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ON CONTINUOUS IMAGE AVERAGING OF PROBABILITY MEASURES

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Let M be a compact space, and X a complete sparable metric space. Let $P(X)$ denote the probability measures on X. Let λ be a probability measure on M. Define a function φ_λ from $C(M, P(X))$ to $P(X)$ by $\varphi_\lambda(T)(f) = \int T(t)(f)d\lambda(t)$ for every $T \in C(M, P(X))$, $f \in C(X)$. We show that φ_λ is an open mapping.

1. Introduction. By a measure on a space X, we mean a regular Borel measure on X. A nonnegative measure is called a probability measure if its total mass is 1.

Let M be a compact space, and let X be a complete separable metric space. Let $P(X)$ denote the collection of all probability measures on X. Let $C(X)$ denote the set of all bounded continuous real-valued functions on X. Give $P(X)$ the weak topology as functionals on $C(X)$. Let $C(M, P(X))$ denote the set of all continuous functions from M into $P(X)$. Give $C(M, P(X))$ the topology of uniform convergence. Let λ be a fixed probability measure on M. For each $T \in C(M, P(X))$, define a functional $\varphi_\lambda(T)$ on $C(X)$ by

$$\varphi_\lambda(T)(f) = \int T(t)(f)d\lambda(t) .$$

By [3, p. 35 and p. 47], $\varphi_\lambda(T)$ may be considered as a measure in $P(X)$. Write $\varphi_\lambda(T) = \int T(t)d\lambda(t)$. Denote the mapping $T \rightarrow \varphi_\lambda(T)$ by φ_λ. Then φ_λ is a continuous function from $C(M, P(X))$ into $P(X)$. This paper is to show that φ_λ is an open mapping. This result contains a result due to Eifler [2, Theorem 2.4] as a special case when M consists of two points.

For a metric space X, we write $x_n \rightarrow x$ if $(x_n)_{n=1}^\infty$ converges to x in X.

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2. Basic lemmas. We will use the following notation in Lemma 2.1: Let X and Y be complete separable metric spaces, and $\pi: Y \rightarrow X$ a continuous function. Then π induces a mapping also denoted by π, from $P(Y)$ to $P(X)$ and defined by $\pi\mu(E) = \mu(\pi^{-1}(E))$.

LEMMA 2.1. *Let X be a complete separable metric space. Then there exist a totally disconnected complete separable metric space G, a continuous function $\varphi: G \to X$, and a continuous function $\tilde{\varphi}: P(X) \to P(G)$ such that $\varphi\tilde{\varphi}(\mu) = \mu$ for all $\mu \in P(X)$. Moreover, $\tilde{\varphi}$ is affine:*

$$\tilde{\varphi}(a\mu + (1-a)\nu) = a\tilde{\varphi}(\mu) + (1-a)\tilde{\varphi}(\nu)$$

for every $0 < a < 1$, and measures $\mu, \nu \in P(X)$.

Proof. Such a space G is constructed by using a sequence $(F_n)_{n=1}^{\infty}$ of partitions of unity on X having the property that each F_n is subordinate to a cover of diameter less than $1/n$. The details of its construction can be found in [1].

Let X be a totally disconnected complete separable metric space. Consider sets of the form

$$(*) \qquad M_{\mu,\varepsilon}(G_1, \cdots, G_n) = \{\nu \in P(X): |\nu(G_i) - \mu(G_i)| < \varepsilon \text{ for } i = 1, \cdots, n\}$$

where $\varepsilon > 0$, $\mu \in P(X)$, and $G_1, G_2, \cdots, G_n$ are mutually disjoint, both open and closed subsets of X such that $\bigcup_{i=1}^{n} G_i = X$.

