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LOCAL AND GLOBAL BIFURCATION FROM NORMAL EIGENVALUES II

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This paper studies the bifurcation of solutions of nonlinear eigenvalue problems of the form $Lu = \lambda u + H(\lambda, u)$, where L is linear and H is $o(\|u\|)$ uniformly on bounded λ intervals. This paper shows that isolated eigenvalues of L having odd multiplicity are bifurcation points if H merely has a "degree" of compactness, but is not necessarily compact (treated in [3], [5]). Moreover, a global alternative theorem follows.

Introduction. In this paper we study the bifurcation of solutions of nonlinear eigenvalue problems. The equations to be studied are of the form

$$Lu = \lambda u + H(u) \tag{0.1}$$

where all operators are defined in a real Banach space $\mathscr{B}$. L is assumed to be linear, bounded or unbounded; I, the identity map, and H, $o(\|u\|)$ near $u = 0$. Clearly, $(\lambda, 0)$ is a solution for each real λ, and these are called the trivial solutions of (0.1). Of more interest are the nontrivial solutions, pairs (λ, u) satisfying (0.1) with $u \neq 0$. In particular, one is interested in how solutions of (0.1) are related to solutions of the linear equation

$$Lu = \lambda u\ . \tag{0.2}$$

The study of this led to the following definition.

DEFINITION. A point $(\lambda_0, 0)$ is a bifurcation point for (0.1) if every neighborhood of $(\lambda_0, 0)$ in $\boldsymbol{R} \times \mathscr{B}$ contains a nontrivial solution of (0.1).

Under quite general conditions, it is easy to show that in order for $(\lambda_0, 0)$ to be a bifurcation point of (0.1), it is necessary that λ_0 be in the spectrum of L. [8].

The first general existence theorem for bifurcation points was obtained by Krasnoseljskii [2]. He considered equations of the type

$$u + \lambda Lu + H(u) \tag{0.3}$$

where L is linear and compact, and H compact. He proved that if λ_0 is a characteristic value of L having odd algebraic multiplicity, then $(\lambda_0, 0)$ is a bifurcation point.

More recently, Rabinowitz [6] studied the same problem as Krasnoseljkii and proved a much stronger result. The bifurcation from such points is a global property, with a continuous branch of solutions joining $(\lambda_0, 0)$ to infinity in $\boldsymbol{R} \times \mathscr{B}$, or if the branch is bounded, containing $(\lambda_1, 0)$ with $\lambda_1 \neq \lambda_0$.

The author ([3] and [5]) eliminated the compactness assumption on L while maintaining the strength of the result. The main result of this paper is that the compactness assumption on H can be relaxed. The proofs of the theorems mentioned involve the use of degree theory.

1. **Preliminaries.** Let $\mathscr{B}$ be a real Banach space and let $\mathscr{E}$ denote $\boldsymbol{R} \times \mathscr{B}$ with the product topology. By a nonlinear eigenvalue problem we mean an equation of the form

$$Lu = \lambda u + H(u) \tag{1.1}$$

where $L: \mathscr{B} \rightarrow \mathscr{B}$ is linear and $H: \mathscr{B} \rightarrow \mathscr{B}$ is a nonlinear operator satisfying hypothesis $H - 1$:

$(H - 1)$ (i) H is continuous, and bounded on each ball centered at 0.
(ii) H is $o(\|u\|)$ for u near 0.

A nontrivial solution of (1.1) is a pair (λ, u) with $u \neq 0$ which satisfies (1.1), and the trivial solutions are the pairs $(\lambda, 0)$.

In what follows, $L: \mathscr{B} \rightarrow \mathscr{B}$ will be a densely defined linear operator (bounded or unbounded) with domain $\mathrm{dom}(L)$. The resolvent set of L, $\rho(L)$, will be all real values of λ for which there exists a bounded linear operator $C: \mathscr{B} \rightarrow \mathscr{B}$ such that

$$\begin{aligned} C(L - \lambda)x &= x\,, \quad x \in \mathrm{dom}\,(L) \\ (L - \lambda)Cx &= x\,, \quad x \in \mathrm{range}\,(L - \lambda)\,. \end{aligned} \tag{1.2}$$

C will be denoted by $(L - \lambda)^{-1}$.

