v) = N(u) + N(v) - N(v\chi_{J'} + u\chi_{J''}) \ ,$$

while (2.40) ensures the vanishing of the last term, since the function $v\chi_{J'} + u\chi_{J''}$ is necessarily constant. The fact that $\tilde{N}$ also satisfies (B'') lies deeper; a proof is given in the appendix.

Utilizing the above remark we deduce from Theorem 2.1 the following result:

THEOREM 2.2. *Let J be a bounded interval and let N be a real functional on $W_1^p(J)$, $1 \leqq p \leqq \infty$, which possesses the properties:*

(A) $N(u) + N(v) = N(u\chi_{J'} + v\chi_{J''}) + N(v\chi_{J'} + u\chi_{J''})$ *whenever u, v are swappable on $\{J', J''\}$,*

(B) $N(u) = N(v)$ *whenever* $u \underset{1}{\approx} v$,

(C) $N(u_m) \to N(u_0)$ *whenever* $\|u_m - u_0\|_{W_1^p(J)} \to 0$, $1 \leqq p < \infty$,

(D) $\lim_{\delta\to 0+} V_M(\delta; \tilde{N}) = 0$ *for each $M > 0$, where $\tilde{N}$ is the envelope additive part of N, as in* (2.38).

Then there exists a unique continuous function $g: R^2 \to R$ such that

$$(2.41) \qquad N(u) = \int_J Gu \quad \textit{for all} \quad u \in W_1^p(J) \ .$$

Moreover, when $1 \leqq p < \infty$ the function g satisfies a growth condition of the form:

$$|g(x_0, x_1)| \leqq K_M(1 + |x_1|)^p \quad \textit{for} \quad |x_0| \leqq M, \quad \textit{for all} \quad M > 0 \ .$$

The argument proceeds as follows. Let $\tilde{N}$ and f be defined as in (2.38) and (2.39). Then f is a continuous function and as is shown in the appendix, $\tilde{N}$ satisfies the assumptions of Theorem 2.1. Let $\tilde{g}: R^2 \rightarrow R$ be a kernel for $\tilde{N}$ as in Theorem 2.1. Then the function $g: R^2 \rightarrow R$ given by

$$g(x_0, x_1) = \tilde{g}(x_0, x_1) + f(x_0)$$

possesses all the properties stated in Theorem 2.2. The uniqueness of g is clear from the proof.

Note that the converse assertion that every functional of the form (2.41) that fulfills the stated conditions, satisfies (A), (B), (C), (D) is immediate from what has gone before.

3. Representation for operators. Here we obtain a characterization for those non-linear operators $G: W_1^p(J) \rightarrow L^q(J)$, $1 \leqq p \leqq \infty$, $1 \leqq q < \infty$, possessing the form (2.5).

It should be noted that each of the conditions (A), (A′), (B), (B′), (C), (D) possesses an analogue which is applicable to such operators. One interprets inequalities in the almost everywhere sense and replaces absolute values by norms. Thus one can formulate the following properties:

(A_G) *G is additively invariant under swapping*:

$$Gu + Gv = G(u\chi_{J'} + v\chi_{J''}) + G(v\chi_{J'} + u\chi_{J''})$$

whenever $u, v \in W_1^p(J)$ are swappable across the partition $\{J', J''\}$.

(A'_G) *G is* 1-*envelope additive*:

$$G(u \oplus v) = Gu + Gv \text{ whenever } u, v \in W_1^p(J)$$

are envelope compatible.

(B_G) *G is invariant, up to equimeasurability, under* 1-*equimeasurability*:

$$Gu \underset{0}{\approx} Gv \quad \text{whenever } u \underset{1}{\approx} v\,, \quad u, v \in W_1^p(J)\,.$$

(B'_G) *G is invariant, up to equimeasurability, under generalized* 1-*equimeasurability*:

$$Gu \underset{0}{\approx} Gv \quad \text{whenever} \quad u \underset{1}{\sim} v\,, \quad u, v \in W_1^p(J)\,.$$

(C_G) *G is continuous*:

$$||G(u_m) - G(u_0)||_{L^q(J)} \longrightarrow 0 \quad \text{whenever} \quad ||u_m - u_0||_{W_1^p(J)} \longrightarrow 0\,, \quad 1 \leqq p < \infty\,.$$

