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S-SPACES IN COUNTABLY COMPACT SPACES USING OSTASZEWSKI'S METHOD

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A method adapted from that used by A. J. Ostaszewski is used to construct S-spaces as subspaces of given spaces. Assuming the set-theoretic principle $\diamond$, it is shown that every countably compact space containing no nontrivial convergent sequences contains a perfect S-space. As a corollary, assuming $\diamond$, if X is a countably compact F-space, then X contains a hereditarily extremally disconnected, hereditarily normal, perfect S-space.

1. Introduction. The set-theoretic principle $\diamond$, due to Jensen **[3]**, has found many interesting applications in topology, particularly the construction of Souslin lines and various S-spaces. The basic technique for constructing S-spaces from $\diamond$ is due to A. J. Ostaszewski **[6]**, and has been modified and applied in constructing other interesting topological spaces, notably in **[5]** and **[8]**. Roughly speaking, the method involves constructing a space having desired properties by defining its topology inductively over more and more of the space (and in some cases refining a given topology) using some principle of enumeration.

Here we will show how the method can be used to construct S-spaces as subspaces of given spaces. That is, rather than building up a space by inductively defining its topology, the desired examples will be obtained by working within a given topological space and extracting a subspace.

Our principal topological references are **[2]**, **[7]** and **[10]**. For set-theoretic notions we refer to **[4]**.

For the reader's convenience we now recall a few notions from topology which we will employ.

A space X is an *S-space* if X is regular, hereditarily separable and not Lindelöf.

X is *countably compact* if every countable covering of X by open sets has a finite subcover.

For a completely regular space X, βX denotes the Stone–Čech compactification of X.

A subset A of X is *C^*-embedded* in X if every bounded, continuous real-valued function on A admits a continuous extension to X. A *cozero-set* in X is a set of the form $\{p \in X: f(p) \neq 0\}$ where f is a continuous real-valued function on X. X is an *F-space* if X is completely regular and every cozero-set in X is C^*-embedded in X. A completely regular space X is *extremally disconnected* if the closure of every open subset of X is open.

For the basic information on F-spaces and extremally disconnected spaces, the reader is referred to **[2]** and **[10]**. We will make use of the following two facts, established in 1.62 and 1.64 of **[10]**.

1.1. If X is σ-compact and locally compact, then $\beta X - X$ is a compact F-space.

1.2. If X is an F-space then every countable subspace of X is C^*-embedded in X.

For the consistency of $\diamondsuit$ with the axioms of set theory the reader is referred to **[3]**. We will not need a precise statement of $\diamondsuit$, rather we will use the following consequence of $\diamondsuit$ derived in **[6]**.

1.3. Let $\lim \omega_1$ denote the set of limit ordinals less than ω_1. Then there is a family $\{S_\gamma : \gamma \in \lim \omega_1\}$ of subsets of ω_1 such that each S_γ is a cofinal subset of γ and such that for every uncountable subset S of ω_1 there is a $\gamma \in \lim \omega_1$ with $S_\gamma \subseteq S$.

It is clear we may assume that each S_γ is a simple ω-sequence increasing to γ in 1.3. This is the form in which we will apply 1.3. (the conclusion of 1.3 is often referred to as "club"; see **[7]**)

2. S-subspaces of countably compact spaces. We now assume the conclusion of 1.3. This assumption will enable us to construct S-spaces in certain countably compact spaces. It is apparently not yet known whether 1.3 is equivalent to $\diamondsuit$ or whether it is strictly weaker. It is known that $\diamondsuit$ is equivalent to the conjunction of 1.3 and the continuum hypothesis, and so this question amounts to whether or not 1.3 implies the continuum hypothesis. (see **[7]**)

All hypothesized spaces are assumed to be infinite.

2.1. THEOREM. *If X is a regular, countably compact Hausdorff space containing no nontrivial convergent sequences, then X contains a perfect S-space.*

Proof. Let $\{S_\gamma : \gamma \in \lim \omega_1\}$ satisfy 1.3 where each S_γ is an ω-sequence increasing to γ. Let X satisfy the hypotheses of the theorem. We inductively select points $(x_\xi : \xi \in \omega_1)$ in X, and open sets $(G_\xi : \xi \in \omega_1)$ in X so that

(i) for all ξ, $x_\xi \in G_\xi$

(ii) $\xi < \eta \rightarrow x_\eta \notin G_\xi$

(iii) for all limit ordinals γ and all $n \in \omega$, $x_{\gamma+n} \in \mathrm{cl}\{x_\xi : \xi \in S_\gamma\}$.