DEFINITION 1.1. The (algebraic) multiplicity of an eigenvalue λ of L is defined to be the dimension of the subspace $\bigcup_{j=1}^{\infty} \ker (L - \lambda)^j$ where $\ker (L - \lambda)^j$ denotes the nullspace of $(L - \lambda)^j$. $\bigcup_{j=1}^{\infty} \ker (L - \lambda)^j$ will be referred to as the principal manifold of L associated with λ.

DEFINITION 1.2. An eigenvalue λ of L is defined to be normal if

(i) the multiplicity of L is finite.

(ii) $\mathscr{B}$ is the direct sum of subspaces $\mathscr{L}_\lambda \oplus \mathscr{N}_\lambda$ where $\mathscr{L}_\lambda = \bigcup_{j=1}^{\infty} \ker (L - \lambda)^j$, $\mathscr{N}_\lambda$ is invariant under L, and $(L - \lambda)$ is invertible on $\mathscr{N}_\lambda$.

The projection of $\mathscr{B}$ onto $\mathscr{L}_\lambda$ along $\mathscr{N}_\lambda$ is denoted by P_λ.

Hence $P_\lambda \mathscr{B} = \mathscr{L}_\lambda$ and $(I - P_\lambda)\mathscr{B} = \mathscr{N}_\lambda$. Let $Q_\lambda = I - P_\lambda$.

An eigenvalue λ of L is isolated if there exists $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda + \varepsilon)$ contains no other members of $\operatorname{sp} L$. The set of isolated normal eigenvalues of L is called the discrete spectrum of L which we denote by $\operatorname{sp}_d(L)$. The remaining part of the spectrum will be called nondiscrete and is denoted by $\operatorname{sp}_{nd}(L)$.

REMARK. If L is self-adjoint, the nondiscrete spectrum is the essential spectrum of L.

DEFINITION 1.3. $(\lambda, 0)$ is a bifurcation point for (1.1) if every neighborhood in $\mathscr{E}$ of $(\lambda, 0)$ contains a nontrivial solution of (1.1).

DEFINITION 1.4. If $\mathscr{Y}$ is a subset of $\mathscr{E}$, $\mathscr{Y}^\lambda$ and $\mathscr{Y}_R$ are defined to be $\mathscr{Y}^\lambda = \{u \mid (\lambda, u) \in \mathscr{Y}\}$ and $\mathscr{Y}_R = \{\lambda \mid (\lambda, u) \in \mathscr{Y}$ for some $u\}$. For $W \subset \boldsymbol{R}$, $\mathscr{B}$, or $\mathscr{E}$, $\bar{W}$ denotes the closure of W in the respective space.

Some of the material that follows in this section was presented in [8], and is repeated here without proof.

DEFINITION. The set measure of compactness of a bounded set Ω, expressed by $\alpha(\Omega)$, is defined to be the infimum of all $\delta > 0$ such that Ω can be covered by a finite number of balls having radius δ.

Some useful results in this area include:

(i) $\alpha(\Omega) = \alpha(\bar{\Omega})$ for all bounded sets Ω.

(ii) If Ω is bounded, Ω is relatively compact if and only if $\alpha(\Omega) = 0$.

(iii) $\alpha(\Omega_1 + \Omega_2) \leqq \alpha(\Omega_1) + \alpha(\Omega_2)$.

(iv) If $\lim_{n\to\infty} x_n = 0$, then $\alpha(\{x_n\}_{n=1,2,\ldots}) = 0$.

DEFINITION. An operator $T: B \to B$ is called a k-set contraction if it is continuous and $\alpha(T(\Omega)) \leqq k\alpha(\Omega)$ for all bounded sets Ω. Let $\gamma(T) = \inf\{k \mid T$ is a k-set contraction$\}$. The following results concerning k-set contractions hold.

(i) T is compact if and only if T is a 0-set contraction.

(ii) If L is a bounded linear operator with operator norm $\|L\|$, then L is a $\|L\|$-set contraction. (This need not be true if L is nonlinear. (See § 4.))

(iii) If L is a bounded, linear, and self-adjoint operator, $\gamma(L) = \rho_e(L)$ where $\rho_e(L)$ is the radius of the essential spectrum of L. [8].

(iv) If $F = GH$ with G linear, $\gamma(F) \leqq \|G\|\gamma(H)$. In general, for all G and H, $\gamma(F) \leqq \gamma(G)\gamma(H)$.