LEMMA 2.2. *The collection of sets of the form (*) is a base for the topology on $P(X)$.*

Proof. For any open subset U of X, let

$$N_{\mu,\varepsilon}(U) = \{\nu \in P(X): \nu(U) + \varepsilon > \mu(U)\} .$$

Since sets of the form $N_{\mu,\varepsilon}(U)$ is a sub-base for the topology on $P(X)$, it suffices to show that

$$N_{\mu,\varepsilon}(U) \cap M_{\mu,\varepsilon}(G_1, \cdots, G_n)$$

contains a set of the form (*). Let $V \subseteq U$ be a both open and closed subset of X such that $\mu(V) + \varepsilon/2 > \mu(U)$. Then $N_{\mu,\varepsilon/2}(V) \subseteq N_{\mu,\varepsilon}(U)$, and it is easy to check that

$$M_{\mu,\varepsilon/2n}(G_1 \cap V, \cdots, G_n \cap V, G_1 \backslash V, \cdots, G_n \backslash V) \subseteq N_{\mu,\varepsilon/2}(V) \cap M_{\mu,\varepsilon}(G_1, \cdots, G_n) .$$

This completes the proof.

3. Main result.

THEOREM 3.1. *Let M be a compact space, and let X be a complete separable metric space. Let λ be a probability measure on M. Then the function $\varphi_\lambda: C(M, P(X)) \to P(X)$ defined by*

$$\varphi_\lambda(T) = \int T(t)d\lambda(t)$$

is open.

Proof. The proof will be accomplished in two steps: (A) We establish the result when X is totally disconnected. (B) We use (A) to complete the proof.

(A) Let X be a totally disconnected complete separable metric space. Let $T \in C(M, P(X))$, and let $\mathscr{U}_T$ be a neighborhood of T. It suffices to show that $\varphi_\lambda(\mathscr{U}_T)$ is a neighborhood of $\varphi_\lambda(T)$. By Lemma 2.2, we may take $\mathscr{U}_T$ to be a set of the form:

$$\mathscr{U}_T = \{S \in C(M, P(X)) \colon S(M_i) \subseteqq \mathscr{V}_i\ , \quad \text{for} \quad i = 1, \cdots, m\}$$

where for each i, M_i is a compact subset of M, and $\mathscr{V}_i$ is a basic open subset of $P(X)$ of the form:

$$\mathscr{V}_i = \{\theta \in P(X) \colon |\theta(G_{ij}) - \theta_i(G_{ij})| < \varepsilon\ , \quad \text{for} \quad j = 1, \cdots, n_i\}$$

where $\theta_i \in P(X)$ and $\{G_{ij} \colon j = 1, \cdots, n_i\}$ is an open cover for X consisting of mutually disjoint open subsets of X.

Let $\mathscr{C}$ be the collection of all nonempty subsets U of X such that $U = G_{1j_1} \cap G_{2j_2} \cap \cdots \cap G_{mj_m}$. Write $\mathscr{C} = \{U_1, \cdots, U_n\}$. Then $\mathscr{C}$ is an open cover for X and $U_i \cap U_j = \varnothing$ if $i \neq j$.

Since each G_{ij} is both open and closed, we have

$$\delta = \operatorname*{Max}_{ij} \operatorname*{Max}_{t \in M_i} |T(t)(G_{ij}) - \theta_i(G_{ij})| < \varepsilon\ .$$

Let $\varepsilon_0 = \varepsilon - \delta > 0$. One sees immediately that if $S \in C(M, P(X))$ is such that $\operatorname{Max}_{t \in M} |S(t)(G_{ij}) - T(t)(G_{ij})| < \varepsilon_0$ for all i, j, then $S \in \mathscr{U}_T$.

Let $\mu = \int T(t)d\lambda(t)$, and $a_i = \mu(U_i)$, $1 \leqq i \leqq n$. Then $\sum a_i = 1$ and we may assume that $a_n > 0$. Let N be an integer such that $N \cdot a_i > n^2$ whenever $a_i > 0$, $1 \leqq i \leqq n$. Define

$$\mathscr{V} = \{\nu \in P(X) \colon |\nu(U_i) - a_i| < \varepsilon_0/2N \quad \text{for} \quad i = 1, \cdots, n\}\ .$$

It suffices to show that $\varphi_\lambda(\mathscr{U}_T) \supseteqq \mathscr{V}$.

