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EXTENDED WEAK-*DIRICHLET ALGEBRAS

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Let $(X, \mathscr{A}, m)$ be a probability measure space and A a subalgebra of $L^\infty(m)$, containing the constant functions. Srinivasan and Wang defined A to be a weak-*Dirichlet algebra if $A+\bar{A}$ (the complex conjugate) is weak-*dense in $L^\infty(m)$ and the integral is multiplicative on A, $\int fgdm = \int fdm \int gdm$ for $f, g \in A$. In this paper the notion of extended weak-*Dirichlet algebra is introduced; A is an extended weak-*Dirichlet algebra if $A+\bar{A}$ is weak-*dense in $L^\infty(m)$ and if the conditional expectation $E^{\mathscr{B}}$ to some sub σ-algebra $\mathscr{B}$ is multiplicative on A. Then most of important theorems proved for weak-*Dirichlet algebras are generalized in the context of extended weak-*Dirichlet algebras, for instance, Szegö's theorem and Beuring's theorem. Besides, our approach will yield several theorems which were not known even for weak-*Dirichlet algebras.

1. Introduction. This paper presents a generalization of a portion of the theory of analytic functions in the unit disc. The theory to be extended consists of some basic theorems related to the Hardy class H^p $(1 \leqq p \leqq \infty)$. For example, (i) the theorem of Szegö, on mean-square approximation of 1 by polynomials which vanish at the origin, (ii) Beurling's theorem on invariant subspaces of H^2, (iii) the factorization of H^p functions into products of "inner" and "outer" functions, (vi) Jensen inequality. The paper was inspired by the work of Srinivasan and Wang [13]. They introduced weak-*Dirichlet algebras for a generalized analytic function theory. Suppose A is an extended weak-*Dirichlet algebra of $L^\infty = L^\infty(m)$, defined in the abstract. The abstract Hardy spaces $H^p = H^p(m)$, $1 \leqq p \leqq \infty$, associated with A are defined as follows. For $1 \leqq p < \infty$, H^p is the $L^p = L^p(m)$-closure of A, while H^∞ is defined to be the weak-*closure of A in L^∞. In operator algebras, A is called a subdiagonal algebra by Arveson [1]. Independently by the author [12], A is called an algebra on which m is quasi-multiplicative, in the study of invariant subspaces of weak-*Dirichlet algebras [12].

Let B be the algebra of continuous, complex-valued functions on the torus $T^2 = \{(z, w) \in C^2 : |z| = |w| = 1\}$ which are uniform limits of polynomials in $z^n w^m$ where $(n, m) \in \{(n, m) \in Z^2; m > 0\} \cup \{(n, 0) \in Z^2 : n \geqq 0\}$. Denote by m the normalized Haar measure on T^2, then B is a weak-*Dirichlet algebra of L^∞. Set $A = \bigcup_{n=0}^\infty \bar{z}^n B$, then A is not a weak-*Dirichlet algebra of L^∞, but it is an extended one. When

$\mathscr{A}$ is the σ-algebra of all Borel sets on T^2, let $\mathscr{B}$ be the sub σ-algebra of $\mathscr{A}$ consisting of all Borel sets of the form $E \times T$ where E is a Borel set on T. Let $E^{\mathscr{B}}$ denote the conditional expectation for $\mathscr{B}$. We show, if $f \in B$, then

$$\int_{T^2} \log |f| \, dm \geqq \int_{T^2} \log |E^{\mathscr{B}}(f)| \, dm \geqq \log \left| \int_{T^2} f dm \right| .$$

There exists f in B such $\int_{T^2} \log |E^{\mathscr{B}}(f)| \, dm \gneqq \log \left| \int_{T^2} f dm \right|$. Let $w \in L^1$, $w \geqq 0$. Even if $\int_{T^2} \log w dm = -\infty$, if $E^{\mathscr{B}}(\log w) > -\infty$ a.e., then there exists f in $H^2(B)$ with $w = |f|^2$ where $E^{\mathscr{B}}(\log w)$ is defined by $\lim_{0<\varepsilon \to 0} E^{\mathscr{B}}\{\log (w + \varepsilon)\}$. Set $I = \bigcap_{n=0}^{\infty} z^n B$, then

$$\inf_{g \in I} \int_{T^2} |1 - g|^2 w dm = \int_{T^2} \exp E^{\mathscr{B}}(\log w) dm .$$

2. **Extended weak-*Dirichlet algebras.** We define an extended weak-*Dirichlet algebras formally.

DEFINITION 1. Let $(X, \mathscr{A}, m)$ be a probability measure space. Let $E^{\mathscr{B}}$ denote the conditional expectation for the sub σ-algebra $\mathscr{B}$ of $\mathscr{A}$. An extended weak-*Dirichlet algebra is an algebra of $L^\infty = L^\infty(m)$ such that (i) the constant functions lie in A; (ii) $A + \bar{A}$ is weak-*dense in L^∞; (iii) for all f and g in A, $E^{\mathscr{B}}(fg) = E^{\mathscr{B}}(f)E^{\mathscr{B}}(g)$; (iv) $E^{\mathscr{B}}(A) \subseteqq A \cap \bar{A}$.