To get the desired sequences $(x_\xi: \xi \in \omega_1)$ and $(G_\xi: \xi \in \omega_1)$ we construct $(x_\xi: \xi < \gamma)$ and $(G_\xi: \xi < \gamma)$ by induction on the limit ordinal γ. To start the construction, we choose a countable discrete subset $(x_n: n \in \omega)$ of X, (X is assumed infinite), and a sequence of open sets $(G_n: n \in \omega)$ in X such that $x_n \in G_n$ and $m \neq n \rightarrow x_m \notin G_n$.[1]

Now suppose $\sigma \in \lim \omega_1$ and for every limit ordinal $\gamma < \sigma$ we have chosen the sequences $(x_\xi: \xi < \gamma)$ and $(G_\xi: \xi < \gamma)$ satisfying (i), (ii), and (iii). If σ is a limit of limits, we simply gather together all the x_ξ's and G_ξ's previously constructed to form $(x_\xi: \xi < \sigma)$ and $(G_\xi: \xi < \sigma)$, clearly satisfying (i), (ii), and (iii). So we need only consider the case where $\sigma = \gamma + \omega$ for some limit ordinal γ. Thus, having the sequences $(x_\xi: \xi < \gamma)$ and $(G_\xi: \xi < \gamma)$ we must define the points $(x_{\gamma+n}: n \in \omega)$ and the open sets $(G_{\gamma+n}: n \in \omega)$. Consider the infinite set $R_\gamma = \{x_\xi: \xi \in S_\gamma\}$. Since X is countably compact, every countable subset of X has a limit point in X. But since X contains no nontrivial convergent sequences, every countable set has infinitely many (in fact uncountably many) limit points. Thus $\mathrm{cl}\, R_\gamma - R_\gamma$ is infinite, and so contains a countable discrete subspace $(x_{\gamma+n}: n \in \omega)$. Choose a sequence of open sets $(G_{\gamma+n}: n \in \omega)$ which witnesses this discreteness, that is, with $x_{\gamma+n} \in G_{\gamma+n}$ and such that $m \neq n \rightarrow x_{\gamma+m} \notin G_{\gamma+n}$.

We now check (i), (ii), and (iii) for $(x_\xi: \xi < \gamma + \omega)$ and $(G_\xi: \xi < \gamma + \omega)$. (i) is clear, as is (iii), by virtue of the induction hypothesis and the selection of the points $x_{\gamma+n}$ in $\mathrm{cl}\, R_\gamma$. To verify (ii), because of the induction hypothesis and the choice of $(x_{\gamma+n}: n \in \omega)$ and $(G_{\gamma+n}: n \in \omega)$, it is sufficient to check the following:

If $\xi < \gamma$ and $n \in \omega$, then $x_{\gamma+n} \notin G_\xi$. But S_γ is an ω-sequence increasing to γ, and so there are at most finitely many ordinals in S_γ which are less than ξ. By property (ii) of the induction hypothesis, this means there are at most finitely many x_η with $\eta \in S_\gamma$ which lie in G_ξ. But $x_{\gamma+n}$ is a limit point of R_γ, so every neighborhood of $x_{\gamma+n}$ contains infinitely many x_η with $\eta \in S_\gamma$. In particular, $x_{\gamma+n} \notin G_\xi$.

This completes the inductive construction, and results in sequences $(x_\xi: \xi \in \omega_1)$ and $(G_\xi: \xi \in \omega_1)$ satisfying (i), (ii), and (iii).