(D_G) *G is locally uniformly continuous in* (interval) *variation*:

$$\lim_{\delta \rightarrow 0+} V_M(\delta; G) = 0 \quad \text{for each} \quad M > 0$$

where $V_M(\delta; G)$ is defined by

$$V_M(\delta; G) = \sup \sum_{i=1}^{l} \| G(u_i^{J_i}) - G(v_i^{J_i}) \|_{L^1(J_i)}$$

with the supremum being taken over all finite partitions $\{J_i\}_{i=1}^{l}$ of J into subintervals and all sets of pairs $u_i, v_i \in W_1^\infty(J_i')$ satisfying:

$$\| u_i \|_{W_1^\infty(J_i)}, \| v_i \|_{W_1^\infty(J_i)} \leqq M , \quad \rho(U, V) \leqq \delta ,$$

where $U = \Sigma u_i \chi_{J_i}$, $Du = \Sigma Du_i \chi_{I_i}$ and similarly for V.

It is easily verified, by arguments similar to those used for functionals N of the form (2.7), that any G of the form (2.5), (2.6) satisfies (A_G), (B_G), (C_G). Similarly, it is seen that when g satisfies (2.8) then (A'_G), (B'_G) and (D_G) also hold. However these conditions do *not* suffice to characterize operators of the form (2.5); indeed the linear transformation $G: W_1^p(J) \to L^q(J)$ (with one-dimensional range) which is given by

$$(Gu)(t) \equiv \int_J Du = \text{const.} ,$$

also satisfies (A_G), (A'_G), (B_G)-(D_G). An additional localization condition is required.

(E_G) *G is local:*

$$K(Gu - Gv) \subset K(u - v) \quad \text{for all} \quad u, v \in W_1^p(J) .$$

The inclusion should be interpreted as inclusion modulo a null set.

We can now give our characterization results for operators of the form (2.5). Again our main efforts will be directed to obtaining the result under the assumption of envelope additivity.

THEOREM 3.1. *Let J be a bounded interval and let G be a transformation from $W_1^p(J)$ to $L^q(J)$, $1 \leqq p \leqq \infty$, $1 \leqq q < \infty$, which satisfies the conditions (A'_G), (B'_G)-(E_G).*

Then there exists a unique continuous real function $g: R^2 \to R$, satisfying

$$(*) \qquad g(x_0, 0) = 0 \quad \textit{for all} \quad x_0 \in R ,$$

such that

$$(3.1) \qquad (Gu)(t) = g(u(t), u'(t)) \text{ a.e.} \quad \textit{for all} \quad u \in W_1^p(J) .$$

Moreover, for $1 \leqq p < \infty$, g satisfies a growth condition of the form:

$$(3.2)\qquad |g(x_0, x_1)| \leqq K_M(1 + |x_1|)^{p/q} \text{ whenever } |x_0| \leqq M\,, \quad \text{for all} \quad M > 0\,.$$

Proof. Note that, in view of (A'_G), the mapping G takes all constants to the zero function. Hence (E_G) ensures the validity of:

(E'_G) $K(Gu) \subseteqq J \backslash A_u$ where A_u denotes the union of all intervals in which u is constant.

First, suppose that the Theorem holds for $q = 1$. In order to establish the result for $q > 1$ we proceed as follows. Consider G as a mapping into L^1. Then G satisfies all the assumptions of the Theorem for the case $q = 1$. Hence there exists a continuous function $g: R_2 \to R$ such that (3.1) and (*) hold. The fact that g satisfies the growth condition (3.2) now follows by the same argument used in the proof of Lemma 2.4. We do not repeat this argument since only minor modifications are needed. Thus in the definition of k_{x_0} and in (2.24) p will be replaced by p/q. In (2.23)(iii) we add the condition

$$|g(v_j(t), v'_j(t))^q - g(c_j, d_j)^q| \leqq \frac{1}{2} g(c_j, d_j)^q\,,$$
$$\text{whenever} \quad v'_j(t) = d_j\,.$$