When $E^{\mathscr{B}}(A) = \{1\}$, the space spanned by 1, then $E^{\mathscr{B}}(f) = \int_X f dm$ for f in A, and hence A is a weak-*Dirichlet algebra. For $1 \leqq p \leqq \infty$, let $I^p = \{f \in H^p : E^{\mathscr{B}}(f) = 0\}$ and let $I = \{f \in A : E^{\mathscr{B}}(f) = 0\}$. Suppose $1 \leqq p \leqq \infty$. For any subset $M \subset L^p$, denote by $[M]_p$ the L^p-closure of M (weak-*closure for $p = \infty$). For any measurable subset E of X, the function χ_E is the characteristic function of E. If $f \in L^p$ $(1 \leqq p \leqq \infty)$, write $E(f)$ for the support set of f. The following lemma is well known [10] and the proof is easy.

LEMMA 1. *For* $1 \leqq p \leqq \infty$,

$$\int_X |E^{\mathscr{B}}(f)|^p \, dm \leqq \int_X |f|^p \, dm \qquad f \in L^p .$$

*For f in L^∞, $\|E^{\mathscr{B}}(f)\|_\infty \leqq \|f\|_\infty$, where $\| \ \|_\infty$ is an essential sup-norm in L^∞. Moreover $E^{\mathscr{B}}$ is a weak-*continuous linear operator from L^∞ into L^∞.*

LEMMA 2. *For $1 \leqq p \leqq \infty$, $E^{\mathscr{B}}(H^p) = [E^{\mathscr{B}}(A)]_p$ and $I^p = [I]_p$.*

The proof is clear by Lemma 1.

PROPOSITION 1. *Suppose* $1 \leqq p \leqq \infty$.

(1) *I is an ideal of A and I^p is a closed (for $p = \infty$ weak-*closed) invariant subspace of L^p.*

(2) *I is a maximum ideal with the property that if J is an ideal of A which contains I, then $J = E^{\mathscr{B}}(J) + I$ and $E^{\mathscr{B}}(J)$ is an ideal of $E^{\mathscr{B}}(A)$.*

(3) *I^p is a maximum invariant subspace with the property that if J^p is a closed invariant subspace of H^p with $I^p \subseteqq J^p \subseteqq H^p$, then $J^p = \chi_E E^{\mathscr{B}}(H^p) \oplus I^p = \chi_E H^p \oplus (1 - \chi_E) I^p$ where χ_E belongs to $[E^{\mathscr{B}}(A)]_\infty$ and $\oplus$ denotes algebraic direct sum.*

(4) *I (or I^∞) is a maximum ideal of A (or H^∞) which is contained in $A_0 = \left\{f \in A: \int_X f dm = 0\right\}\left(or\ H_0^\infty = \left\{f \in H^\infty: \int_X f dm = 0\right\}\right)$.*

Proof. Since $E^{\mathscr{B}}(fg) = E^{\mathscr{B}}(f)E^{\mathscr{B}}(g)$ for all f and g in A, (1) is clear.

(2) It is clear that if J is an ideal of A which contains I, then $J = E^{\mathscr{B}}(J) + I$ and $E^{\mathscr{B}}(J)$ is an ideal of $E^{\mathscr{B}}(A)$. Suppose I' is an ideal with the above property, then $\ker E^{\mathscr{B}}|_{I'} \subseteqq I$. $E^{\mathscr{B}}(I') + I \supseteqq I'$ and $E^{\mathscr{B}}(I') + I$ is an ideal of A. By the assumption on I', $E^{\mathscr{B}}(I') + I = E^{\mathscr{B}}(I') + I'$ and hence $E^{\mathscr{B}}(I') + I = I'$. Thus $I' \supseteqq I$.

(3) can be shown as in the proof of (2), using Lemma 2. For $E^{\mathscr{B}}(A) \cdot E^{\mathscr{B}}(J^p) \subseteqq E^{\mathscr{B}}(J^p) \subseteqq [E^{\mathscr{B}}(A)]_p = L^p(X, \mathscr{B}, m)$ and so $E^{\mathscr{B}}(J^p) = \chi_E [E^{\mathscr{B}}(A)]_p$ for some χ_E in $[E^{\mathscr{B}}(A)]_\infty = L^\infty(X, \mathscr{B}, m)$.

(4) Set $J = \left\{f \in A: \int_X fg dm = 0 \text{ for all } g \text{ in } A\right\}$, then J is a maximum ideal of A which is contained in A_0. We shall show $J = I$. Since $J \supseteqq I$, by (2), $J = E^{\mathscr{B}}(J) + I$. If $f \in E^{\mathscr{B}}(J)$, then $\bar{f} \in A$ and hence $\int_X |f|^2 dm = 0$. Thus $E^{\mathscr{B}}(J) = \{0\}$ and $I = J$. The proof for I^∞ is similar to the above.

LEMMA 3. *$E^{\mathscr{B}}(A) = A \cap \bar{A}$ and for $p \geqq 2$, $E^{\mathscr{B}}(H^p) = H^p \cap \bar{H}^p$ and hence $[A \cap \bar{A}]_p = H^p \cap \bar{H}^p$.*

Proof. By Lemma 2, $E^{\mathscr{B}}(H^p) \subseteqq H^p \cap \bar{H}^p$. We shall show that $H^p \cap \bar{H}^p \subseteqq E^{\mathscr{B}}(H^p)$. If $f \in H^p \cap \bar{H}^p$, then both $f - E^{\mathscr{B}}(f)$ and $\overline{f - E^{\mathscr{B}}(f)}$ lie in I^p. Since $p \geqq 2$,

$$\int_X |f - E^{\mathscr{B}}(f)|^2 dm = \int_X E^{\mathscr{B}}\{(f - E^{\mathscr{B}}(f)(\overline{f - E^{\mathscr{B}}(f)})\} dm$$

$$= \int_X E^{\mathscr{B}}(f - E^{\mathscr{B}}(f)) E^{\mathscr{B}}(\overline{f - E^{\mathscr{B}}(f)}) dm = 0$$

and so $f = E^{\mathscr{B}}(f)$ a.e.. The proof for $E^{\mathscr{B}}(A) = A \cap \bar{A}$ is similar to the above.

