PACIFIC JOURNAL OF MATHEMATICS
Vol. 74, No. 1, 1978

# QUOTIENTS OF COMPLETE INTERSECTIONS BY $C^*$ ACTIONS

RICHARD RANDELL

**We consider complete intersections $V$ in $C^m$ which have an isolated singularity at $\underline{0}$. When $V$ admits a $C^*$ action, one has the orbit space $V^* = V - \{\underline{0}\}/C^*$. In this paper we determine when $V^*$ is a topological manifold, or in some cases, the precise dimension of the set $\Sigma$ along which $V^*$ is not a manifold. For proper actions we consider a natural complex structure on the space $V^*$ and determine some equivalences among $V^*$ for different $V$. Our methods are topological; the results are expressed numerically in terms of weighted degrees of the polynomials defining $V$.**

1. Introduction. Let $f^{(1)}, \cdots, f^{(k)}$ be complex polynomials in $\underline{z} = (z_1, \cdots, z_m)$. Let $V(f^{(j)}) = \{\underline{z} \in C^m \mid f^{(j)}(\underline{z}) = 0\}$, and suppose that $V$ has an isolated singularity at $\underline{0}$ and is the complete intersection of the $V(f^{(j)})$; $\dim_C V = m - k$. We set $n = m - k$. Further suppose that there is an action of $C^* = C - \{0\}$ on $C^m$ of the form

$$\sigma(t; z_1, \cdots, z_m) = (t^{q_1}z_1, \cdots, t^{q_m}z_m) \tag{1.1}$$

leaving $V$ invariant, with (i) $q_i \in Z$, $i = 1, \cdots, m$ and (ii) g.c.d. $(q_1, \cdots, q_m) = 1$. Such an action will be called a *diagonal action* of type $(q_1, \cdots, q_m)$. We assume that $V$ is not contained in any hyperplane so that (ii) implies the $C^*$ action is effective. Also, $q_i \neq 0$ implies that $\underline{0}$ is the only fixed point, while $q_i > 0$ implies that the action is proper (i.e., the map $\psi: C^* \times C^m \to C^m \times C^m$ given by $\psi(t, \underline{z}) = (\underline{z}, \sigma(t; \underline{z}))$ is proper.) We shall call such actions *fixed-point free* and *proper*, respectively.

Results of Holmann [5] show that for proper actions there is a unique complex structure on $V^* = V - \{0\}/C^*$ such that the orbit map is holomorphic. Later we will describe this structure in more detail.

By [10, Proposition (1.1.3)], any algebraic variety $V$ admitting a $C^*$ action given by a morphism of algebraic varieties may be embedded in some $C^m$ so that the given action is induced by a diagonal action on $C^m$. By the above, the action is proper and without fixed-points on $V - \{0\}$ precisely when $q_i > 0$. Actions with $q_i \leqq 0$ are also of interest, as they arise when considering $C^*$ actions on versal deformations.

We next note that [10, Proposition (1.1.2)] allows us to assume that $V$ is defined by weighted (or quasi-) homogeneous polynomials.

Recall that given an $m$-tuple $\underline{w} = (w_1, \cdots, w_m)$ of positive rationals we say that a polynomial is *weighted homogeneous* with weights $\underline{w}$ (or, $f$ is of type $\underline{w}$) if $a_1/w_1 + \cdots + a_m/w_m = 1$ for every monomial $\alpha z_1^{a_1} \cdots z_m^{a_m}$ of $f$. Write $w_i = u_i/v_i$, $(u_i, v_i) = 1$ and let $d = \text{l.c.m.}(u_1, \cdots, u_m)$, $q_i = d/w_i$. Then

$$f(t^{q_1}z_1, \cdots, t^{q_m}z_m) = t^d f(z_1, \cdots, z_m) .$$

We call $d$ the *polynomial degree* of $f$ and $q_i$, $i = 1, \cdots, m$ the *coordinate degrees* of $f$. The coordinate degrees are related to the $q_i$ of (1.1).

We may thus restate our situation: $V$ is a complete intersection of varieties $V(f^{(j)})$, $j = 1, \cdots, k$, where $f^{(j)}$ is a weighted homogeneous polynomial with degree $d^{(j)}$ and coordinate degrees $q_i^{(j)}$, $i = 1, \cdots, m$. Furthermore, there are integers $\lambda^{(j)}$ with g.c.d. $(\lambda^{(1)}, \cdots, \lambda^{(k)}) = 1$ so that $(q_1, \cdots, q_m) = \lambda^{(j)}(q_1^{(j)}, \cdots, q_m^{(j)})$, $j = 1, \cdots, k$.

Since $V$ is a complete intersection we may conclude from work of Hamm [3] that $K = V \cap S^{2m-1}$ is a $(2n - 1)$-dimensional manifold with an effective action of $S^1 \subset C^*$. It is easily seen that $K^* = K/S^1$ is homeomorphic to $V^*$, and we will often work with $K^*$.