A degree theory for nonlinear operators of the form $I - T$, where $T: B \to B$ is a k-set contraction with $k < 1$, was developed by

Nussbaum in his thesis. The results of Nussbaum's to be used are given below, together with a theorem of Stuart.

Let $T: B \to B$ be a k-set contraction ($k < 1$). Then an integer-valued function, denoted by deg, can be defined so as to have the following properties.

(1) $\deg(\Omega, I - T, 0)$ is well defined for each open, bounded subset $\Omega \subset B$ such that T has no fixed points on the boundary $\partial\Omega$ of Ω.

(2) If $\deg(\Omega, I - T, 0) \neq 0$, then there is a point $x \in \Omega$ such that $x = Tx$.

(3) If Ω_1 and Ω_2 are open subsets of Ω, itself a bounded, open subset of such that T has no fixed points in $[\bar{\Omega} | (\Omega_1 \cup \Omega_2)] \cup \overline{(\Omega_1 \cap \Omega_2)}$, then $\deg(\Omega, I - T, 0) = \deg(\Omega_1, I - T, 0) + \deg(\Omega_2, I - T, 0)$.

(4) If T is compact, then $\deg(\Omega, I - T, 0) = d(\Omega, I - T, 0)$, where d denotes the Leray-Schauder degree, whenever the left-hand side is defined. [8].

The arguments of this paper will closely follow those of [5]. Thus, a notation of index is helpful. Define

$$\operatorname{index}(T, x_0) = \deg(B, I - T, 0)$$

where B is an open ball in B centered at x_0 with a radius small enough so that x_0 is the only fixed point of T in $\bar{B}$.

In [5], critical use was made of a theorem in Leray-Schauder degree theory which has been extended to the Nussbaum degree theory by Toland and Stuart [8].

THEOREM. *Let $T: X \to X$ be a k-set contraction ($k < 1$) and let x_0 be a fixed point of T. Suppose that T has Frechet derivative $T'(x_0)$ at x_0 and that unity is not an eigenvalue of $T'(x_0)$.*

Then x_0 is an isolated fixed point of T, and

$$\operatorname{ind}(T, x_0) = (-1)^{\nu},$$

where ν is the sum of the multiplicities of the eigenvalues greater than unity of $T'(x_0)$.

Proof. See [8].

2. Local bifurcation theorem. The first theorem shows that bifurcation from an isolated eigenvalue λ_0 of L having odd multiplicity is not dependent upon H being compact, but rather on how "close" H is to being compact.

THEOREM 2.1. *Let L be as above and let H satisfy $H - 1$. λ_0*

is an isolated normal eigenvalue of L having odd multiplicity. Assume that for $|\lambda - \lambda_0| < \varepsilon'$, $\|(L-\lambda)^{-1}Q_{\lambda_0}\|\gamma(H) \leqq K < 1$. *Then,* $(\lambda_0, 0)$ *is a bifurcation point for* (1.1).

Proof. In order to prove this theorem, (1.1) will be rewritten in the form $u - C(\lambda, u) = 0$. Split (1.1) by

$$
\begin{aligned}
LP_{\lambda_0}u &= \lambda P_{\lambda_0}u + P_{\lambda_0}H(\lambda, u) \\
LQ_{\lambda_0}u &= \lambda Q_{\lambda_0}u + Q_{\lambda_0}H(\lambda, u)\,.
\end{aligned}
\tag{2.1}
$$

A solution of (1.1) is equivalent to a simulation solution of the two equations in (2.1). Select $\mu_0 \in \rho(L)$. Instead of (2.1) we may write

$$
\begin{aligned}
P_{\lambda_0}u &= \frac{(L-\mu_0)P_{\lambda_0}u}{\lambda - \mu_0} - \frac{P_{\lambda_0}H(\lambda, u)}{\lambda - {}_0\mu} \\
Q_{\lambda_0}u &= (L-\lambda)^{-1}Q_{\lambda_0}H(\lambda, u)
\end{aligned}
\tag{2.2}
$$

where $(L-\lambda)^{-1}$ is to be interpreted as $(L-\lambda)^{-1}|\mathscr{N}_{\lambda_0}$. Thus, (2.2) is valid for $\lambda \in \{\lambda_0\} \cup \{\rho(L)\backslash\{\mu_0\}\}$. Adding these equations we get

$$
\begin{aligned}
u &= C_1(\lambda, u) + C_2(\lambda, u) \\
C_1(\lambda, u) &= \frac{(L-\mu_0)P_{\lambda_0}u}{\lambda - \mu_0} \\
C_2(\lambda, u) &= \left((L-\lambda)^{-1}Q_{\lambda_0} - \frac{P_{\lambda_0}}{\lambda - \mu_0}\right)H\,.
\end{aligned}
\tag{2.3}
$$