Let $\mathscr{L}^\infty$ be a commutative von Neumann algebra of operators on L^2 which is contained in L^∞ and let $\mathscr{B}$ be the σ-algebra of measurable subsets E of X for which the characteristic functions χ_E lie in $\mathscr{L}^\infty$. Then $\mathscr{B}$ is a sub σ-algebra of $\mathscr{A}$ and $\mathscr{L}^\infty = L^\infty(\mathscr{B}) = L^\infty(X, \mathscr{B}, m)$. We say $E^{\mathscr{B}}$ is the conditional expectation for $\mathscr{L}^\infty$ (or $\mathscr{B}$).

PROPOSITION 2. *Let A be a weak-*closed algebra of L^∞ such that (i) the constant functions lie in A; (ii) $A + \bar{A}$ is weak-*dense in L^∞. Let $E^{\mathscr{B}}$ be the conditional expectation for $A \cap \bar{A}$ and let $K = L^2 \ominus H^2$ where '$\ominus$' denotes the orthogonal complement of H^2 in L^2. Then $E^{\mathscr{B}}$ is multiplicative on A if and only if $H^2 \cap \bar{H}^2 = [A \cap \bar{A}]_2$ and $\bar{K} \subset H^2$.*

Proof. Suppose $E^{\mathscr{B}}$ is multiplicative on A. Then Lemma 3 implies $H^2 \cap \bar{H}^2 = [A \cap \bar{A}]_2$. Since $H^2 = H^2 \cap \bar{H}^2 \oplus I^2$ and $A + \bar{A}$ is weak-*dense in L^∞, $L^2 = H^2 \oplus \bar{I}^2$ and so $K = \bar{I}^2$.

Suppose $H^2 \cap \bar{H}^2 = [A \cap \bar{A}]_2$ and $\bar{K} \subset H^2$. Then $H^2 = H^2 \cap \bar{H}^2 \oplus \bar{K}$. Since $H^2 \cap \bar{H}^2 = [A \cap \bar{A}]_2$ and $E^{\mathscr{B}}(A) = A \cap \bar{A}$, $E^{\mathscr{B}}(H^2) = [E^{\mathscr{B}}(A)]_2 = [A \cap \bar{A}]_2 = H^2 \cap \bar{H}^2$ and hence $\ker E^{\mathscr{B}}|_{H^2} = \bar{K}$. By the definition of K, $\bar{K} \cap L^\infty$ and so $(\ker E^{\mathscr{B}}|_{H^2}) \cap L^\infty$ is an ideal of $B = H^2 \cap L^\infty$. Since $\ker E^{\mathscr{B}}|_B = (\ker E^{\mathscr{B}}|_{H^2}) \cap L^\infty$, $\ker E^{\mathscr{B}}|_B$ is an ideal and hence $E^{\mathscr{B}}$ is multiplicative on A.

Later in § 5 we shall use this proposition to show that an algebra, consists of analytic functions defined by a flow, is an (extended) weak-*Dirichlet algebra.

DEFINITION 2. By Jensen's inequality, we mean the following statement:

$$E^{\mathscr{B}}(\log|f|) \geqq \log|E^{\mathscr{B}}(f)|$$

for every f in A, where $E^{\mathscr{B}}(\log|f|)$ is defined by $\lim_{0<\varepsilon\to 0} E^{\mathscr{B}}\{\log(|f|+\varepsilon)\}$.

If $E^{\mathscr{B}}(A) = \{1\}$, then $E^{\mathscr{B}}(w) = \int_X w\,dm$ and hence $\int_X \log|f|\,dm \geqq \log\left|\int_X f\,dm\right|$. Then it is known [15, Corollary 2.4.6.] that m is a Jensen measure.

LEMMA 4. *Let B be $\{1\} + I$, then B is a subalgebra of A and for all f and g in B,*

$$\int_X fg\,dm = \int_X f\,dm \int_X g\,dm\,.$$

The proof is clear.

PROPOSITION 3. *$E^{\mathscr{B}}(A) = A \cap \bar{A}$ and for $p \geqq 1$, $E^{\mathscr{B}}(H^p) = H^p \cap \bar{H}^p$ and hence $[A \cap \bar{A}]_p = H^p \cap \bar{H}^p$.*

Proof. If $f \in B = \{1\} + 1$, then $E^{\mathscr{B}}(f) = \int_X f dm$. By Theorem 2 in § 3, Jensen's inequality is valid for A. By the definition of Jensen's inequality, it follows that for all f in B,

$$\int_X \log |f| \, dm \geqq \log \left| \int_X f dm \right| .$$

If g in $[B]_1$ is a real-valued function, then it must be a constant [5, p. 140]. Hence if $f \in H^p \cap \bar{H}^p$, then both $f - E^{\mathscr{B}}(f)$ and $\overline{f - E^{\mathscr{B}}(f)}$ lie in $I^p(\subseteqq [B]_1)$ and so $f = E^{\mathscr{B}}(f)$ a.e.. Thus $E^{\mathscr{B}}(H^p) \supseteqq H^p \cap \bar{H}^p$ and by Lemma 2 $E^{\mathscr{B}}(H^p) = H^p \cap \bar{H}^p$.