In § 2 we state some results on $S^1$ actions due to Neumann [8] which we use in § 4, where we determine necessary conditions for $K^*$ to be a topological manifold. The most easily stated result is (with $q_i \neq 0$).

COROLLARY 4.4. *Suppose $n > 3$ and $K^*$ is a manifold. If the weights $\underline{w}^{(j)}$ are the same for all $j$, then the weights are integers, and $V$ is therefore equivariantly homeomorphic to a variety defined by Pham-Brieskorn polynomials.*

In § 5 we determine number-theoretic conditions sufficient to ensure that certain $K^*$ are manifolds, and in fact we determine precisely the dimension of the singular set. The final section studies the complex structure of $V^*$ if $q_i > 0$. We show that $V^*$ is nonsingular as a complex space precisely when $K^*$ is a topological manifold. We also give a general criterion to determine when different $V$ yield biholomorphically equivalent $V^*$.

Many authors have studied these varieties. J. Milnor [7] was perhaps the first to notice that weighted homogeneous polynomials are topologically pleasant to work with. W. Neumann [8] considered many of the same problems for the Pham-Brieskorn polynomials $\Sigma z_i^{a_i}$; we often use his techniques. G. Edmunds [2] gave an explicit embedding of $V^*$ into projective space. Finally, P. Orlik and P. Wagreich have contributed extensively to the study of varieties with $C^*$ actions [10, 11, 12, 13]; it is a pleasure to thank them for many useful conversations and comments.

**2. Slices and $S^1$ actions.** It is convenient to work with the action of $S^1$ on $K$. In this section we briefly recall some language of slice diagrams (see Jänich [6] for more details) and state some results of Neumann [8] for quotients of linear actions of $S^1$ and finite cyclic groups.

Let $G$ be a compact Lie group. At every point $x$ of a $G$-manifold $X$ there is a slice $W_x$ transverse to the orbit $G(x)$ of $G$ at $x$. $W_x$ is a real vector space and the isotropy group $G_x = \{g \in G \mid gx = x\}$ acts effectively on $W_x$ via a representation $\sigma$. The slice theorem [6, 1.3] yields the following easy result.

THEOREM. *Suppose $G$ is a compact Lie group acting effectively on a smooth manifold $X$. Then $X/G$ is a manifold if and only if $W_x/G_x$ is a manifold for every $x \in X$.*

We will write $[G_x, \sigma]$ to indicate the action of $G_x$ on $W_x$ via $\sigma$, and we will call $[G_x, \sigma]$ the slice type at $x$. If $W_x/G_x$ is a manifold we say $[G_x, \sigma]$ has $QM$.

In our situation we have an effective action of $S^1$ on $K$. Possible isotropy groups are $\{1\}$, cyclic groups $\boldsymbol{Z}_q$, and $S^1$ (possible only if some $q_i = 0$). For $W = \boldsymbol{R}^2$ or $\boldsymbol{C}$, we denote by $\sigma_p$ the real or complex representation of $S^1$ or $\boldsymbol{Z}_q$ on $W$ given by

$$\exp(i\theta) \longrightarrow \exp(i\theta p) .$$

Every representation of $S^1$ or $\boldsymbol{Z}_q$ as an isotropy group of the $S^1$ action on $K$ on the vector space $W_x$ is equivalent to one of the form $\sigma_{p_1} \oplus \cdots \oplus \sigma_{p_r} \oplus j$, where $j$ denotes a $j$-dimensional trivial representation.

Thus the following result of Neumann [8, Theorem 2.2] is crucial. As usual, we write $[G_x, \sigma_{p_1} \oplus \cdots \oplus \sigma_{p_r} \oplus j]$ for a slice with $G_x = \boldsymbol{Z}_q$ or $S^1$ and indicated linear action of $G_x$ on $W_x$.

THEOREM (Neumann's criterion).

(i) *Let* g.c.d. $(p_1, \cdots, p_r, q) = 1$ *and let* $\bar{p}_i =$ g.c.d. $(p_1, \cdots, \hat{p}_i, \cdots, p_r, q)$. *Then for $r \geqq 1$, $[\boldsymbol{Z}_q, \sigma_{p_1} \oplus \cdots \oplus \sigma_{p_r} \oplus j]$ has $QM$ if and only if $\bar{p}_1 \cdots \bar{p}_r = q$.*

(ii) *Let* g.c.d. $(p_1, \cdots, p_r) = 1$. *Then $[S^1, \sigma_{p_1} \oplus \cdots \oplus \sigma_{p_r} \oplus j]$ has $QM$ if and only if $r \leqq 2$.*

Thus, $[\boldsymbol{Z}_6, \sigma_2 \oplus \sigma_3]$ has $QM$, while $[\boldsymbol{Z}_6, \sigma_2 \oplus \sigma_5]$ does not.