Note that $C_1\colon \mathscr{E} \to \mathscr{B}$ is compact and linear in u for each fixed λ $C_2\colon \mathscr{E} \to \mathscr{B}$ satisfies $H-1$. Define

$$
\Phi(\lambda, \cdot) = I - C_1(\lambda, \cdot) - C_2(\lambda, \cdot)\,.
\tag{2.4}
$$

Clearly, (2.3) or $\Phi(\lambda, u) = 0$ is equivalent to (1.1) for the specified values of λ when L is bounded. If L is unbounded, the question arises as to wheter u is in dom(L) if (λ, u) is a zero of Φ. Noting (2.2), which is obtained from (2.3) by projecting onto $\mathscr{L}_{\lambda_0}$, $\mathscr{N}_{\lambda_0}$ respectively, we see that $Q_{\lambda_0}u$ is in dom(L). Since $P_{\lambda_0}u$ is in an eigenspace of L, $u = P_{\lambda_0}u + Q_{\lambda_0}u$ is in dom(L).

If the assertion of the theorem is not true we can find a neighborhood $\mathscr{O}$ of $(\lambda_0, 0)$ such that the only solutions of (1.1) in $\mathscr{O}$ are trivial solutions, $\rho(L)\backslash\mathscr{O}_R \neq \varnothing$, and $\mathscr{O}_R \cap \operatorname{sp} L = \{\lambda_0\}$. Select $\mu_0 \in \rho(L)\backslash\mathscr{O}_R$ such that (1.1) is equivalent to (2.3) for all $\mathscr{O}_R$. Select $\varepsilon > 0$, $0 < \varepsilon < \varepsilon'$, that $[-\varepsilon + \lambda_0, \lambda_0 + \varepsilon] \times \{0\} \subset \mathscr{O}$. Applying the homotopy property of degree theory we obtain

$$
\deg(\Phi(\lambda, \cdot), \mathscr{O}^\lambda, 0) = \text{constant}, \quad |\lambda - \lambda_0| < \varepsilon\,.
\tag{2.5}
$$

Select $\underline{\lambda}$ and $\bar{\lambda}$ such that $\lambda_0 - \varepsilon < \underline{\lambda} < \lambda_0 < \bar{\lambda} < \lambda_0 + \varepsilon$. Then

$$\begin{aligned}(2.6)\qquad &\deg(\Phi(\underline{\lambda}, \cdot), \mathcal{O}^{\underline{\lambda}}, 0) = \operatorname{index}(I - C_1(\underline{\lambda}, \cdot), (\underline{\lambda}, 0))\\ &\deg(\Phi(\bar{\lambda}, \cdot), \mathcal{O}^{\bar{\lambda}}, 0) = \operatorname{index}(I - C_1(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)) .\end{aligned}$$

Thus, using (2.5) and (2.6),

$$\begin{aligned}(2.7)\qquad &\operatorname{index}(I - C_1(\underline{\lambda}, \cdot), (\underline{\lambda}, 0))\\ &\quad = \operatorname{index}(I - C_1(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)) .\end{aligned}$$

However, since the multiplicity of λ_0 is odd,

$$\begin{aligned}(2.8)\qquad &\operatorname{index}(I - C_1(\underline{\lambda}, \cdot), (\underline{\lambda}, 0))\\ &\quad = -\operatorname{index}(I - C_1(\bar{\lambda}, \cdot), (\bar{\lambda}, 0)) .\end{aligned}$$

Since the indices in (2.7) and (2.8) are either $+1$ or -1, we have a contradiction. Thus, such a neighborhood can never be found. This proves that $(\lambda_0, 0)$ is a bifurcation point.

REMARK 1. If $\lambda_0 \neq 0$ is an eigenvalue of L having odd multiplicity, then the hypotheses of Theorem 1 are satisfied if L is compact or if L is self-adjoint with λ_0 isolated in $\operatorname{sp} L$.

