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SOBOLEV APPROXIMATION BY A SUM OF SUBALGEBRAS ON THE CIRCLE

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Let ψ be an orientation-reversing homeomorphism of the unit circle onto itself. We consider approximation in certain Sobolev norms by functions of the form $f(z) + g(\psi)$, where f and g are polynomials. The methods involve conformal welding and Hardy space theory. We construct a Jordan arc of positive continuous analytic capacity such that the harmonic measures for the two complementary domains are mutually absolutely continuous.

Let $C(S)$ denote the space of continuous, complex-valued functions on the unit circle S. For $f \in C(S)$, let $P(f)$ denote the space of all polynominals in f with complex coefficients. Let z denote the identity function. Browder and Wermer proved the following theorem [3, p. 551].

Let ψ be a direction-reversing homeomorphism of S onto S. Then the vector space sum $P(z) + P(\psi)$ is uniformly dense in $C(S)$.

In the first part of this paper we prove a result that partially extends this theorem to the C^1 norm. We say that a direction-reversing homeomorphism $\psi: S \to S$ is an *involution* if $\psi \circ \psi = z$. Let $C^1(S)$ denote the space of continuously-differentiable, complex-valued functions on S, with the norm

$$||f||_{C^1} = ||f||_\infty + \left|\left|\frac{df}{d\theta}\right|\right|_\infty .$$

If $\alpha > 0$, let $C^{1+\alpha}(S)$ denote the space of functions f in $C^1(S)$ such that $f' = df/d\theta$ satisfies a Lipschitz condition with exponent α:

$$|f'(a) - f'(b)| \leqq K|a - b|^\alpha$$

for all points a and b in S.

THEOREM. *Let $\alpha > 0$, and let $\psi \in C^{1+\alpha}(S)$ be an involution. Then $P(z) + P(\psi)$ is dense in $C^1(S)$.*

Our proof of this theorem is radically different from Browder and Wermer's proof of their result. By means of duality and conformal welding, we reduce the theorem to a statement about removable singularities for a certain Hardy class, which we then prove.

For $p \geqq 1$, let W^p denote the Sobolev space

$$\left\{f \in C(S): \frac{df}{d\theta} \in L^p(d\theta)\right\} .$$

The norm on the algebra W^p is

$$||f||_{W^p} = ||f||_\infty + \left\|\frac{df}{d\theta}\right\|_p .$$

In the second part of the paper we construct an involution $\psi \in W^1$ such that the sum $P(z) + P(\psi)$ is not dense in W^1. This same ψ provides the answer to a problem raised by Wermer. He asked for an absolutely-continuous involution ψ such that there exists a nonconstant function f, analytic on the open unit disc D, and continuous on $D \cup S$, such that $f \circ \psi = f$ on S.

We wish to thank John Wermer for suggesting the problem discussed here. The idea of using welding in this way is due to Wermer. We are grateful to Lennart Carleson for suggesting the argument of section (1.4).

1. Proof of the theorem.

1.1. Let $\alpha > 0$, and let $\psi \in C^{1+\alpha}(S)$ be an involution. We wish to prove that $P(z) + P(\psi)$ is dense in $C^1(S)$. Each linear fractional transformation that fixes D is approximable in $C^1(S)$ by polynominals. Hence we may assume that the fixed points of ψ are -1 and 1.

Clearly, $P(z) + P(\psi)$ is dense in $C^1(S)$ if and only if $P(z) + \psi' P(\psi)$ is dense in $C(S)$ (in the uniform norm). By a duality argument and the F. and M. Riesz theorem, this happens if and only if

$$H^1(z) \cap H^1(\psi) = \boldsymbol{C} , \tag{1}$$

where $H^1(z)$ denotes the usual Hardy class on the circle $\left(\text{that is, } H^1(z)\right.$ consists of the functions $f \in L^1(d\theta)$ such that $\int fz^n d\theta = 0,\ n = 1, 2, 3, \cdots\Big)$, $H^1(\psi)$ denotes

$$\{f \circ \psi : f \in H^1(z)\} ,$$

and $\boldsymbol{C}$ denotes the constant functions.

To finish the proof, we need to prove (1).

1.2. Fix $f \in H^1(z) \cap H^1(\psi)$. Since ψ is an involution, the functions

$$g = \frac{1}{2}(f + f \circ \psi) ,$$

$$h = \frac{1}{2}(f - f\circ\psi)$$

belong to $H^1(z)$, and furthermore

$$g\circ\psi = g\ , \qquad h\circ\psi = -h\ ,$$

almost everywhere on S.