Recall that, by assumption $g(c_j, d_j) > 0$ ($j = 1, 2, \cdots$) and condition (2.29)(iii), as stated in Section 2, ensures that $g(v_j(t), v'_j(t)) > 0$ whenever $v'_j(t) = d_j (j = 1, 2, \cdots)$. In the final part of the argument we obtain a contradiction by considering the inequalities

$$\begin{aligned}\int_J |Gu_n|^q dt &= \sum_{j=1}^{n} \left(\int_{J'_j} |g(v_j(t), v'_j(t))|^q dt + \int_{J''_j} |g(w_j(t), w'_j(t)|^q dt \right) \\ &\geqq \sum_{j=1}^{n} \frac{1}{2} g(c_j, d_j)^q m(A_j) - \sup_{|x_0| \leqq M_1} |g(x_0, \pm 1)|^q m(J) \\ &\geqq \left(\sum_{j=1}^{n} D_j \theta_j - \sup_{|x_0| \leqq M^1} |g(x_0, \pm 1)| \right) m(J) \longrightarrow \infty\,.\end{aligned}$$

We turn now to the proof of the theorem in the case $q = 1$. For any interval $I \subset J$ we can utilize the operator G to define a functional $N^I: W_1^p(I) \to R$ as follows.

$$(3.3)\qquad N^I(u) = \int_J G(u^I) = \int_I G(u^I) \quad \text{for all} \quad u \in W_1^p(I)\,,$$

where $u^I \in W_1^p(J)$ is obtained by extending u to all of J by the use of constants to the right and left of I so that the resulting function is continuous on J, and where the second equality follows from (E'_G). Now the fact that G satisfies (A'_G)-(E_G) implies that N^I satisfies (A′)-(D) of Theorem 2.1. Hence there exists a unique continuous function

$g^I: R^2 \to R$ such that (*) holds and (if $p < \infty$) (3.2) holds and, in addition,

$$N^I(u) = \int_I g^I(u(t), u'(t))dt \quad \text{for all} \quad u \in W_1^p(I) . \tag{3.4}$$

Next it will be shown that the functions $\{g^I\}_{I\subset J}$ are identical. By (3.3) and (3.4):

$$N^I(u) = N^J(u^I) = \int_I g^J(u(t), u'(t))dt \quad \text{for all} \quad u \in W_1^p(I) , \tag{3.5}$$

where the second equality follows from (*). On comparing (3.3) and (3.5) and applying the uniqueness statement of Theorem 2.1 we deduce that

$$g^I = g^J \quad \text{whenever} \quad I \subset J . \tag{3.6}$$

Finally, given $u \in W_1^p(J)$ and any $I \subset J$ let $\{I', I, I''\}$ denote the partition of J (into intervals) which is induced by I. Then by (A'_G)

$$Gu = Gu^{I'} + Gu^I + Gu^{I''}$$

and hence by the use of (E'_G) and (3.6)

$$\int_I Gu = \int_I Gu^I = N^I(u) = \int_I g^J(u(t), u'(t))dt , \quad u \in W_1^p(J) . \tag{3.7}$$

The validity of (3.7) for all $I \subset J$ clearly implies

$$(Gu)(t) = g^J(u(t), u'(t)) \text{ a.e.} \quad \text{for all} \quad u \in W_1^p(J) .$$

This completes the argument.

REMARK 3.1. In order to extend our results to more general operators G we decompose G as follows. Supposing that G takes constant functions to constant functions, we let $f: R \to R$ be given by:

$$(Gc)(t) \equiv f(c) = \text{const. a.e.,} \quad \text{for all} \quad c \in R . \tag{3.8}$$

Then condition (C_G) implies that $f: R \to R$ is continuous.

Now if we decompose

$$\begin{aligned}(Gu)(t) &= [(Gu)(t) - f(u(t))] + f(u(t)) =: (\tilde{G}u)(t) + (Fu)(t) , \\ & u \in W_1^p(J) ,\end{aligned} \tag{3.9}$$

then F satisfies (A_G)-(E_G). Hence $\tilde{G}$ is easily seen to satisfy (A_G)-(E_G) and, by construction, $\tilde{G}$ takes all constants to the zero function. It follows from this that $\tilde{G}$ also satisfies (A'_G) and (B'_G) and hence Theorem 3.1 is applicable. In this way we could obtain as a corollary of Theorem 3.1 a result pertaining to transformations $G: W_1^p(J) \to L^q(J)$ which are not envelope additive but instead satisfy (A_G) and map constants to constants.

4. **Appendix.** Here it will be shown that, under the hypotheses of Theorem 2.2, the functional $\tilde{N}$ defined in (2.38) satisfies condition (B″) in addition to (A)′, (B), (C), (D). By Theorem 2.1 (see Remark 2.1) $\tilde{N}$ then possesses the representation (2.10), (2.11), which yields Theorem 2.2.