**3. The slice representation.** From the preceding section it is clear that we must determine the various slice types of the action (1.1) on $S^{2m-1}$ and $K$.

On $S^{2m-1}$ the problem is trivial. At a point $\underset{\sim}{z}$ with precisely the first $r$ coordinates nonzero the slice type is $[\boldsymbol{Z}_q, (r-1)\oplus\sigma_{q_{r+1}}\oplus\cdots\oplus\sigma_{q_m}]$, where $q = \text{g.c.d.}\,(q_1, \cdots, q_r)$.

On $K$ the problem is slightly less trivial. Given an $r$-element subset $I_r$ of $\{1, \cdots, m\}$, we will write $q(I_r) = \text{g.c.d.}\,\{q_i, i \in I_r\}$ and we will denote by $T(I_r)$ the slice type of $K$ at a point $\underset{\sim}{z}$ whose nonzero coordinates are precisely those with subscripts in $I_r$. $O(I_r)$ will denote the orbit bundle of $T(I_r)$, that is, the set of the points of $K$ with slice type $T(I_r)$. It is easily seen that $\dim_{\boldsymbol{R}} O(I_r) \geqq 2(r-k)-1$.

LEMMA 3.1. *If* $\dim_{\boldsymbol{R}} O(I_r) = 2(r-k)-1$, *then* $T(I_r) = [\boldsymbol{Z}_{q(I_r)}, (r-k-1)\oplus\sigma_{q_{r+1}}\oplus\cdots\oplus\sigma_{q_m}]$.

*Proof.* As in [8], this lemma is a consequence of the following general fact: Suppose $Y$ is an invariant submanifold of $X$, and suppose that at some point $y \in Y$ the codimension of $Y$ in $X$ is the same as the codimension of the orbit bundle of $y$ in $Y$ in the orbit bundle of $y$ in $X$. Then the slice type of $y$ in $Y$ is the same, up to trivial factors, as the slice type of $x$ in $X$.

REMARK 3.2. In general, the slice representation at $\underset{\sim}{z}$ in $K$ is a subrepresentation of the slice representation at $\underset{\sim}{z}$ in $S^{2m-1}$.

**4. Bounds of the dimension of the singular set.** Let $\Sigma$ be the subset of $K^*$ consisting of points where $K^*$ is not locally homeomorphic to $\boldsymbol{R}^{2n-2}$. We will call $\Sigma$ the *singular set* of $K^*$. Suppose $q_i \neq 0$ for all $i$. Recall that we have weights $w_i^{(j)} = u_i^{(j)}/v_i^{(j)}$, $(u_i^{(j)}, v_i^{(j)}) = 1$, for $i = 1, \cdots, m$; $j = 1, \cdots, k$. Let $t(I_r) = \dim_{\boldsymbol{C}}(V \cap \{z_i = 0, i \notin I_r\})$.

THEOREM 4.1. *Suppose $V$ is a complete intersection with isolated singularity at the origin, and suppose $V$ is defined by weighted homogeneous polynomials $f^{(j)}$ with weights $\underset{\sim}{w}^{(j)}$. Further suppose that $V$ is invariant under a fixed-point free diagonal action of type $(q_1, \cdots, q_m)$. If* (i) *there are sets $I_r \subset \{1, \cdots, m\}$ and $J_s \subset \{1, \cdots, k\}$ with $r$ and $s$ elements respectively, so that some prime $p$ divides $v_i^{(j)}$ for $i \in I_r$, $j \in J_s$ and if* (ii) $n - 2(k-s) > 3$, *then*

$$(*) \qquad \dim_{\boldsymbol{R}} \Sigma \geqq 2(t(I_r)-1) \geqq 2(r-(k-s)-1)\,.$$

Before proving this we state several corollaries and give some examples. As shown in [12] (and certainly to be expected), one is particularly interested in the question of when $K^*$ is a manifold.

COROLLARY 4.2. *Suppose $K^*$ is a manifold and $n > 2k + 1$. Then for every $j \in \{1, \cdots, k\}$ and any $k$-element set $I_k$, one has* g.c.d. $\{v_i^{(j)}, i \in I_k\} = 1$.

COROLLARY 4.3. *Suppose $n > 3$. If there is a set $I_r$ so that $p$ divides $v_i^{(j)}$ for $i \in I_r$, $j = 1, \cdots, k$ then $\dim_R \Sigma \geqq 2(r-1)$.*

COROLLARY 4.4. *Suppose $n > 3$, $K^*$ is a manifold, and the weights $\underline{w}^{(j)}$ are the same for all $j$. Then the weights are integers and $V$ is equivariantly homeomorphic to a variety defined by a complete intersection of Brieskorn varieties.*

The last statement of 4.4 follows from the straightforward generalization of [10, Theorem 3.1.4].