We claim that there exists a $C^{1+\alpha/2}$ arc Γ on the Riemann sphere Σ, and a conformal map $\varphi\colon D \to \Sigma \sim \Gamma$, such that the continuous extension of φ to S satisfies $\varphi\circ\psi = \varphi$ on S. By a $C^{1+\alpha/2}$ arc we mean the image of the closed unit interval under some $C^{1+\alpha/2}$ injective map with non-vanishing derivative.

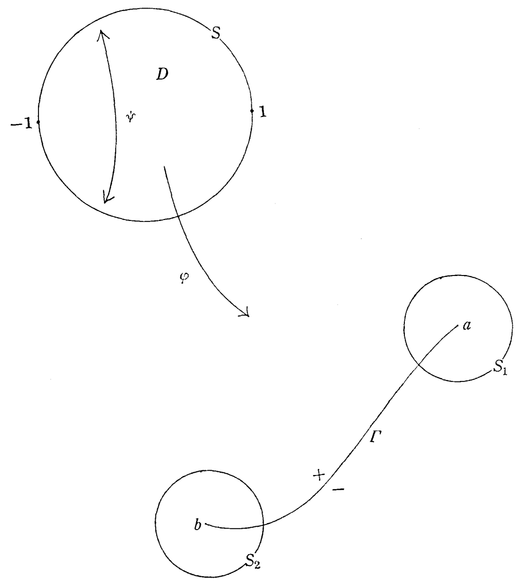

Figure 1.

To prove this claim it is sufficient to rewrite the classical welding argument [6, 7], keeping track of the smoothness of the various functions involved. The Beurling-Ahlfors solution [2, p. 135, equation (14)] of the quasi-conformal extension problem yields a $C^{1+\alpha}$ extension, given $C^{1+\alpha}$ data. Bers has proved [1] that the solution F of the Beltrami equation

$$\frac{\partial F}{\partial \bar\zeta}(\zeta) = G(\zeta)\frac{\partial F}{\partial \zeta}(\zeta)\ , \qquad (\zeta \in \boldsymbol{C})$$

$$F(\infty) = \infty\ ,$$

belongs to $C^{1+\alpha}(\boldsymbol{C})$ if G belongs to $\mathrm{Lip}\,(\alpha, \boldsymbol{C})$, and $\|G\|_\infty < 1$. So the welding argument [6, Lemma 1, Lemma 3, Theorem 1] shows that

Γ exists, and is locally $C^{1+\alpha}$, except possibly at the endpoints. A special argument is needed for the endpoints (make a preliminary map of the disc to the complement of the negative axis, and use the fact that $\psi'(1) = \psi'(-1) = 1$), and the conclusion is that Γ must be $C^{1+\alpha/2}$ near these points. We also find that, to second order,

$$\varphi(\zeta) \cong a + c(1-\zeta)^2 \tag{2}$$

for small $|\zeta - 1|$, and

$$\varphi(\zeta) \cong b + d(1+\zeta)^2 \tag{2'}$$

for small $|\zeta + 1|$, where a, b, c, and d are certain complex constants.

1.3. Define

$$g_1(\omega) = g(\varphi^{-1}(\omega)) , \qquad h_1(\omega) = h(\varphi^{-1}(\omega)) ,$$

whenever $\omega \in \Sigma \sim \Gamma$. Then g_1 and h_1 belong to the Hardy class $H^1(\Sigma \sim \Gamma)$ [6]. In view of [5, p. 169, Corollary], it is clear that $H^1(\Sigma \sim \Gamma)$ is contained in the class $E^1(\Sigma \sim \Gamma)$. Hence g_1 and h_1 have normal limits from either side at almost all points of Γ, and the various limit functions are integrable with respect to arc length. Let us denote the limit functions by g_1^+, g_1^-, h_1^+, and h_1^-. We have

$$g_1^+ = g_1^- , \qquad h_1^+ = -h_1^-$$

almost everywhere on Γ. An application of Morera's theorem shows that g_1 extends analytically across Γ, and hence is constant. (To do the argument at the endpoints simply continue Γ a little further.) A similar argument shows that on any small disc Δ, separated by Γ into U^+ and U^-, the function $h_1 \mid U^+$, $-h_1 \mid U^-$ extends analytically across $\Gamma \cap \Delta$. Hence h_1^2 extends analytically to $\Sigma \sim \{a, b\}$, where a and b are the endpoints of Γ.