Let $\nu \in \mathscr{V}$. Then $\nu = \nu_1 + \cdots + \nu_n$, where ν_i is a measure on X defined as $\nu_i(A) = \nu(A \cap U_i)$. Let $b_i = \nu(U_i)$. Then $|a_i - b_i| < \varepsilon_0/2N$, and $b_i > 0$ whenever $a_i > 0$.

Now, go back to the function T. Let $f_i(t) = T(t)(U_i)$. Then all f_i, $i = 1, \cdots, n$, are continuous functions on M, and $\int f_i(t)d\lambda(t) = a_i$. We will construct continuous functions $g_1, \cdots, g_n$ on M such that

(1) $\int g_i(t)d\lambda(t) = b_i$,

(2) $\mathrm{Max}_{t\in M}|g_i(t)-f_i(t)|<\varepsilon_0/n$, and

(3) $0\leqq g_i(t)\leqq 1$ and $\sum_{i=1}^n g_i(t)=1$ for all t.

Given $i=1,\cdots,n-1$, define g_i as follows:

(a) If $b_i=a_i$, let $g_i(t)=f_i(t)$ for all t.

(b) If $b_i>a_i$, set $\delta_i=b_i-a_i<\varepsilon_0/2N$. Let $g_i(t)=f_i(t)+(\delta_i/a_n)f_n(t)$. Then,

$$\begin{aligned} f_i(t)\leqq g_i(t)&\leqq f_i(t)+(\varepsilon_0/2N\cdot a_n)f_n(t)\\ &\leqq f_i(t)+(\varepsilon_0/2n^2)f_n(t)\,. \end{aligned}$$

(c) If $b_i<a_i$, set $\delta_i=a_i-b_i<\varepsilon_0/2N$. Since $a_i>0$, so that $b_i>0$. Define $h_i(t)=0$, if $f_i(t)\leqq\delta_i$; $h_i(t)=f_i(t)-\delta_i$, otherwise. Then $b_i\leqq\int h_i(t)d\lambda(t)\leqq a_i$. Let $b_i'=\int h_i(t)d\lambda(t)$ and $g_i(t)=(b_i/b_i')h_i(t)$. Then $g_i(t)\leqq f_i(t)$ and

$$\begin{aligned} f_i(t)-g_i(t)&\leqq\delta_i+h_i(t)(1-b_i/b_i')\\ &\leqq\delta_i+\varepsilon_0/2N\cdot a_i<\varepsilon_0/n^2\,. \end{aligned}$$

Thus for $i=1,\cdots,n-1$, $0\leqq g_i\leqq 1$, $\int g_i(t)d\lambda(t)=b_i$, and

$$\operatorname*{Max}_{t\in M}|g_i(t)-f_i(t)|<\varepsilon_0/n^2\,.$$

Moreover, $g_i(t)\leqq f_i(t)+(\varepsilon_0/2n^2)f_n(t)$. Hence, $g_1(t)+\cdots+g_{n-1}(t)\leqq 1$ for all t. Let $g_n(t)=1-g_1(t)-\cdots-g_{n-1}(t)$. Then the functions $g_1,\cdots,g_n$ are as required. This completes the construction.

Now let I, J be subsets of $\{1,2,\cdots,n\}$ such that $I=\{i\colon b_i>0\}$, $J=\{j\colon b_j=0\}$. For each $j\in J$, pick a measure $\alpha_j\in P(U_j)$. Define a continuous function $S\colon M\to P(X)$ by $S(t)=\sum_{i\in I}(g_i(t)/b_i)\nu_i+\sum_{j\in J}g_j(t)\alpha_j$. Clearly,

$$\begin{aligned} \varphi_\lambda(S)&=\sum_{i\in I}\Big(\int\frac{g_i(t)}{b_i}d\lambda(t)\Big)\nu_i+\sum_{j\in J}\Big(\int g_j(t)d\lambda(t)\Big)\alpha_j\\ &=\sum_{i\in I}\nu_i=\nu,\quad\text{and}\quad\operatorname*{Max}_{t\in M}|S(t)(U_i)-T(t)(U_i)|<\varepsilon_0/n \end{aligned}$$

for all i. Since each G_{ij} is a disjoint union of U_k, it follows that $\mathrm{Max}_{t\in M}|S(t)(G_{ij})-T(t)(G_{ij})|<\varepsilon_0$. Therefore, $S\in\mathscr{U}_T$. This completes the proof of (A).