PROPOSITION 4. *Suppose $1 < p \leqq \infty$. Then*

$$H^p \oplus \bar{I}^p = H^p \cap \bar{H}^p \oplus I^p \oplus \bar{I}^p = L^p .$$

Proof. Since $A + \bar{A}$ is weak-*dense in L^∞, by Lemma 3, $E^{\mathscr{B}}(A) + I + \bar{I}$ is weak-*dense in L^∞. By Lemma 1, $[E^{\mathscr{B}}(A)]_p \oplus [I + \bar{I}]_p = L^p$. By Theorem 2 in § 3, m is a Jensen measure for $B = \{1\} + I$. Hence by [9] and Lemma 4, $[I + \bar{I}]_p = I^p \oplus \bar{I}^p$.

$[E^{\mathscr{B}}(A)]_\infty$ is a commutative von Neumann algebra as operators on L^2.

LEMMA 5. *Let $E^{\mathscr{C}}$ be an conditional expectation for $[E^{\mathscr{B}}(A)]_\infty$, then $E^{\mathscr{C}} = E^{\mathscr{B}}$. Hence $\mathscr{B} = \{X, \phi\}$ if and only if $E^{\mathscr{B}}(A) = \{1\}$.*

Proof. For all f in A $E^{\mathscr{C}}(f) = E^{\mathscr{C}}(E^{\mathscr{B}}(f)) = E^{\mathscr{B}}(f)$. For $E^{\mathscr{B}}(f) \in [E^{\mathscr{B}}(A)]_\infty$. Since $A + \bar{A}$ is weak-*dense in L^∞, it follows that $E^{\mathscr{C}} = E^{\mathscr{B}}$.

Now we shall show the main lemma which is used later and is trivial for weak-*Dirichlet algebras, i.e., $E^{\mathscr{B}}(A) = \{1\}$. We do not use Jensen's inequality to show it.

LEMMA 6. *Suppose $1 \leqq p \leqq \infty$ and $v \in L^p$. If for all f and g in I, $\int_X v(f + \bar{g}) dm = 0$, then v lies in $E^{\mathscr{B}}(H^p) = [E^{\mathscr{B}}(A)]_p$.*

Proof. Since $A + \bar{A}$ is weak-*dense in L^∞ and so $E^{\mathscr{B}}(A) + I + \bar{I}$ is weak-*dense in L^∞, by Lemma 1, it follows that $[E^{\mathscr{B}}(A)]_p \oplus [I + \bar{I}]_p = L^p$. Let $E^{\mathscr{C}}$ be a conditional expectation for $[E^{\mathscr{B}}(A)]_\infty$ then $E^{\mathscr{C}} = E^{\mathscr{B}}$ by Lemma 5. Hence $E^{\mathscr{B}}(L^p) = E^{\mathscr{B}}(H^p)$ and so $\ker E^{\mathscr{B}} |_{L^p} = [I + \bar{I}]_p$.

If $v \in L^p$ annihilates $I + \bar{I}$, i.e., $\int_X v(g + \bar{f})dm = 0$ for all f and g in I, then

$$\int_X (v - E^{\mathscr{B}}(v))(g + \bar{f})dm = -\int_X E^{\mathscr{B}}(v)(g + \bar{f})dm$$
$$= -\int_X E^{\mathscr{B}}(v)E^{\mathscr{B}}(g + \bar{f})dm = 0\ ,$$

i.e., $v - E^{\mathscr{B}}(v)$ annihilates $I + \bar{I}$ too. Since $v - E^{\mathscr{B}}(v)$ lies in $[I + \bar{I}]_p$, it follows that $v = E^{\mathscr{B}}(v)$ a.e.. For if $k \in L^q$ with $1/p + 1/q = 1$, since $v - E^{\mathscr{B}}(v)$ annihilates $I + \bar{I}$ and it lies in $[I + \bar{I}]_p$,

$$\int_X k(v - E^{\mathscr{B}}(v))dm = \int_X E^{\mathscr{B}}(k)(v - E^{\mathscr{B}}(v))dm = 0\ .$$

Thus for any k in L^q, $\int_X k(v - E^{\mathscr{B}}(v))dm = 0$ and so $v = E^{\mathscr{B}}(v)$ a.e.

3. Invariant subspaces and Jensen's inequality. Let A be an extended weak-*Dirichlet algebra of L^∞ with respect to $E^{\mathscr{B}}$. For $1 \leqq p \leqq \infty$, a closed subspace M of L^p is called invariant if $f \in M$ and $g \in A$, then $fg \in M$.