1.4. Since $h \in H^1(z)$, $(h(\zeta) = 0((1 - |\zeta|)^{-1})$ for $|\zeta| < 1$, and since h_1^2 is analytic on $S_1^2\backslash\{a, b\}$, we see from (2) and (2′) that there is a constant $\kappa_1 > 0$ such that,

$$|h_1(\omega)| \leqq \kappa_1 \operatorname{dist}(\omega, \Gamma)^{-2} ,$$

whenever $\omega \in \Sigma \sim \Gamma$. Thus there exists a constant $\kappa_2 > 0$ such that, for all ρ with $0 < \rho < |a - b|/2$ we have the bound

$$\int_{S_i} |h_1(\omega)|^{1/4} \, |d\omega| \leqq \kappa_2 \rho^{1/2}$$

for $i = 1, 2$, where

$$S_1 = \{\omega : |\omega - a| = \rho\} ,$$
$$S_2 = \{\omega : |\omega - b| = \rho\} .$$

Let μ_ξ denote harmonic measure for the point ξ on $S_1 \cup S_2$ with respect to

$$\Sigma \sim \{\zeta : |\zeta - a| \leqq \rho \quad \text{or} \quad |\zeta - b| \leqq \rho\} .$$

Suppose

$$0 < \rho < |a - b|/4 , \quad \text{and} \quad |\xi - a| = 2\rho .$$

Then

$$d\mu_\xi(\omega) \leqq 3\rho^{-1} |d\omega| ,$$

and since $|h_1|^{1/4}$ is subharmonic,

$$|h_1(\xi)|^{1/4} \leqq \int |h_1|^{1/4} d\mu_\xi \leqq 6\kappa_2 \rho^{-1/2} .$$

Hence,

$$|h_1(\omega)| \leqq 6^4 \kappa_2^4 \max\{|\omega - a|^{-2}, |\omega - b|^{-2}\}$$

whenever $\omega \neq a$ or b. Thus a and b are at worst poles of h_1^2.

If h_1^2 had a pole at a, then $h(\zeta)$ would have growth at least $|1 - \zeta|^{-1}$ at 1, and this is impossible, since $h \in L^1(d\theta)$. Thus h_1^2 extends analytically over a, and similarly over b. Hence h_1^2 is constant, h_1 is constant, h is constant, and $f = g + h$ is constant. We are done.

1.5. The obvious question is whether $P(z) + P(\psi)$ is dense in $C^1(S)$ whenever $\psi \in C^1(S)$ is a direction-reversing homeomorphism. We do not know. A result with a somewhat weaker hypothesis than our theorem is the following.

If $1 \leqq p \leqq 2$, if $\psi \in C^1(S)$ is an involution, and if there exists $\kappa > 0$ such that

$$\int \left| \frac{\psi'(\theta) - \psi'(\varphi)}{\theta - \varphi} \right| d\theta \leqq \kappa \tag{2}$$

for all φ, then $P(z) + P(\psi)$ is dense in W^p.

Proof. It suffices to prove the case $p = 2$.

The sum $P(z) + P(\psi)$ is dense in W^2 if and only if the sum $P(z) + \psi' P(\psi)$ is dense in L^2, and this happens if and only if

$$H^2(z) \cap H^2(\psi) = \boldsymbol{C} ,$$

where $H^2(z)$ is the usual Hardy space on the circle, and

$$H^2(\psi) = \{f \circ \psi : f \in H^2(z)\} .$$

Now weld, as in the proof of the theorem. Volkoviskii [7] has shown that the welded arc Γ is Lipschitzian (his argument has to be modified a little to cover the endpoints).

Let $f \in H^2(z) \cap H^2(\psi)$, and form g, h, g_1, and h_1 as before. The functions g_1 and h_1 lie in $H^2(\Sigma \sim \Gamma)$, hence g_1 and h_1^2 belong to $H^1(\Sigma \sim \Gamma)$. Since Γ is Lipschitz, g_1 and h_1^2 have normal limits almost everywhere on Γ, from either side, the limit functions are integrable with respect to arc length on Γ, and the limits of each function from either side agree almost everywhere. Applying Morera's theorem, we conclude that g_1^2 and h_1 extend analytically across Γ, and hence are constant. Hence f is constant.

2. The Example.

2.1. Let $\psi \in W^1$. Then $P(z) + P(\psi)$ is dense in W^1 if and only if $P(z) + \psi' P(\psi)$ is dense in L^1. If there exists a nonconstant function $f \in C(S) \cap H^\infty(z)$ such that $f = f \circ \psi$, then f, regarded as an element of $L^\infty = (L^1)^*$, annihilates $P(z) + \psi' P(\psi)$, hence $P(z) + P(\psi)$ is not dense in W^1.