These results are essentially the best possible: If $n = 2$, $K^*$ is always a manifold. If $n = 3$ we have the following

EXAMPLE 4.5. Let $n = 3$, $k = 1$, and define $V$ by $f(z_1, \cdots, z_4) = z_1^5 + z_1 z_2^6 + z_3^3 + z_3 z_4^5$. Then the weights are $(5, 15/2, 3, 15/2)$, but one may compute slice types and apply Neumann's criterion to see that $K^*$ is a 4-manifold.

EXAMPLE 4.6. The variety $V'$ defined by the equations

$$\begin{aligned} z_1^4 + z_2^6 + z_3^{20} + z_4^{28} + z_5^{44} + z_6^{52} &= 0 \\ z_1^3 + z_1 z_2^3 + z_3^{15} + z_4^{21} + z_5^{33} + z_6^{39} &= 0 \end{aligned} \tag{4.6.1}$$

has $n = 4$, $k = 2$, and $w_2^{(2)} = 9/2$. The reader may use Neumann's criterion and 3.1 to verify that $K^*$ is a manifold. (This will also follow from 5.3.) This example should be compared with 4.4.

*Proof of* 4.1. Suppose we have $I_r$ and $J_s$ satisfying (i) so that the first inequality of (*) fails. Then we will show that (ii) also fails. In the course of doing this we will show that (i) implies $t(I_r) \geqq r - (k - s)$, giving the second inequality.

For convenience we will assume $I_r = \{1, \cdots, r\}$ and $J_s = \{1, \cdots, s\}$. Then for any monomial $a z_1^{a_1} \cdots z_r^{a_r}$ of $f^{(j)}$, $j \in J_s$, one has

$$a_1/w_1^{(j)} + \cdots + a_r/w_r^{(j)} = 1 .$$

But since $p$ divides $v_i^{(j)}$, $i \in I_r$, $j \in J_s$, the above equation implies that $p$ divides $u_i^{(j)}$, $i \in I_r$, $j \in J_s$. Since $(u_i^{(j)}, v_i^{(j)}) = 1$ this is a contradiction, and no such monomial appears in $f^{(j)}$, $j \in J_s$.

Therefore the set $S = \{\underline{z} \in C^m | z_i = 0, i > r\}$ is contained in $\{f^{(1)} = \cdots = f^{(s)} = 0\}$, so that $\dim_C V \cap S = t(I_r) \geqq r - (k - s)$.

Now let $S^* = S \cap K/S^1$. Then $\dim_R S^* \geqq 2(t(I_r) - 1)$, so that if we let $\underline{z} \in S \cap K$ be a point with precisely the first $r$ coordinates nonzero, and if we assume that (*) fails, then the slice type at $\underline{z}$ must have $QM$.

Let this slice type be $[Z_q, \sigma]$. Then $q = \text{g.c.d.}\,(q_1, \cdots, q_r)$. Since $p$ divides $v_i^{(j)}$, $i = 1, \cdots, r$, $p$ divides $q_i^{(j)} = d^{(j)} v_i^{(j)} u_i^{(j)}$, and thus $p$ divides $q$. Since $\sigma$ has $QM$ it follows easily from Neumann's criterion and 3.2 that $p$ must divide at least $n - 1$ of the $q_i$, say $p$ divides $q_i$, $i \in I_{n-1}$, where $I_r \subset I_{n-1}$. We may assume $I_n = \{1, \cdots, n-1\}$.

We next claim that in fact, $p$ divides $v_i^{(j)}$, $i \in I_{n-1}$, $j \in J_s$. By assumption $p$ divides $v_i^{(j)}$, $i \in I_r \subset I_{n-1}$, $j \in J_s$. For $i \in I_{n-1}$, $j \in J_s$, $p$ divides $q_i = \lambda^{(j)} d^{(j)} v_i^{(j)} / u_i^{(j)}$. If $p$ does not divide $v_i^{(j)}$, then $p$ divides $\lambda^{(j)} d^{(j)}$. This implies $p^2$ divides $q$ which in turn implies that $p^2$ divides $\lambda^{(j)} d^{(j)}$, etc. Thus $p$ divides $v_i^{(j)}$, $i \in I_{n-1}$, $j \in J_s$.