REMARK 2. The condition on $\gamma((L - \lambda)^{-1}Q_{\lambda_0}H)$ can be relaxed. If one restricted the operators to a ball B_0 centered at $(\lambda_0, 0)$ and then extended them to all of $R \times B_0$ in a linear manner, one could apply Theorem 1.1 if $\gamma((L - \lambda)^{-1}Q_{\lambda_0}H | B_0) = K < 1$, for $|\lambda - \lambda_0| < \varepsilon'$. This would handle the case that H is well behaved near $u = 0$ but grows too large for u far from 0.

$(H - 2)$ $H\colon \mathscr{E} \to \mathscr{B}$ satisfies:

(i) H is continuous, and bounded on each ball centered at 0.

(ii) H is $o(\|u\|)$ uniformly on bounded λ intervals.

REMARK 3. The theorem remains true if H satisfies hypothesis $H - 2$ rather than the more restricted $H - 1$. The proof is very similar.

3. A global alternative theorem. In this section we will show that the local bifurcation exhibited in Theorem 2.1 is a global property with an alternative-type result.

For $\mathscr{V} \subset \mathscr{E}$, a subcontinuum of $\mathscr{V}$ is a subset of $\mathscr{V}$ which is closed and connected in $\mathscr{E}$. $\mathscr{S}$ will denote the closure of the set of nontrivial solutions of (1.1) in $\mathscr{E}$. Let $\mathscr{C}_{\lambda_0}$ denote the maximal subcontinuum of $\mathscr{S} \cup (\lambda_0, 0)$ containing $(\lambda_0, 0)$. B_ρ will denote the open ball in $\mathscr{B}$ centered at 0 and having radius ρ. L and H will be as in § 2.

LEMMA 3.1. *Suppose* λ_0 *and* λ_1 *are distinct normal eigenvalues of* L. *Then* $\mathscr{B} = \mathscr{L}_{\lambda_0} \oplus \mathscr{L}_{\lambda_1} \oplus \mathscr{N}$, *a direct sum of subspaces, where* $\mathscr{N} = \mathscr{N}_{\lambda_0} \cap \mathscr{N}_{\lambda_1}$, *and* $P = P_{\lambda_0} + P_{\lambda_1}$ *projects onto* $\mathscr{L}_{\lambda_0} \oplus \mathscr{L}_{\lambda_1}$ *along* $\mathscr{N}$. [5].

LEMMA 3.2. *Let* K *be a compact metric space and* A *and* B *disjoint closed subsets of* K. *Then either there exists a subcontinuum of* K *meeting both* A *and* B, *or* $K = K_A \cup K_B$ *where* K_A *and* K_B *are disjoint compact subsets of* K *containing* A *and* B *respectively.*

Proof. See [9].

For λ_0 as before, define

$$\alpha_1(\lambda_0) = \sup \{\lambda \mid \lambda < \lambda_0, \lambda \in \mathrm{sp}_{nd}(L)\}$$
$$\beta_1(\lambda_0) = \inf \{\lambda \mid \lambda > \lambda_0, \lambda \in \mathrm{sp}_{nd}(L)\} .$$

These values will be $\pm\infty$ respectively if the vacuous case results. For $\varepsilon_j > 0$, consider $I(\varepsilon_1, \varepsilon_2) = [\alpha_1(\lambda_0) + \varepsilon_1, \beta_1(\lambda_0) - \varepsilon_2]$. (Here assume both are finite.) Let $P_{\varepsilon_1,\varepsilon_2} = \sum P_\lambda$ where the summation is over all eigenvalues of L in $I(\varepsilon_1, \varepsilon_2)$, and let $Q_{\varepsilon_1,\varepsilon_2} = I - P_{\varepsilon_1,\varepsilon_2}$. Select $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\|(L-\lambda)^{-1}Q_{\varepsilon_1,\varepsilon_2}\|\gamma(H) < 1$ on $I(\varepsilon_1, \varepsilon_2)$. Let $[\alpha(\lambda_0), \beta(\lambda_0)] = I(\varepsilon_1, \varepsilon_2)$. If $\alpha_1(\lambda_0)$ or $\beta_1(\lambda_0)$ are infinite, select $\alpha(\lambda_0)$ or $\beta(\lambda_0)$ to be any appropriate finite number.