We now claim that $Y = \{x_\xi: \xi \in \omega_1\}$ is a perfect S-space. The verification of this is essentially identical with the argument given in **[6]**, so we will be content to sketch that argument here. That Y is not Lindelöf is immediate from (ii) and (i). Any countable subspace of Y is separable, and if $\{x_\xi: \xi \in S\}$ is an uncountable subspace of Y, there is, by 1.3, a $\gamma \in \lim \omega_1$ such that $S_\gamma \subseteq S$. Using (iii) we see that $\{x_\xi: \xi \in S$ and $\xi < \gamma\}$ is a countable dense subset of $\{x_\xi: \xi \in S\}$. This proves Y is hereditarily separable. Since $\gamma < \eta \rightarrow x_\eta \in \mathrm{cl}\{x_\xi: \xi \in S_\gamma\}$, the same ar-

[1] The fact that every infinite Hausdorff space contains a countably infinite discrete subspace is well-known and easy to prove. A proof may be found in 0.13 of **[2]**.

gument shows that every closed subset of Y is either countable or co-countable, from which it is immediate that every closed subset of Y is a G_δ in Y, that is, Y is perfect.

2.2. COROLLARY. *If X is a countably compact F-space then X contains a hereditarily extremally disconnected, hereditarily normal, perfect S-space.*

Proof. Using 1.2 it is easy to see there are no nontrivial convergent sequences in an F-space, so the hypotheses of 2.1 apply. We show that the S-space Y obtained in 2.1 is hereditarily extremally disconnected and hereditarily normal under the present assumptions on X. Now, as is well-known, a space is extremally disconnected if and only if each of its open subsets is C^*-embedded (see 1H in **[2]**), and a space is normal if and only if each of its closed subsets is C^*-embedded (see 3D in **[2]**). So to verify that Y is normal and extremally disconnected hereditarily, it is sufficient to prove that every subspace of Y is C^*-embedded in Y. So, let $Z \subseteq Y$, and let f be a bounded, continuous real-valued function on Z. Since Y is hereditarily separable, Z contains a countable dense subset D. By 1.2, D is C^*-embedded in X, and so the function $f|D$ admits a continuous extension F to all of X. Clearly $F|Y$ is the desired extension of f.

REMARK. 2.3. There is a large number of spaces to which these results can be applied. One class of such spaces is furnished by 1.1. So assuming 1.3 we see for example that $\beta\mathbf{R} - \mathbf{R}$ and $\beta N - N$ contain interesting S-spaces.

REMARK. 2.4. The fact that $\diamond$ implies the existence of S-spaces which are extremally disconnected was previously observed by M. Wage **[9]**. Wage's construction, like Ostaszewski's original method, involves inductively defining a topology to get the desired example.

One significant difference between the S-spaces obtained in 2.2 and the original S-space described in **[6]** is countable compactness. The S-space in **[6]** is, in addition, countably compact, while the S-spaces in 2.2 are never countably compact. If CH is true this follows from the results in **[11]** which imply that, assuming CH, every countably compact, separable normal F-space is compact, and therefore Lindelöf. If CH is false, we argue as follows: A slight modification of the argument in **[1]** shows that a countably compact space of cardinality $< c$ is sequentially compact. Since our S-spaces have cardinality ω_1 and contain no convergent sequences, they cannot be countably compact if CH fails either. Thus our S-spaces constructed using 1.3 are not countably compact.

REFERENCES

1. S. P. Franklin, *On two questions of Moore and Mrowka*, Proc. Amer. Math. Soc. 21 (1969), 597–599.

2. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N.J., (1960).

3. R. B. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic, **4** (1972), 229–308.

4. I. Juhász, *Cardinal functions in topology*, Math. Centre Tracts 34, Mathematical Centre, Amsterdam, (1971).

5. I. Juhász, K. Kunen, and M. E. Rudin, *Two more hereditarily separable non-Lindelöf spaces*, Canad. J. Math., **28** No. 5 (1976), 998–1005.

6. A. J. Ostaszewski, *On countably compact, perfectly normal spaces*, J. London Math. Soc., (2), **14** (1976), 505–516.

7. M. E. Rudin, *Lectures on set-theoretic topology*, CBMS series, **23**, Amer. Math. Soc., 1975.

8. E. K. Van Douwen, *A technique for constructing honest locally compact, submetrizable examples*, to appear.

9. M. Wage, *Extremally disconnected S-spaces*, to appear.

10. R. C. Walker, *The Stone–Čech compactification.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 83, Berlin-Heidelberg-New York, Springer 1974.

11. R. G. Woods, *The structure of small normal F-spaces*, Proc. Auburn Top. Conf. (1976).

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