The first stage of the argument involves showing that in any event there exists a continuous function $\gamma: R \to R$ such that $N^* = \tilde{N} - P_\gamma$ satisfies (A′), (B″) and (C), where

$$\begin{aligned} P_\gamma(u) &= \int_J t d(\gamma \circ u)(t) = t\gamma(u(t))\Big|_{\inf J}^{\sup J} - \int_J (\gamma \circ u)(t) dt \\ &= t\gamma(u(t))\Big|_{\inf K(Du)}^{\sup K(Du)} - \int_I (\gamma \circ u)(t) dt , \\ &\quad \text{where } I = [\inf K(Du), \sup K(Du)] . \end{aligned} \tag{4.1}$$

Let $u \in W_1^p(J)$, $h \in R$ be such that $T_h u \underset{1}{\sim} u$ and $K(Du) \subset \mathring{J}$. Put

$$\alpha_u(h) = \tilde{N}(T_h u) - \tilde{N}(u) . \tag{4.2}$$

Now there exists a function $v \in W_{1,p}(J)$, $K(Dv) \subset \mathring{J}$, satisfying $v(\inf J) = u(\sup J)$, $v(\sup J) = u(\inf J)$, $u \oplus v$ exists,

$$T_h v \underset{1}{\sim} v . \tag{4.3}$$

It follows that $T_h u \oplus T_h v \underset{1}{\approx} u \oplus v$, so that properties (A′) and (B) of $\tilde{N}$ imply

$$\alpha_u(h) = -(\tilde{N}(T_h v) - \tilde{N}(v)) . \tag{4.4}$$

Hence, denoting $u(\inf J) = a$, $u(\sup J) = b$ we see that the dependence of α_u on u is described by

$$\alpha_u(h) = \alpha(a, b; h) . \tag{4.5}$$

Condition (C) implies that $\alpha: R^3 \to R$ is continuous. Moreover, (4.4) and the definition of v yield the identity

$$\alpha(b, a; h) = -\alpha(a, b; h) . \tag{4.6}$$

Next we observe that the value of α is proportional to h. This follows from the relations

$$\tilde{N}(T_h u) - \tilde{N}(u) = \sum_{j=1}^{n} [\tilde{N}(T_{(j/n)h}u) - \tilde{N}(T_{(j-1/n)h}u)] ,$$
$$T_{rh}u \underset{1}{\sim} u \quad \text{for} \quad r \in [0, 1] ;$$

these relations imply

$$\alpha\left(a, b; \frac{j}{n}h\right) = \frac{j}{n}\alpha(a, b; h) , \quad 1 \leqq j \leqq n , \qquad n = 1, 2, \cdots .$$

Hence there exists a continuous function $\beta: R^2 \to R$ such that

$$\alpha(a, b; h) = \beta(a, b)h . \tag{4.7}$$

Moreover, since for any c between a and b we can decompose

$$u = w \oplus z, \text{ where } w(\inf J) = a, w(\sup J) = z(\inf J) = c , \quad z(\sup J) = b ,$$

it follows that

$$\beta(a, c)h + \beta(c, b)h = \beta(a, b)h . \tag{4.8}$$

Together (4.6)-(4.8) imply the existence of a unique continuous function $\gamma: R \to R$ satisfying

$$\gamma(0) = 0 , \quad \beta(a, b) = \gamma(b) - \gamma(a) .$$

That is

$$\tilde{N}(T_h u) - \tilde{N}(u) = [\gamma(b) - \gamma(a)]h , \tag{4.9}$$

which justifies the earlier assertion that $N^* = \tilde{N} - P_\gamma$ satisfies (B''):

$$N^*(u) = \tilde{N}(u) - P_\gamma(u) = \tilde{N}(T_h u) - P_\gamma(T_h u) = N^*(T_h u) .$$

Since (4.1) clearly implies that P_γ satisfies (A'), (C), the claim that N^* satisfies (A'), (B''), (C) is proved.

We now proceed by a series of propositions. For convenience we hereafter put $J = [0, 1]$.