DEFINITION 3. Let M be a closed invariant subspace of L^p for $1 \leqq p \leqq \infty$. (i) M is called type I if

$$\chi_E M \supsetneqq \chi_E[IM]_p$$

for every nonzero $\chi_E \in [E^{\mathscr{B}}(A)]_\infty$ so that $\chi_E M \neq \{0\}$. (ii) M is called type II if $M^\perp$ is type I where $M^\perp = \left\{f \in \chi_F L^s \colon \int_X fg dm = 0 \text{ for all } g \in M\right\}$ and F is a support set of M and $1/p + 1/s = 1$, and if M contains no nontrivial invariant subspace of type I. (iii) M is called type III if $M = [IM]_p$ and $M^\perp = [IM^\perp]_s$ where $1/p + 1/s = 1$.

If $\mathscr{B} = \{X, \phi\}$ or $E^{\mathscr{B}}(A) = \{1\}$, then an invariant subspace of type I is a simply invariant subspace [15], for then $[E^{\mathscr{B}}(A)]_\infty$ is the complex field.

PROPOSITION 5. *Suppose $1 \leqq p \leqq \infty$ and M is an invariant subspace of L^p. Then*

$$M = \chi_{E_1}M \oplus \chi_{E_2}M \oplus \chi_{E_3}M$$

where χ_{E_1}, χ_{E_2}, and χ_{E_3} belongs to $[E^{\mathscr{B}}(A)]_\infty$, $\chi_{E_1} + \chi_{E_2} + \chi_{E_3} = 1$. $\chi_{E_1}M$ is type I, *$\chi_{E_2}M$ is type* II *and $\chi_{E_3}M$ is type* III. *This decomposition is unique.*

The proof is parallel to [12, Theorem 1] and we omit it.

THEOREM 1. *Let M be an invariant subspace of L^2.*

(1) *M is type I if and only if*

$$M = \chi_E q H^2$$

where χ_E belongs to $[E^{\mathscr{B}}(A)]_\infty$ and q is unimodular. If $M = \chi_E q' H^2$ with another unimodular q', then $\chi_E q' = \chi_E F q$ where F is a unimodular function in $[E^{\mathscr{B}}(A)]_\infty$.

(2) *If M is type II, then*

$$M = \chi_E q I^2$$

where χ_E belongs to $[E^{\mathscr{B}}(A)]_\infty$ and q is unimodular.

The proof is almost parallel to [12, Theorem 2] if we use Lemma 6. The proof of the part of 'only if' is only nontrivial by that $\bar{I}^2 = L^2 \ominus H^2$. We shall give a sketch of the proof.

Let M be type I and let $R = M \ominus [IM]_2$. Observe that for any f in R,

$$\int_X g\,|f|^2\,dm = 0 \qquad (g \in I)\ .$$

Then by Lemma 6, it follows that $|f|^2$ lies in $E^{\mathscr{B}}(H^1)$. By Lemma 2 and Lemma 5, $E^{\mathscr{B}}(H^1) = L^1(X, \mathscr{B}, m)$. Hence $|f|$ lies in $E^{\mathscr{B}}(H^1)$ and $\chi_{E(f)} \in [E^{\mathscr{B}}(A)]_\infty$. Let E be the support set of R, then there exists f_0 in R with $E(f_0) = E$. Define

$$q(x) = \begin{cases} f_0(x)/|f_0(x)| & x \in E \\ 1 & x \notin E\ , \end{cases}$$

then $\chi_E q$ lies in M. By the assumption on M and that $H^2 \oplus \bar{I}^2 = L^2$, it follows that $M = \chi_E q H^2$.

COROLLARY 1. [15, Theorem 2.2.1]. *Suppose $\mathscr{B} = \{X, \phi\}$, M is a simply invariant subspace in L^2 if and only if $M = qH^2$, where q is unimodular and the q is unique up to multiplication by a constant of absolute value* 1.

In the proofs of Propositions 3 and 4, we used Jensen's inequality for A. We now prove it. Let $w \in L^1$, $w \geqq 0$ and ε is any positive number. Define $E^{\mathscr{B}}(\log w)$ by $\lim_{\varepsilon \to 0} E^{\mathscr{B}}\{\log (w + \varepsilon)\}$.

THEOREM 2. *Jensen's inequality is valid for H^∞.*

Proof. Let f be an invertible element in H^∞, then $\log |f| \in L^\infty$. Let $E^{\mathscr{C}}$ be an conditional expectation for $[E^{\mathscr{B}}(A)]_\infty$, then by Lemma 5 $E^{\mathscr{C}} = E^{\mathscr{B}}$. Since $L^\infty = E^{\mathscr{B}}(L^\infty) \oplus [I + \bar{I}\,]_\infty$, $E^{\mathscr{B}}(\log |f|) \in E^{\mathscr{B}}(L^\infty)$ and

$\log|f| - E^{\mathscr{B}}(\log|f|) \in [I + \bar{I}]_\infty$. Hence $\log|f| - E^{\mathscr{B}}(\log|f|)$ lies in the uniqueness subspace of $[B]_\infty = [\{1\} + I]_\infty$ by Lemma 4 and [5, p. 103]. By [5, p. 103], there exists f_2 in $[B]_\infty$ such that $\log|f| - E^{\mathscr{B}}(\log|f|) = \log|f_2|$. Set $f_1 = \exp E^{\mathscr{B}}(\log|f|)$, then $f_1 \in E^{\mathscr{B}}(H^\infty)$. Since both f_1 and f_2 are invertible in H^∞, $f_1 f_2$ is in H^∞ too and

$$\begin{aligned}\log|f| &= E^{\mathscr{B}}(\log|f|) + \log|f| - E^{\mathscr{B}}(\log|f|)\\ &= \log|f_1| + \log|f_2| = \log|f_1 f_2|\ .\end{aligned}$$