Now consider the $k \times m$ matrix $D = (d_{\alpha\beta})$, where $d_{\alpha\beta} = \partial f^{(\alpha)} / \partial z_\beta$. We have seen that every monomial in $f^{(j)}$, $j \in J_s$, which contains a variable $z_i$, $i = 1, \cdots, n-1$, must also contain some $z_i$, $i > n - 1$. Let $P_0 = \{z_n = \cdots = z_m = 0\}$, and let $P = P_0 \cap V$. Then $f^{(j)}(z) = 0$, for $\underline{z} \in P_0$, $j \in J_s$, so $\dim_C P \geqq (n-1) - (k - s)$. On $P$ we clearly have $d_{\alpha\beta} = 0$, $1 \leqq \alpha \leqq s$, $1 \leqq \beta \leqq n - 1$.

Of course, $V$ as a complete intersection is singular wherever $D$ has rank less than $k$. Let $D_s$ be the $s \times m$ matrix consisting of the first $s$ rows of $D$, and let $D_s'$ be the $s \times (m - (n-1)) = s \times (k+1)$ matrix consisting of the last $k + 1$ columns of $D_s$. If the rank of $D_s'$ is less than $s$ at any point $\underline{z}_0$ of $P$, then $V$ is singular at $\underline{z}_0$.

But $D_s'$ will have rank less than $s$ if $k - s + 2$ minors of size $s \times s$ vanish. Thus $V$ will be singular on a set of complex dimension at least $\dim_C P - (k - s + 2) \geqq (n-1) - (k-s) - (k-s+2) = n - 2(k-s) - 3$. Since $V$ has an isolated singularity, $n - 2(k-s) - 3 \leqq 0$, contradicting (ii) and thus completing the proof.

We conclude this section with two trivial consequences of Neumann's criterion.

PROPOSITION 4.7. *Suppose $V$ is a complete intersection with isolated singularity and diagonal $C^*$ action, and suppose $q_1 = \cdots = q_r = 0$, $q_i \neq 0$, $i > r$. If $K^*$ is a manifold, $n - \dim_C V \cap \{z_i = 0, i > r\} \leqq 2$.*

*Proof.* The $S^1$ action on $K$ fixes $K \cap \{z_i = 0, i > r\}$.

The next proposition is a topological analogue of a phenomenon noticed by G. Edmunds [2, § 5].

PROPOSITION 4.8. *The real codimension of $\Sigma$ in $K$ is at least* 4.

*Proof.* This is a trivial consequence of Neumann's criteria, as at any point the isotropy is $S^1$ or $Z_q$, and $K^*$ can fail to be a manifold at the point only if the slice representation has at least three or two nontrivial summands, respectively.

## 5. Totally complete intersections.

In general one needs to know the form of the polynomials defining $V$ in order to determine the exact dimension of $\Sigma$. There is, however, one class of complete intersections for which a knowledge of the polynomial and coordinate degrees will suffice.

DEFINITION 5.1. $V^n \subset C^m$ is called a *totally complete intersection* if the intersection of $V$ with all coordinate subspaces of $C^m$ has minimal dimension.

An example is an intersection of Brieskorn varieties with suitable coefficients (see Hamm [4]). The complete intersection $V'$ of 4.6 is another such example.

DEFINITION 5.2. Given a complete intersection $V^n$ with diagonal $C^*$ action of type $(q_1, \cdots, q_m)$, $q_i \neq 0$, we define $t_i = \text{g.c.d.}\,(q_1, \cdots, \hat{q}_i, \cdots, q_m)$, and $s_i = q_i/t_1 \cdots \hat{t}_i \cdots t_m$.

Since $\text{g.c.d.}\,(q_1, \cdots, q_m) = 1$ we easily see that $(t_i, t_j) = 1$, $i \neq j$, $s_i \in Z$, and $(s_i, t_i) = 1$, $i = 1, \cdots, m$. Let $\gamma$ be the largest integer such that there exist $\gamma$ of the $s_i$ with common divisor greater than one.

THEOREM 5.3. *Suppose $V^n \subset C^m$ is a totally complete intersection with isolated singularity at $\underline{0}$ admitting a diagonal $C^*$ action of type $(q_1, \cdots, q_m)$, with $q_i \neq 0, i = 1, \cdots, m$. Then the real dimension of the singular set $\Sigma$ of the orbit space $V - \{0\}/C^*$ is* $\max\{-1, 2(n - m - 1 + \gamma)\}$.

*Proof.* We consider the associated $S^1$ action on $K$. At a point $\underline{z}$ of $K$ with precisely the first $\gamma$ coordinates nonzero, we have cyclic isotropy of order $q = \text{g.c.d.}\,(q_1, \cdots, q_\gamma)$. By 3.1, the slice representation is $\sigma = \sigma_{q_{\gamma+1}} \oplus \cdots \oplus \sigma_{q_m} \oplus (n + \gamma - m - 1)$.