2.2. Let Γ be an arc on Σ, and let ψ be the involution on S induced by the conformal map $\varphi \colon D \rightarrow \Sigma \sim \Gamma$. Suppose φ maps 1 and -1 to the endpoints of Γ, and $\varphi(0) = \alpha$, say. Then harmonic measure with respect to α for $\Sigma \sim \Gamma$ on Γ may be regarded as made up of two pieces, μ^+ and μ^-, where μ^+ is the image under φ of $d\theta/2\pi$ measure on the upper half-circle, and μ^- is the image of $d\theta/2\pi$ measure on the lower half-circle. The map ψ belongs to W^1 if and only if μ^+ and μ^- are mutually absolutely-continuous. If F is a nonconstant analytic function on $\Sigma \sim \Gamma$ that extends continuously to Γ, then $f = F \circ \varphi$ is analytic on D, continuous up to S, and satisfies $f \circ \psi = f$.

So what we wish to construct is an arc $\Gamma \subset \Sigma$ such that the continuous analytic capacity of Γ is positive, and the harmonic measures for the two sides (i.e. μ^+ and μ^-) are mutually absolutely-continuous.

LEMMA 2.3. *Let γ be a simple closed Jordan curve, enclosing a region U. Let Γ be a Jordan arc that separates U into two nonempty Jordan regions V and V'. Let α and β be closed subsets of $\gamma \cap \operatorname{clos} V$, both at positive distance from Γ. For $a \in U$, let $\omega_U(a, E)$ denot the harmonic measure of E with respect to U, evaluated at a. Suppose α and β have positive harmonic measure with respect to U. Then there exists a constant $\kappa \geqq 1$ such that*

$$\frac{1}{\kappa} \leqq \frac{\omega_U(a, \alpha)}{\omega_U(a, \beta)} \leqq \kappa$$

whenever $a \in V'$. *Moreover, the constant* κ *may be chosen to depend on* V, α, *and* β, *and not on the shape or size of* bdy $V' \sim \Gamma$.

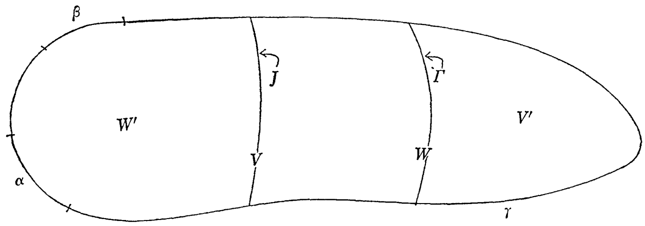

Figure 2.

Proof. Let J be a Jordan arc that separates U into two Jordan domains W and W', such that $\alpha \cup \beta \subset$ bdy W', $U \cap J \subset V$, $U \cap \Gamma \subset W$, and dist $(J, \Gamma \cup \alpha \cup \beta) > 0$.

Pick a point $p \in$ bdy $V \sim (\alpha \cup \beta \cup \Gamma \cup J)$, and map V conformally to the upper half-plane, so that p goes to ∞. By comparing the Poisson kernel on the images of α, β, and Γ, we see that there exist constants κ_1 and κ_2, depending only on V, α, β, and J, such that

$$\omega_V(a, \alpha) \leqq \kappa_1 \omega_V(a, \beta) ,$$
$$\omega_V(a, \Gamma) \leqq \kappa_2 \omega_V(a, \beta) ,$$

for all $a \in J \cap U$.

Hence, for $a \in J$ we have

$$\begin{aligned}\omega_U(a, \alpha) &\leqq \omega_V(a, \alpha) + \omega_V(a, \Gamma) \\ &\leqq (\kappa_1 + \kappa_2)\omega_V(a, \beta) \\ &\leqq (\kappa_1 + \kappa_2)\omega_U(a, \beta) .\end{aligned}$$

By the maximum principle,

$$\omega_U(a, \alpha) \leqq (\kappa_1 + \kappa_2)\omega_U(a, \beta)$$

for all $a \in W$. The reverse inequality is similar.

2.3. The construction.

The basic idea is to pass an arc Γ through a Cantor set C with positive continuous analytic capacity, in such a way as to ensure that C has harmonic measure zero with respect to $\Sigma \sim \Gamma$, while $\Gamma \sim C$ is a union of line segments.