Hence $f = q f_1 f_2$ for some unimodular q in $E^\infty(H^\infty)$, $\log|f_1| = \log|qf_1|$ and $E^{\mathscr{B}}(f) = qf_1 E^{\mathscr{B}}(f_2)$. Since $E^{\mathscr{B}}(\log|f_2|) = 0$,

$$\int_X \log|f_2|\, dm = \log\left|\int_X f_2 dm\right| = 0$$

and so $f_2 = c + f_{2,0}$ for a constant c of absolute value 1 and for $f_{2,0} \in [I]_\infty$. Thus for any invertible f in H^∞,

$$\begin{aligned}E^{\mathscr{B}}(\log|f|) &= \log|f_1| = \log|cqf_1|\\ &= \log|E^{\mathscr{B}}(f)|\ .\end{aligned}$$

For all f in H^∞ and for any $\varepsilon > 0$, $E^{\mathscr{B}}\{\log(|f| + \varepsilon)\} \geqq \log|E^{\mathscr{B}}(f)|$. For $\log(|f| + \varepsilon) \in L^\infty$ and so there exists an invertible g in H^∞ with $\log(|f| + \varepsilon) = \log|g|$, using Theorem 1 as in the proof of [15, Lemma 2.4.3]. Now we can use the method of Hoffman [6, Theorem 4.1]. Let $h = fg^{-1}$, then $|h| = |f|/|g| = |f|/(|f| + \varepsilon) \leqq 1$. By Lemma 1, $|E^{\mathscr{B}}(h)| \leqq 1$ and so $|E^{\mathscr{B}}(f)|\,|E^{\mathscr{B}}(g)|^{-1} \leqq 1$,

$$\log|E^{\mathscr{B}}(f)| \leqq \log|E^{\mathscr{B}}(g)|\ .$$

Since g is invertible in H^∞, by the first half of this proof, $\log|E^{\mathscr{B}}(g)| = E^{\mathscr{B}}(\log|g|) = E^{\mathscr{B}}\{\log(|f| + \varepsilon)\}$. Thus

$$E^{\mathscr{B}}\{\log(|f| + \varepsilon)\} = \log|E^{\mathscr{B}}(g)| \geqq \log|E^{\mathscr{B}}(f)|\ .$$

COROLLARY 2. *For every f in A,*

(1) $\displaystyle\int_X \log|f|\, dm \geqq \int_X \log|E^{\mathscr{B}}(f)|\, dm$

(2) $\displaystyle\int_X \exp E^{\mathscr{B}}(\log|f|) dm \geqq \int_X |E^{\mathscr{B}}(f)|\, dm$

(1) *of this corollary is known* [1, *Corollary* 4.4.6]. *Our proof is different.*

COROLLARY 3. *For every f in H^1,*

$$E^{\mathscr{B}}(\log|f|) \geqq \log|E^{\mathscr{B}}(f)$$

and so

$$\int_X \log |f|\, dm \geqq \int_X \log |E^{\mathscr{B}}(f)|\, dm$$

$$\int_X \exp E^{\mathscr{B}}(\log |f|)\, dm \geqq \int_X |E^{\mathscr{B}}(f)|\, dm .$$

Proof. Using Fatou's lemma for the conditionol expectation (easily shown), as in the proof of [3, p. 122], we can show this corollary.

4. Szegö's theorem and factorization theorems. Let A be an extended weak-*Dirichlet algebra of L^∞ with respect to $E^{\mathscr{B}}$. In this section we shall show Szegö's theorem which is different from that in Arveson [1, p. 611].

DEFINITION 4. A function h in H^1 is called outer if $[hA]_1 = H^1$.

If h is outer, then $|h| > 0$ a.e. and $|E^{\mathscr{B}}(h)| > 0$ a.e.; in particular, $\chi_E h \notin [hI]_1$ for every nonzero χ_E in $[E^{\mathscr{B}}(A)]_\infty$. If h, h' are outer and $|h| = |h'|$, then $h = qh'$ for some unimodular q in $[E^{\mathscr{B}}(A)]_\infty$.

LEMMA 7. *If $f \in L^2$ and $\chi_E f \notin [fI]_2$ for every χ_E in $[E^{\mathscr{B}}(A)]_\infty$ with $\chi_E f \neq 0$, then $f = \chi_{E(f)} qh$ where h is outer and q is unimodular.*

Proof. Our assumption implies that $[fA]_2$ is an invariant subspace of type I, and hence by Theorem 1, $[fA]_2 = \chi_{E(f)} qH^2$ for some unimodular q. Now this lemma is clear.

As we noted in the proof of Theorem 2, H^∞ is a logmodular algebra on the maximal ideal space of L^∞ by Lemma 7. In general, m is not multiplicative on H^∞. However $E^{\mathscr{B}}$ is multiplicative on H^∞. Moreover if we use the method of Srinivasan and Wang [15, pp. 230-231], it is easy to show the following.

(a) $H^1 = \left\{ f \in L^1 : \int_X fg\,dm = 0 \text{ for all } g \text{ in } I \right\}$.

(b) $H^\infty = H^1 \cap L^\infty$.