PROPOSITION 4.1. *The function γ is of class C^1.*

Proof. Given $I_M = [-M, M]$, there exists, by condition (D), for each $\varepsilon > 0$, a $\delta > 0$ satisfying $V_M(\delta; \tilde{N}) < \varepsilon$. Select $U = \Sigma u_i \chi_{J_i}$, $V = \Sigma v_i \chi_{J_i}$ subject only to the following restrictions:

$$\begin{cases} J_i = [\underline{t}_i, \bar{t}_i] \text{ are non-overlapping subintervals of } [0, 1/2] , \\ \|u_i\|_{W_{1,\infty}}(J_i), \ \|v_i\|_{W_{1,\infty}}(J_i) \leqq M , \\ \rho(U, V) < \delta/2 . \end{cases} \tag{4.10}$$

Denote

$$u_i(\inf J_i) =: a_i,\ v_i(\inf J_i) =: a_i',\ u_i(\sup J_i) =: b_i\ ,\quad v_i(\sup J_i) =: b_i'$$

and define $\tilde{U}$, $\tilde{V}$ by

$$\tilde{U} = U + T_{1/2}U,\ \tilde{V} = V + T_{1/2}V\ .$$

Then $\rho(\tilde{U}, \tilde{V}) < \delta$ and it follows that

$$\begin{aligned}
\varepsilon &> \Sigma|\tilde{N}(T_{1/2}u_i^{J_i}) - \tilde{N}(T_{1/2}v_i^{J_i})| + \Sigma|\tilde{N}(u_i^{J_i}) - \tilde{N}(v_i^{J_i})| \\
&\geqq \Sigma|\tilde{N}(T_{1/2}u_i^{J_i}) - \tilde{N}(u_i^{J_i}) - (\tilde{N}(T_{1/2}v_i^{J_i}) - \tilde{N}(v_i^{J_i}))| \\
&= \Sigma\left|\frac{1}{2}(\gamma(b_i) - \gamma(a_i)) - \frac{1}{2}(\gamma(b_i') - \gamma(a_i'))\right| ,
\end{aligned} \tag{4.11}$$

where the last equation follows from (4.9).

Consider first the following case:

$v_i \equiv 0\,\forall i$, $\Sigma \operatorname{mes} J_i < \delta/2$, $u_i|_{J_i}$ has constant slope of magnitude $\pm M$.

Equation (4.11) then implies that for every family of (possibly overlapping) subintervals $[a_i, b_i] \subset [-M, M] = I_M$,

$$\Sigma|b_i - a_i| < M\delta/2 \Longrightarrow \Sigma|\gamma(b_i) - \gamma(a_i)| < 2\varepsilon\ . \tag{4.12}$$

This condition ensures that γ is absolutely continuous (in fact, Lipschitz continuous) on I_M. Hence the derivative γ' is defined on a subset E of total measure in $\mathring{I}_M = (-M, M)$. We proceed to show that γ' is uniformly continuous on E. Thus γ' is equivalent to a *continuous* function on I_M, from which it follows that the absolutely continuous function γ is actually C^1.

Given $a^*, \bar{a}^* \in E$ we show

$$|\bar{a}^* - a^*| < \delta \Longrightarrow |\gamma'(a^*) - \gamma'(\bar{a}^*)| \leqq 4\varepsilon/M\ . \tag{4.13}$$

Let us define U, V by means of $U = \Sigma u_i\chi_{J_i}$, $V = \Sigma v_i\chi_{J_i}$, where

$$J_i = [\underline{t}_i, \bar{t}_i] = \left[\frac{i-1}{2n}, \frac{i}{2n}\right],\ 1 \leqq i \leqq n,\ a_i = a^*,\ b_i = a^* + M/2n$$
$$a_i' = \bar{a}^*,\ b_i' = \bar{a}^* + M/2n\ .$$

For n sufficiently large the functions u_i, v_i will satisfy $\|u_i\|_{W_1^\infty(J_i)}$, $\|V_i\|_{W_1^\infty(J_i)} \leqq M$. The inequality (4.11) now reads

$$\begin{aligned}
\varepsilon &> n/2|\gamma(a^* + M/2n) - \gamma(a^*) - (\gamma(\bar{a}^* + M/2n) - \gamma(\bar{a}^*))| \\
&> M/4\left|\frac{\gamma(a^* + M/2n) - \gamma(a^*)}{M/2n} - \frac{\gamma(\bar{a}^* + M/2n) - \gamma(\bar{a}^*)}{M/2n}\right|\ .
\end{aligned}$$

Proceeding to the limit as $n \to \infty$, we obtain the final inequality in (4.13), which completes the argument.

PROPOSITION 4.2. *P_γ satisfies condition* (D).