LEMMA 3.3 *Suppose* λ_0 *is an isolated normal eigenvalue of* L *having finite multiplicity. Assume* $\mathscr{C}_{\lambda_0}$ *is bounded,* $(\overline{\mathscr{C}_{\lambda_0}})_R \cap \{\alpha(\lambda_0), \beta(\lambda_0)\} = \phi$, *and* $\mathscr{C}_{\lambda_0} \cap \{R \times \{0\}\} = (\lambda_0, 0)$. *Then* $\mathscr{C}_{\lambda_0}$ *is compact and there exists a bounded open set* $\mathscr{O}$ *such that* $\mathscr{C}_{\lambda_0} \subset \mathscr{O}$, $\partial\mathscr{O} \cap \mathscr{S} = \varnothing$, $(\overline{\mathscr{O}_R}) \cap \mathrm{sp}_{nd}(L) = \varnothing$, *the trivial solutions contained in* $\mathscr{O}$ *are the points* $(\lambda, 0)$ *where* $|\lambda - \lambda_0| < \varepsilon$ *for some* $\varepsilon < \varepsilon_0 = \mathrm{dist}\,(\lambda_0, \mathrm{sp}\, L\backslash\{\lambda_0\})$, *and* $\|(\lambda, u) - (\mu, 0)\| \geqq 2\varepsilon_1$ *for some positive* ε_1 *whenever* $(\lambda, u) \in \partial\mathscr{O}$ *and* $\mu \in \mathrm{sp}\, L$.

Proof. $\mathscr{C}_{\lambda_0}$ is compact. Indeed, let $\{(\lambda_n, u_n)\}_{n\in\mathscr{N}}$ be elements of $\mathscr{C}_{\lambda_0}$. Since $\mathscr{C}_{\lambda_0}$ is bounded, we may find a λ and a subsequence $\mathscr{N}_1$ such that $\lim_{\substack{n\to\infty \\ n\in\mathscr{N}_1}} \lambda_n = \lambda$. Let P be the projection for $(\overline{\mathscr{C}_{\lambda_0}})_R$ and $Q = I - P$. Consider $\{u_n\}_{n\in\mathscr{N}_1}$.

$$\begin{aligned}\alpha(\{u_n\}_{n\in\mathscr{N}_1}) &= \alpha(\{C_1(\lambda_n, u_n) + C_2(\lambda_n, u_n)\}_{n\in\mathscr{N}_1}) \\ &\leqq \alpha(\{C_1(\lambda_n, u_n)\}_{n\in\mathscr{N}_1} + \{C_2(\lambda_n, u_n)\}_{n\in\mathscr{N}_1}) \\ &\leqq \alpha(\{C_2(\lambda_n, u_n)\}_{n\in\mathscr{N}_1})\end{aligned}$$

$$
\begin{aligned}
(\text{similarly}) &\leqq \alpha(\{(L-\lambda_n)^{-1}QH(u_n)\}_{n\in\mathscr{N}_1}) \\
&\leqq \alpha(\{(L-\lambda)^{-1}QH(u_n)\}_{n\in\mathscr{N}_1}) \\
&\quad + \alpha(\{((L-\lambda_n)^{-1}-(L-\lambda)^{-1})QH(u_n)\}_{n\in\mathscr{N}_1}) \\
&= \alpha(\{(L-\lambda)^{-1}QH(u_n)\}_{n\in\mathscr{N}_1}) \\
&\leqq \|(L-\lambda)^{-1}Q\|\alpha(\{H(u_n)\}_{n\in\mathscr{N}_1}) \\
&\leqq \|(L-\lambda)^{-1}Q\|\gamma(H)\alpha(\{u_n\}_{n\in\mathscr{N}_1}) \\
&< \alpha(\{u_n\}_{n\in\mathscr{N}_1}) .
\end{aligned}
$$

Thus $\alpha(\{u_n\}_{n\in\mathscr{N}_1}) = 0$ meaning the set is compact, meaning it has a convergent subsequence. Thus, C_{λ_0} is compact.

The remainder of the proof follows from [5] and [8] using Lemma 3.1.

The following theorem is modeled after one in [5] given for the case when H is compact.