If D is a subalgebra such that $D \supseteqq H^\infty$ and it is an extended weak-*Dirichlet algebra with respect to $E^{\mathscr{B}}$, then $D = H^\infty$. For $I^\infty \subseteqq \ker E^{\mathscr{B}}|_D$ and by Proposition 4 $[\ker E^{\mathscr{B}}|_D]_2 \subseteqq I^2$. So $[\ker E^{\mathscr{B}}|_D]_2 = I^2$ and $[D]_2 = H^2$ by Proposition 4. By (b), it follows that $D = H^\infty$.

THEOREM 3. *Let $w \in L^1$, $w \geqq 0$. Then*

$$\inf_{g \in I} \int_X |1 - g|^2 w\,dm = \int_X \exp E^{\mathscr{B}}(\log w)\,dm ,$$

where $E^{\mathscr{B}}(\log w)$ *is defined by* $\lim_{\varepsilon\to 0} E^{\mathscr{B}}\{\log(w+\varepsilon)\}$.

Proof. We shall use the method of Srinivasan and Wang [15, Theorem 2.5.5]. We can show the inequality of arithmetic and geometric means for conditional expectation. So if v is a real function in L^1 and $\exp v \in L^1$, then $\exp E^{\mathscr{B}}(v) \leqq E^{\mathscr{B}}(\exp v)$. Fix $w \in L^1$, $w \geqq 0$. Hence for any g in I and any $\varepsilon > 0$,

$$\begin{aligned}\int_X |1-g|^2 (w+\varepsilon)dm &\geqq \int_X \exp E^{\mathscr{B}}\{\log |1-g|^2 (w+\varepsilon)\}dm \\ &= \int_X \exp E^{\mathscr{B}}(\log|1-g|^2) \exp E^{\mathscr{B}}\{\log(w+\varepsilon)\}dm\,.\end{aligned}$$

By Corollary 3,

$$\int_X |1-g|^2 (w+\varepsilon)dm \geqq \int_X \exp E^{\mathscr{B}}\{\log(w+\varepsilon)\}dm\,.$$

As $\varepsilon \to 0$

$$\begin{aligned}\int_X |1-g|^2\, w dm &\geqq \int_X \exp \lim_{\varepsilon\to 0} E^{\mathscr{B}}\{\log(w+\varepsilon)\}dm \\ &= \int_X \exp E^{\mathscr{B}}(\log w)dm\end{aligned}$$

for all g in I, which is one half of theorem.

Fix any $\varepsilon > 0$.

$$\inf_{g\in I} \int_E |1-g|^2 (w+\varepsilon)dm > 0$$

for all nonzero χ_E in $[E^{\mathscr{B}}(A)]_\infty$. For by the first half of theorem,

$$\begin{aligned}&\inf_{g\in I}\int_X |1-g|^2 \chi_E(w+\varepsilon)dm \\ &\qquad \geqq \int_X \exp E^{\mathscr{B}}\{\log \chi_E(w+\varepsilon)\}dm \gneqq 0\,.\end{aligned}$$

For let $E^{\mathscr{B}_1}$ be a conditional expectation for $\chi_E[E^{\mathscr{B}}(A)]_\infty$ and let $E^{\mathscr{B}_2}$ be a conditional expectation for $(1-\chi_E)[E^{\mathscr{B}}(A)]_\infty$. Then

$$\begin{aligned}&E^{\mathscr{B}}\{\log \chi_E(w+\varepsilon)\} \\ &\quad = \lim_{0<\delta\to 0} E^{\mathscr{B}}[\log\{\chi_E(w+\varepsilon)+\delta\}] \\ &\quad = \lim_{\delta\to 0} (\chi_E E^{\mathscr{B}}[\log\{\chi_E(w+\varepsilon)+\delta\}] + (1-\chi_E)E^{\mathscr{B}}[\log\{\chi_E(w+\varepsilon)+\delta\}]) \\ &\quad = \lim_{\delta\to 0} [E^{\mathscr{B}_1}\{\log(w+\varepsilon)+\delta\} + E^{\mathscr{B}_2}(\log\delta)] \\ &\quad = \chi_E E^{\mathscr{B}_2}\{\log(w+\varepsilon)\} + \lim_{\delta\to 0}(1-\chi_E)\log\delta \not\equiv -\infty.\end{aligned}$$

So $\chi_E(w+\varepsilon)^{1/2} \notin [(w+\varepsilon)^{1/2} I]_2$ for all nonzero χ_E in $[E^{\mathscr{B}}(A)]_\infty$ and hence by Lemma 6, there exists an outer function h_ε in H^2 with $|h_\varepsilon|^2 = w + \varepsilon$. Hence if $w \in L^1$, by Corollary 3,

$$\begin{aligned}
&\inf_{g \in I} \int_X |1 - g|^2 w dm \\
&\quad \leqq \inf_{g \in I} \int_X |1 - g|^2 (w + \varepsilon) dm \\
&\quad = \inf_{g \in I} \int_X |1 - g|^2 (w + \varepsilon) dm = \int_X |E^{\mathscr{B}}(h_\varepsilon)|^2 dm \\
&\quad \leqq \int_X \exp E^{\mathscr{B}}(\log |h_\varepsilon|^2) dm = \int_X \exp E^{\mathscr{B}}\{\log (w + \varepsilon)\} dm .
\end{aligned}$$

This completes the proof as $\varepsilon \to 0$.