Proof. Given M and $\varepsilon > 0$, select $\delta > 0$ and let $U = \Sigma u_i \chi_{J_i}$, $V = \Sigma v_i \chi_{J_i}$ be chosen arbitrarily subject to:

$$J_i \subset [0, 1];\ \|u_i\|_{W_{1,\infty}(J_i)},\ \|v_i\|_{W_{1,\infty}(J_i)} \leqq M;\ \rho(U, V) < \delta .$$

We can assume, by partitioning the intervals J_i if necessary, that mes $J_i = \bar{t}_i - \underline{t}_i \leqq \delta/M$, $\forall i$. It then follows that

$$\begin{aligned} |a_i' - a_i| > 2\delta &\Longrightarrow |u_i(t) - v_i(t)| > \delta\ , \quad \forall t \in J_i \\ |a_i' - a_i| \leqq 2\delta &\Longrightarrow \|u_i - v_i\|_{L^\infty(J_i)} < 3\delta\ . \end{aligned} \tag{4.14}$$

Now applying a well-known chain rule we obtain:

$$\begin{aligned} \Sigma\,|P_\gamma(u_i^{J_i}) - P_\gamma(v_i^{J_i})| &= \Sigma \left| \int_{J_i} t[\gamma'(u_i(t))u_i'(t) - \gamma'(v_i(t))v_i'(t)]dt \right. \\ &\leqq \sum_{|a_i - a_i'| \leqq 2\delta} \int_{J_i} t[|\gamma'(u_i(t)) - \gamma'(v_i(t))|\,|u_i'(t)| \\ &\quad + |\gamma'(v_i(t))|\,|u_i'(t) - v_i'(t)|]dt \\ &\quad + \sum_{|a_i' - a_i| > 2\delta} \int_{J_i} tM[|\gamma'(u_i(t))| + |\gamma'(v_i(t))|]dt \\ &\leqq \sum_{|a_i' - a_i| \leqq 2\delta} \left[M \sup_{\substack{|s - s^*| \leqq 3\delta \\ |s|, |s^*| \leqq M}} |\gamma'(s) - \gamma'(s^*)| + \delta \sup_{|s| \leqq M} |\gamma'(s)| \right] [\bar{t}_i - \underline{t}_i] \\ &\quad + 2M\delta \sup_{|s| \leqq M} \gamma'(s) + \sum_{|a_i' - a_i| > 2\delta} 2M \sup_{|s| \leqq M} |\gamma'(s)|\,[\bar{t}_i - \underline{t}_i] \\ &\leqq M \sup_{\substack{|s - s^*| \leqq 3\delta \\ |s|, |s^*| \leqq M}} |\gamma'(s) - \gamma'(s^*)| + (4M + 1)\delta \sup_{|s| \leqq M} |\gamma'(s)|\ . \end{aligned}$$

Clearly for δ sufficiently small the right side will be less than ε, which yields the proof.

By Proposition 4.2 it follows that $N^* = \tilde{N} - P_\gamma$ satisfies (D) as well as (A′), (B″), (C). Hence by Theorem 2.1 (see Remark 2.1) there exists a unique continuous function $g^*: R^2 \to R$ such that

$$N^*(u) = \int_J g^*(u(t), \dot{u}(t))dt\ . \tag{4.15}$$

PROPOSITION 4.3. *The function γ is identically zero, so that P_γ is the zero functional and $\tilde{N} = N^*$.*

Proof. Given $a \neq b$, select $u \in W_{1,p}(J)$ satisfying

$$u(\inf J) = u(\sup J) = a,\ u(t_0) = b \quad \text{for some} \quad t_0 \in \mathring{J}\ .$$

Let the function $\bar{u} \in W_{1,p}(J)$ be defined by

$$\bar{u}(t) = u(\bar{t}) \quad \text{where} \quad \bar{t} = t - t_0 (\mathrm{mod}\ 1)\ .$$

Clearly $\bar{u} \underset{1}{\approx} u$ so that condition (B) implies $\tilde{N}(\bar{u}) = \tilde{N}(u)$ while (4.15) implies $N^*(\bar{u}) = N^*(u)$, and hence $P_\gamma(\bar{u}) = P_\gamma(u)$. On the other hand, we deduce by (4.1) that

$$P_\gamma(\bar{u}) - P_\gamma(u) = \gamma(b) - \gamma(a)\ .$$

This completes the argument.

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