We now apply Neumann's criterion: $K^*$ will be a manifold if

$$(5.3.1) \qquad \prod_{s=1}^{m-\gamma} \text{g.c.d.}\,(q_1, \cdots, q_\gamma, q_{\gamma+1}, \cdots \hat{q}_{\gamma+s}, \cdots, q_m) = q$$

(5.3.1) holds, by definition, if $t_{\gamma+1} \cdots t_m = \text{g.c.d.}\,(q_1, \cdots, q_\gamma)$. The latter equation easily is seen to hold if and only if $\text{g.c.d.}\,(s_1, \cdots, s_\gamma) = 1$.

Since the set $V \cap \{\underline{z} \mid \text{precisely } z_1, \cdots, z_\gamma \text{ are nonzero}\}$ has complex dimension $n - (m - \gamma)$, the result follows.

In particular, $K^*$ is a manifold if and only if no collection of $(k + 1)$ of the $s_i$ has a common divisor.

For various applications (cf. [12, § 4]) one wishes to construct $V$ with $C^*$ action so that $K^*$ is a manifold.

PROPOSITION 5.4. *Suppose integers* $t_i, s_i, i = 1, \cdots, m$ *and* $c_j$, $j = 1, \cdots, k$ *are given such that* $(t_i, t_j) = 1$, $i \neq j$, *and* $(s_i, t_i) = 1$, *for all* $i$. *Define* $a_{ij} = (c_j t_i)[\text{l.c.m.}(s_1, \cdots, s_m)]/s_i$. *Then a totally complete intersection* $V$ *defined by the equations*

$$\sum_{i=1}^{m} \alpha_{ij} z_i^{a_{ij}} = 0 , \quad j = 1, \cdots, k \tag{5.4.1}$$

*has a* $C^*$ *action. The associated* $K^*$ *is a manifold if and only if no* $k + 1$ *of the numbers* $s_1, \cdots, s_m$ *have a common divisor.*

*Proof.* This follows from 5.3 and easy computations which yield

$$d^{(j)} = c_j[\text{l.c.m.}(s_1, \cdots, s_m)]t_1 \cdots t_m$$
$$q_i = t_1 \cdots t_{i-1} s_i t_{i+1} \cdots t_m .$$

Neumann proved 5.4 for $k = 1$. We should emphasize that 5.3 does not depend on the polynomials themselves, but only on the polynomial and coordinate degrees.

**6. The complex spaces $V^*$.** We now change our viewpoint somewhat and require $q_i > 0$, $i = 1, \cdots, m$, so that the action (1.1) is proper.

We give $V^* = V - \{0\}/C^*$ a complex structure as in Brieskorn and Van de Ven [1]: Define a holomorphic operation of $C$ on $V - \{0\}$ by

$$t(z_1, \cdots, z_m) = (\exp(tq_1)z_1, \cdots, \exp(tq_m)z_m) . \tag{6.0.1}$$

Notice that an orbit of the $C$ action on $V$ intersects $K$ in an orbit of the $S^1$ action on $K$. In fact the imaginary axis from 0 to $2\pi i$ moves any point of $K$ through its $S^1$ orbit. Thus $V - \{0\}/C \cong V^* \cong K^*$.

Consider $Z \subset C$ as an additive subgroup and let $H = V - \{0\}/Z$. It is easily seen that $H \cong K \times S^1$. Let $\Gamma$ be the discrete subgroup of $C$ generated by 1 and $2\pi i$. The torus $T = C/\Gamma$ acts on $H$ by (6.0.1), and by results of Holmann [5], $H/T$ is a complex space homeomorphic to $V^*$ or $K^*$.

THEOREM 6.1 (Neumann [8] for Brieskorn varieties). *Suppose* $V$ *is a complete intersection with proper diagonal* $C^*$ *action, and* *suppose $V$ has an isolated singularity at $\underline{0}$. Then $K^*$ is a manifold if and only if the complex structure on $V^*$ is nonsingular.*

*Proof.* A theorem of Prill [14] asserts that the complex structure is nonsingular if and only if the isotropy group at every point $p$ is generated by elements of $T$ with complex codimension one fixed-point sets passing through $p$.

Let $K$ denote the intersection of $V$ with the *unit* sphere and let $\underline{z} \in K$, $\theta \in S^1$. Then the $T$ action on $H$ is given by $(a, b)(\underline{z}, \theta) = (b(\underline{z}), a(\theta))$, where $(a, b)$ are coordinates of $T$ in the direction of 1 and $2\pi i$. Clearly the isotropy at $(\underline{z}, \theta) \in H$ is the same as the isotropy of the $S^1 \subset T$, $S^1 = \{(0, 2\pi i\theta) \mid 0 \leqq \theta < 1\}$ and this in turn is the same as the isotropy of the $S^1$ action on $K$ at $\underline{z}$.

The result then follows by direct comparison of the criterion of Neumann for $K^*$ to be a manifold with the above criterion of Prill.