THEOREM 3.1. *Suppose λ_0 is an isolated eigenvalue of L of odd multiplicity. L is as before and H satisfies $H-1$. Furthermore, let $\|(L-\lambda_0)^{-1}Q_{\lambda_0}\|\gamma(H) < 1$. Then $(\lambda_0, 0)$ is a bifurcation point of* (1.1) *possessing a maximal continuous branch $\mathscr{C}_{\lambda_0}$ such that exactly one of the following alternatives occurs.*

(i) *$\mathscr{C}_{\lambda_0}$ is unbounded.*

(ii) *$\mathscr{C}_{\lambda_0}$ is bounded and $(\overline{\mathscr{C}_{\lambda_0}})_R \cap \{\alpha(\lambda_0), \beta(\lambda_0)\} \neq 0$.*

(iii) *$\mathscr{C}_{\lambda_0}$ is compact, $(\mathscr{C}_{\lambda_0})_R \cap \{\alpha(\lambda_0), \beta(\lambda_0)\} = \varnothing$ and $\mathscr{C}_{\lambda_0} \cap \{\boldsymbol{R}\times\{0\}\} = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}\times\{0\}$ where $\lambda_1, \cdots, \lambda_n$ are normal eigenvalues of L distinct from λ_0, and the sum of the multiplicities of $\lambda_0, \lambda_1, \cdots, \lambda_n$ is even.*

Proof. With the use of Lemma 3.3, the proof is similar to Theorem 2.1 and Theorem 2.2 [5].

REMARK 1. The hypotheses of this theorem are unnecessarily stringent. The same results hold with $H(\lambda, u)$ if $\|(L-\lambda_0)^{-1}Q_{\lambda_0}\|\gamma(H(\lambda_0, \cdot)) < 1$, where H satisfies $H-2$. The preceding proofs, however, become a little more complicated mainly due to notation.

REMARK 2. Suppose that $H(\lambda, \cdot)$, when restricted to a ball centered at $u=0$, has $\|(L-\lambda_0)^{-1}Q_{\lambda_0}\|\gamma(H_{(\lambda_0,\cdot)}) < 1$ (but this hypothesis fails on the entire space). One can do the degree work on these balls (by reworking all previous proofs) and obtain a theorem similar to that in [4]. (It was necessary to make a change in that theorem due to an error committed in [4] (see the next section).)

Assign $F(\varepsilon) = [\alpha(\lambda_0)+\varepsilon, \beta(\lambda_0)-\varepsilon]$. Let P_ε correspond to $F(\varepsilon)$, and $Q_\varepsilon = I - P_\varepsilon$. When restricted to a ball of radius r centered at

0, let $H_{(\lambda,\cdot)}$ be a $\gamma_r(H(\lambda, \cdot))$-set contraction, and define $\gamma_r(H)$ to be strictly monotone increasing.

THEOREM 3.2. *Let λ_0 be an isolated eigenvalue of L having odd algebraic multiplicity. L is as before and H satisfies $H-2$. Then $(\lambda_0, 0)$ is a bifurcation point of (1.1) and emanating from it is a maximal continuous branch $\mathscr{C}_{\lambda_0}$ which obeys exactly one of the following alternatives for each suitably small $\varepsilon > 0$.*

(i) *$\mathscr{C}_{\lambda_0}$ is unbounded.*

(ii) *$\mathscr{C}_{\lambda_0}$ is bounded and $\overline{\mathscr{C}_{\lambda_0}}$ meets $S_\varepsilon = \{(\lambda, u) \mid \lambda \in F(\varepsilon)$ and $\|u\| = r$ where $\gamma_r(H(\lambda, \cdot)) = \|(L-\lambda)^{-1}Q_\varepsilon\|^{-1}\} \cup \{(\alpha(\lambda_0)+\varepsilon) \times \mathscr{B}\} \cup \{(\beta(\lambda_0)-\varepsilon) \times \mathscr{B}$.*

(iii) *$\mathscr{C}_{\lambda_0}$ is compact, $\overline{\mathscr{C}_{\lambda_0}}$ does not meet S_ε, and $\mathscr{C}_{\lambda_0} \cap \{0 \times B\} = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}$, each a distinct normal eigenvalue of L, and the sum of their algebraic multiplicities is even.*

REMARK 1. In the case where L is self-adjoint, $\|(L-\lambda)^{-1}Q_\varepsilon\| = 1/\mathrm{dist}\,(\lambda, \mathrm{sp}\,(L)/F(\varepsilon))$ where dist $(\cdot)$ is the standard distance function in $\boldsymbol{R}$. This simplifies the statement of (ii).