Let C be the product of Cantor's tertiary set with itself. Then $C = \bigcap_{n=0}^{\infty} C_n$, where C_n consists of 4^n squares of side 3^{-n}. Denjoy [4] has shown that C has continuous analytic capacity.

Let $\Gamma_0 = C_0$.

Let Γ_1 be the union of the four solid squares in C_1 and the three line segments shown in Figure 3.

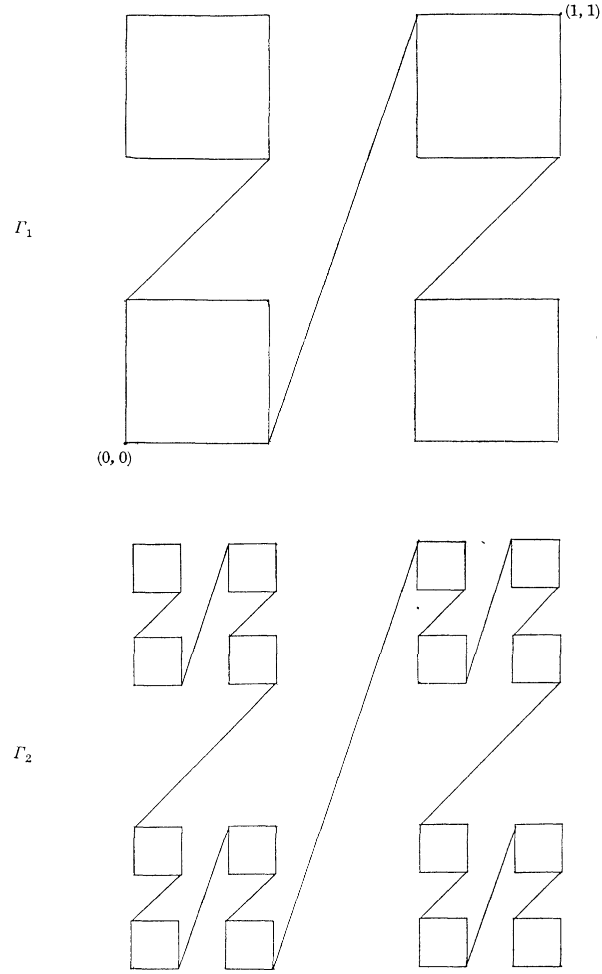

Figure 3.

Let Γ_2 be obtained from Γ_1 by leaving the three line segments alone, and replacing the four squares by four copies of Γ_1, each one reduced in scale by one-third, as in Figure 3.

Let Γ_{n+1} be obtained from Γ_n in the analogous way.

Then $\{\Gamma_n\}$ is a decreasing sequence of compact connected sets, and $\Gamma = \bigcap_n \Gamma_n$ is a Jordan arc that contains C. If $\mu = \mu_+ + \mu_-$ is harmonic measure for $\Sigma \sim \Gamma$, then μ_+ and μ_- are mutually absolutely-continuous with respect to arc length on the union of line segments $\Gamma \sim C$. Thus, to prove that μ_+ and μ_- are mutually absolutely-continuous it suffices to show that $\mu_+(C) = \mu_-(C) = 0$.

Let γ denote the union of Γ with

$$\tau = \{x + iy \in \boldsymbol{C}: x + y = 1, x \leqq 0 \quad \text{or} \quad y \leqq 0\} .$$

Then γ separates the plane into two domains. Let U be that domain to which the point 2 belongs. To prove that $\mu_+(C) = 0$, it suffices to prove that $\omega(2, C) = 0$, where ω is harmonic measure with respect to U.

Let $\omega^{(n)}(a, E)$ be the harmonic measure for the subdomain of U bounded by $\tau \cup \Gamma_n$. By applying Lemma 2.3 to six different choices of (V, α, β) we see that there is a constant κ so that

$$\omega^{(n+1)}(2, C_{n+1}) \leqq \kappa\omega^{(n+1)}(2, C_n \sim C_{n+1}) .$$

But

$$\omega^{(n+1)}(2, C_{n+1}) + \omega^{(n+1)}(2, C_n \sim C_{n+1}) \leqq \omega^{(n)}(2, C_n) .$$

Hence,

$$\omega^{(n+1)}(2, C_{n+1}) \leqq (1 + \kappa)^{-1}\omega^{(n)}(2, C_n) ,$$

and

$$\omega(2, C) \leqq \lim_{n\to\infty} \omega^{(n)}(2, C_n) = 0 .$$

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