We now generalize the concept of the cone over $V$, [10]. We no longer assume that $V$ is a complete intersection or has an isolated singularity at $\underline{0}$. We do of course continue to assume that $V$ is invariant under a proper diagonal $C^*$ action.

DEFINITION 6.2. Suppose $V$ is defined by polynomials $f^{(j)} = \Sigma \alpha z_1^{a_1} \cdots z_m^{a_m}$. The variety $V_0$ defined by $g^{(j)} = \Sigma \alpha z_1^{a_1 r_1} \cdots z_m^{a_m r_m}$ is called the *weighted cone over $V$ with weights* $(r_1, \cdots, r_m) \in (\boldsymbol{Z}^+)^m$.

Note that $\phi(z_1, \cdots, z_m) = (z_1^{r_1}, \cdots, z_m^{r_m})$ defines a map $\phi: V_0 \to V$, and that $\phi$ has degree $r_1 \cdots r_m$ so long as $V$ is contained in no coordinate hyperplane. In [10], the weighted cone with weights $(q_1, \cdots, q_m)$ was called simply the cone over $V$. We will call this special case the *minimal homogeneous cone* over $V$. $V_0$ admits a proper diagonal $C^*$ action which commutes with $\phi$, so that one obtains a map $\psi: V_0^* \to V^*$ of complex spaces. Thus if $V_0$ is the minimal homogeneous cone, $V^*$ is branch covered by a projective variety.

We next ask for the degree of $\psi$, and in particular, when is $\psi$ biholomorphic?

THEOREM 6.3. *Let $V_0$ be a variety with proper diagonal $C^*$ action of type $(q_1, \cdots, q_m)$. Define $t_i = \text{g.c.d.}\,(q_1, \cdots, \hat{q}_i, \cdots, q_m)$. Suppose $V_0$ is the weighted cone over $V$ of type $(r_1, \cdots, r_m)$, and define $e_i = \text{g.c.d.}\,(r_i, t_i)$. Then the degree of $\psi: V_0^* \to V^*$ is $r_1 \cdots r_m / e_1 \cdots e_m$.*

*Proof.* The finite group $G = \boldsymbol{Z}_{r_1} \oplus \cdots \oplus \boldsymbol{Z}_{r_m}$ acts on $V_0$ and $V_0/G \cong V$. Similarly, $G$ acts on $V_0^*$, and $V_0^*/G = V^*$. However, the latter action is not effective in general. Setting $G' = \{g \in G \mid gz^* = z^*$, for all $z^* \in V_0^*\}$, we must show that the order of $G'$ is $e_1 \cdots e_m$.

Let $\beta_i$ generate $\boldsymbol{Z}_{r_i}$, so that $\beta^{(i)} = (1, \cdots, \beta_i, \cdots, 1) \in G$ acts on

$V_0$ by fixing all coordinates except the $i$th, which is multiplied by $\exp(2\pi i/r_i)$. Let $\gamma_i = \beta_i^{\tau_i/e_i}$ and $\gamma^{(i)} = (1, \cdots, \gamma_i, \cdots, 1)$. We claim that $G'$ is generated by the $\gamma^{(i)}$.

We show first that $\gamma^{(i)}z^* = z^*$. That is, we show that $\underline{z}$ and $\gamma^{(i)}(\underline{z})$ are in the same orbit of the $C^*$ action on $V_0$. Let $\zeta = \exp(2\pi i/e_i)$. Then $\zeta(z_1, \cdots, z_m) = (z_1, \cdots, z_{i-1}, \zeta^{q_i}z_i, z_{i+1}, \cdots, z_m)$, since $\zeta^{q_j} = \exp(2\pi i q_j/e_i) = 1$ because $e_i$ divides $t_i$ and $t_i$ divides $q_j$, $i \neq j$. Further, since g.c.d. $(q_i, e_i) = 1$, some power of $\zeta$ maps $\underline{z}$ to $\gamma^{(i)}(\underline{z})$. Thus $\gamma^{(i)}z^* = z^*$.

A similar argument shows that any element of $G'$ must be a product of $\gamma^{(i)}$, and the result follows.

COROLLARY 6.4. *Let $V_0$ be the minimal homogeneous cone over $V$. Then* $\deg \phi = \deg \psi = q_1 \cdots q_m$.

*Proof.* g.c.d. $(q_i, t_i) = 1$, $i = 1, \cdots, m$.

COROLLARY 6.5. *$\psi$ is biholomorphic if and only if $r_i$ divides $t_i$, $i = 1, \cdots, m$.*

This was proved by Neumann for Brieskorn varieties.

REMARK 6.6. The restriction of $\psi$ to coordinate hyperplanes may not have the expected degree. For instance, if $V_0$ is defined by $z_1^6 + z_2^6 + z_3^6$ and $V$ is defined by $z_1^2 + z_2^3 + z_3^6$, $\deg \psi = 6$ but $\deg \psi|_{z_1=0} = 2$, since the restricted $C^*$ action is not effective.