(B) Let X be a complete separable metric space. To show that the mapping φ_λ is open, it is equivalent to show the following: Let $T\in C(M,P(X))$, and $\mu=\varphi_\lambda(T)$. Let μ_n be a sequence converging to μ in $P(X)$. Then there is a sequence $T_n\to T$ in $C(M,P(X))$ such that $\varphi_\lambda(T_n)=\mu_n$.

For this purpose, we use Lemma 2.1 to pick a totally disconnected space G, continuous functions $\varphi\colon G\to X$ and $\tilde{\varphi}\colon P(X)\to P(G)$, such that $\varphi\tilde{\varphi}(\mu) = \mu$, and that $\tilde{\varphi}$ is affine. Let $\tilde{\mu}_n = \tilde{\varphi}\mu_n$, $\tilde{\mu} = \tilde{\varphi}\mu$. Then $\tilde{\mu}_n \rightarrow \tilde{\mu}$ in $P(G)$. Let $\tilde{T}(t) = \tilde{\varphi}T(t)$ for each t. Then $\tilde{T} \in C(M, P(G))$. It is easy to check that $\varphi_\lambda(\tilde{T}) = \tilde{\varphi}\varphi_\lambda(T)$. In fact, this is obvious if there is a finite subset $\{t_1, \cdots, t_n\} \subseteqq M$ with $\lambda\{t_1, \cdots, t_n\} = 1$. In general, we may pick a net $\lambda_\alpha \rightarrow \lambda$ in $P(M)$ such that for each α, $\lambda_\alpha(F_\alpha) = 1$ for some finite subset F_α of M. Thus, $\varphi_{\lambda_\alpha}(\tilde{T}) = \tilde{\varphi}\varphi_{\lambda_\alpha}(T)$. Let $\alpha \rightarrow \infty$, then we obtain

$$\varphi_\lambda(\tilde{T}) = \tilde{\varphi}\varphi_\lambda(T) .$$

Hence $\varphi_\lambda(\tilde{T}) = \tilde{\mu}$. Since by (A), the function

$$\varphi_\lambda \colon C(M, P(G)) \longrightarrow P(G)$$

is open, hence, we may pick $\tilde{T}_n \rightarrow \tilde{T}$ in $C(M, P(G))$ such that $\varphi_\lambda(\tilde{T}_n) = \tilde{\mu}_n$. Let $T_n(t) = \varphi\tilde{T}_n(t)$. Then $T_n \rightarrow \varphi\tilde{T} = T$ in $C(M, P(X))$, and the same argument in proving $\varphi_\lambda(\tilde{T}) = \tilde{\varphi}\varphi_\lambda(T)$ will give $\varphi_\lambda(T_n) = \varphi\varphi_\lambda(\tilde{T}_n)$. Therefore,

$$\varphi_\lambda(T_n) = \varphi\tilde{\mu}_n = \mu_n .$$

This proves (B), and so completes the proof of this theorem.

As a special case of Theorem 3.1, we let $M = \{1, 2\}$ with the discrete topology. We obtain Eifler's result [2]:

COROLLARY 3.2. *Let X be a complete separable metric space, and let $0 < \lambda < 1$. Then the function*

$$\lambda \colon P(X) \times P(X) \longrightarrow P(X)$$

defined by $(\mu, \nu) \rightarrow \lambda\mu + (1 - \lambda)\nu$ is open.

REFERENCES

1. R. M. Blumenthal and H. H. Corson, *On continuous collections of measures,* Ann. Inst. Fourier, Grenoble, **20** (1970), 193-199.

2. L. Q. Eifler, *Open mapping theorem for probability measures on metric spaces,* (to appear).

3. K. R. Parthasarathy, *Probability Measures on Metric Spaces,* Academic Press (1967).

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