REMARK 2. If $(\alpha(\lambda_0), \beta(\lambda_0)) \cap \mathrm{sp}\,(L)$ consists of a finite list of eigenvalues, there is an $\varepsilon_0 > 0$ such that whenever $0 < \varepsilon_1 < \varepsilon_2 \leqq \varepsilon_0$, S_{ε_1} and S_{ε_2} are identical in $F(\varepsilon_2) \times B$. This is because $\|(L-\lambda)^{-1}Q_\varepsilon\|$ is constant in ε for $0 < \varepsilon < \varepsilon_0$. This leads to an improvement in (ii) and (iii).

(ii)′ $\mathscr{C}_{\lambda_0}$ is bounded and $\overline{\mathscr{C}_{\lambda_0}}$ meets $S = \{(\lambda, u) \mid \lambda \in (\alpha(\lambda_0), \beta(\lambda_0))$ and $\|u\| = r$ where $\gamma_r(H(\lambda, \cdot)) = \|(L-\lambda)^{-1}Q_{\varepsilon_0}\|^{-1}\} \cup \{\alpha(\lambda_0) \times B\} \cup \{\beta(\lambda_0) \times B\}$.

(iii)′ $\mathscr{C}_{\lambda_0}$ is compact, $\overline{\mathscr{C}_{\lambda_0}}$ does not meet S, and $\mathscr{C}_{\lambda_0} \cap \{0 \times B\} = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}$, each a distinct normal eigenvalue of L, and the sum of their algebraic multiplicities is even.

4. **Other results.** The theorems I proposed in [4] are unfortunately incorrect as stated and require modification as in §3 of this paper. The hypothesis of continuity on H had to be strengthened. My error was in a proof that if one restricted H to a ball centered at 0 in B and on that ball $\|H\| = k$, then H was a k-set contraction on the ball. This is true for linear operators.

This error was found by Professor Norman Dancer, The University of New England, Armidale N.S.W., Australia. He constructed a counterexample to Theorem I of [4], which I present here. There is an operator $V: c_0 \to c_0$ such that if $x = \lambda V(x)$, then $x = 0$ and $\lambda = 0$. Set $B = c_0 \times R$, $L: B \to B$ is defined by $L(w, t) = (2w, t)$ and $H: R \times B \to B$ is defined by $H(\lambda, (w, t) = (0, \lambda t^2 V(w))$.

$\lambda = 1$ is an eigenvalue of L of multiplicity 1. If λ is near 1 and $Lu = \lambda u + H(\lambda, u)$ where $u = (w, t)$, then $t = \lambda t$ and $2w = \lambda w + \lambda t^2 V(w)$. $w = \lambda t^2 V(w)/(2 - \lambda)$ which implies $w = 0$, and together with λ being near 1 imply $t = 0$. Thus, for λ near 1, the only solution is $u = 0$. Many thanks to Professor Dancer. The operator V is due to Ana and Vasile Istratescu and appeared in the Proceeding of the Amer. Math. Soc., Vol. 48, No. 1, page 197.

References

1. N. Dunford and J. T. Schwarz, *Linear Operators*, Part II, Interscience, New York, 1963).
2. M. A. Krasnoseljskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, New York, 1964.
3. J. A. MacBain, *Global bifurcation theorems for noncompact operators*, Bulletin Amer. Math. Soc., **80** (1974),
4. ———, *Global bifurcation theorems for nonlinearly perturbed operator equation*, Bulletin Amer. Math. Soc., **82** (1976).
5. ———, *Local and global bifurcation from normal eigenvalues*, Pacific J. Math., **63** (1976), 445–466.
6. P. H. Rabinowitz, *Some aspects of nonlinear eigenvalue problems*, Rocky Mountain J. Math., **3** (1973).
7. F. Riesz, B. Sz.-Nagy, *Functional Analysis*, tr. L. Boron, Ungar, New York, 1971.
8. C. A. Stuart, *Some bifurcation theory for k-set contractions*, Proc. London Math. Soc., (1973).
9. G. T. Whyburn, *Topological Analysis*, Princeton University Press, Princeton, 1958.

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