Corollary 6.5 shows that one cannot obtain biholomorphic complex spaces by considering weighted cones between $V$ and the minimal homogeneous cone. One *can* obtain biholomorphic complex spaces by dividing the exponents of the defining polynomial by $t_i$, assuming that such division yields a polynomial. Our final result shows that one does get a polynomial.

PROPOSITION 6.7. *Suppose $V$ is a hypersurface with an isolated singularity at $\underline{0}$ and suppose $V$ admits a proper diagonal $C^*$ action of type $(q_1, \cdots, q_m)$. If $V$ is defined by $f$, with*

$$f(z_1, \cdots, z_m) = \Sigma \alpha z_1^{a_1} \cdots z_m^{a_m} .$$

*Then $t_i$ divides $a_i$ for every monomial of $f$.*

*Proof.* Let $z_1^{a_1} \cdots z_m^{a_m}$ be a monomial of $f$, with polynomial degree $d$. Then, since $w_i = d/q_i$, we have

$$a_1q_1 + \cdots + a_mq_m = d .$$

Since $t_i$ divides $q_j$ for $i \neq j$, and $(t_i, q_i) = 1$ we see that $t_i$ divides $a_i$ if and only if $t_i$ divides $d$. Since $t_i$ divides $q_j$, $t_i$ divides $dv_j/u_j$, so if $t_i$ does not divide $d$, $t_i$ must divide $v_j$, $i \neq j$. Then, as in the proof of 4.1, we see that $f$ has at most 2 variables, if the singularity is isolated. So we are done for $m > 2$. For $m = 1$ the result is trivial, and for $m = 2$ it may be checked by direct computation.

EXAMPLE. $z_1^{a_1} + z_1z_2^{a_2} + z_2z_3^{a_3}$, with $(a_1 - 1, a_2) = 1$, $(a_2a_3, a_1a_2 - a_1 + 1) = 1$. The weights, in reduced form, are

$$w_1 = a_1 , \quad w_2 = a_1a_2/(a_1 - 1) , \quad w_3 = a_1a_2a_3/(a_1a_2 - a_1 + 1) .$$

Thus, $q_1 = a_2a_3$, $q_2 = a_3(a_1 - 1)$, $q_3 = a_1a_2 - a_1 + 1$. Then

$$t_1 = \text{g.c.d.}\,((a_1 - 1), (a_1a_2 - a_1 + 1)) = 1$$
$$t_2 = \text{g.c.d.}\,(a_2a_3, a_1a_2 - a_1 + 1) = 1$$
$$t_3 = \text{g.c.d.}\,(a_2a_3, a_3(a_1 - 1)) = a_3 .$$

## REFERENCES

1. E. Brieskorn and A. Van de Ven, *Some complex structures on products of homotopy spheres*, Topology, **7** (1968), 389-393.
2. G. Edmunds, *On the orbit space of a weighted homogeneous hypersurface*, Proc. London Math. Soc., **29** (1974), 684-698.
3. H. Hamm, *Lokale topologische Eigenschaften komplexer Räume*, Math. Ann., **191** (1971), 235-252.
4. ———, *Exotische Sphären als Umgebungsränder in speziellen komplexen Räumen*, Math. Ann., **197** (1972), 44-56.
5. H. Holmann, *Quotienträume komplexer Mannigfaltigkeiten nach komplexen Liescher Automorphismengruppen*, Math. Ann., **139** (1960), 383-402.
6. K. Jänich, *Differenzierbare G-Mannigfaltigkeiten*, Springer lecture notes in math., No. **59**, 1968.
7. J. Milnor, *Singular Points of Complex Hypersurfaces*, Princeton University Press, 1968.
8. W. Neumann, *$S^1$-actions and the $\alpha$-invariant of their involutions*, Bonn. Math. Schr. **44**, 1970.
9. P. Orlik and R. Randell, *The classification and monodromy of weighted homogeneous singularities*, preprint.
10. P. Orlik and P. Wagreich, *Isolated singularities of algebraic surfaces with $C^*$ action*, Ann. of Math., **93** (1971), 205-228.
11. ———, *Singularities of algebraic surfaces with $C^*$ action*, Math. Ann., **193** (1971), 121-135.
12. ———, *Seifert n-manifolds*, Invent. Math., **28** (1975), 137-159.
13. ———, *Algebraic surfaces with $k^*$-action*, Acta Math., **138** (1977), 43-81.
14. D. Prill, *Local classification of quotients of complex manifolds by discontinuous groups*, Duke Math. J., **34** (1967), 375-386.

Received February 14, 1977.

THE INSTITUTE FOR ADVANCED STUDY
PRINCETON, NJ 08540