REMARK. We shall state Szegö's theorem in Arveson [1, pp. 611-615]. Let $w \in L^1$, $w \geqq 0$. Then

$$\begin{aligned}
&\inf \left\{\int_X |u - g|^2 w dm; g \in I, u \in E^{\mathscr{B}}(A) \text{ and } \int_X \log |u| dm \geqq 0\right\} \\
&\quad = \exp \int_X \log w dm .
\end{aligned}$$

COROLLARY 4. [15, Theorem 2.5.5.] *Suppose* $\mathscr{B} = \{X, \phi\}$. *Let* $w \in L^1$, $w \geqq 0$. *Then*

$$\inf_{g \in I} \int_X |1 - g|^2 w dm = \exp \int_X \log w dm .$$

Proof. Since $[E^{\mathscr{B}}(A)]_\infty$ is the complex field, $\int_X \exp E^{\mathscr{B}}(\log w) dm = \exp \int_X \log w dm$ and so Theorem 3 implies this corollary. This corollary can be shown by Szegö's theorem in Arveson, too.

COROLLARY 5. *$h \in H^1$ is outer if and only if $|E^{\mathscr{B}}(h)| > 0$ and*

$$\int_X \exp E^{\mathscr{B}}(\log |h|) dm = \int_X |E^{\mathscr{B}}(h)| dm .$$

In particular, if $\mathscr{B} = \{X, \phi\}$, then $h \in H^1$ is outer if and only if

$$\exp \int_X \log |h| dm = \left|\int_X h dm\right| > 0 .$$

Proof. If $h \in H^1$ is outer, then there exists h_1 in H^2, which is outer, such that $h = h_1^2$. Then by Theorem 3,

$$\int_X |E^{\mathscr{B}}(h)|\,dm = \int_X |E^{\mathscr{B}}(h_1)|^2\,dm = \inf_{g\in I}\int_X |1-g|^2\,|h_1|^2\,dm$$
$$= \int_X \exp E^{\mathscr{B}}(\log|h_1|^2)dm = \int_X \exp E^{\mathscr{B}}(\log|h|)dm\ .$$

To prove the 'if' part, if $|E^{\mathscr{B}}(h)|>0$, a.e. then $h = qh_1^2$ by Lemma 7 for $h_1\in H^2$ is outer and $q\in H^\infty$ is uni- modular. Then our condition gives

$$\int_X \exp E^{\mathscr{B}}(\log|h|)dm = \int_X |E^{\mathscr{B}}(q)|\,|E^{\mathscr{B}}(h_1^2)|\,dm$$
$$\leqq \int_X |E^{\mathscr{B}}(h_1^2)|\,dm = \int_X \exp E^{\mathscr{B}}(\log|h_1^2|)dm\ .$$

Thus $|E^{\mathscr{B}}(q)| = E^{\mathscr{B}}(q)$ a.e.. Since $|q|=1$ a.e.,

$$E^{\mathscr{B}}(|q-E^{\mathscr{B}}(q)|^2)=0\ ,$$

and hence $q = E^{\mathscr{B}}(q)$. This shows that h is outer.

If $f\in H^\infty$, by (2) in Corollary 2

$$\int_X \exp E^{\mathscr{B}}(\log|f|)dm \geqq \exp\int_X \log|f|\,dm$$
$$\geqq \exp\int_X \log|E^{\mathscr{B}}(f)|\,dm$$

and

$$\int_X \exp E^{\mathscr{B}}(\log|f|)dm \geqq \int_X |E^{\mathscr{B}}(f)|\,dm$$
$$\geqq \exp\int_X \log|E^{\mathscr{B}}(f)|dm\ .$$

If f is invertible in H^∞, then

$$\int_X \exp E^{\mathscr{B}}(\log|f|)dm = \int_X |E^{\mathscr{B}}(f)|\,dm$$
$$\geqq \exp\int_X \log|f|\,dm = \exp\int_X \log|E^{\mathscr{B}}(f)|\,dm\ .$$

Moreover if $|E^{\mathscr{B}}(f)| =$ constant a.e., then

$$\int_X \exp E^{\mathscr{B}}(\log|f|)dm = \int_X |E^{\mathscr{B}}(f)|\,dm = \exp\int_X \log|f|\,dm$$
$$= \exp\int_X \log|E^{\mathscr{B}}(f)|\,dm\ .$$

In general,

$$\int_X \exp E^{\mathscr{B}}(\log|f|)dm \gneqq \exp\int_X \log|f|\,dm$$

and

$$\int_X |E^{\mathscr{B}}(f)|\,dm \gneqq \exp \int_X \log |E^{\mathscr{B}}(f)|\,dm\ .$$

THEOREM 4.

(1) *Every f in H^1 with* $\int_E \exp E^{\mathscr{B}}(\log |f|)dm > 0$, *for any* $\chi_E \in [E^{\mathscr{B}}(A)]_\infty$ *so that* $\chi_E f \neq 0$, *is a product of two* H^2 *functions.*

(2) *A function f in H^1 is a product $\chi_{E(f)}qF$ of an inner function q* (i.e., $q \in H^\infty$ *with* $|q| = 1$ a.e.) *and an outer function F if and only if* $\int_E \exp E^{\mathscr{B}}(\log |f|)dm > 0$ *for any* $\chi_E \in [E^{\mathscr{B}}(A)]_\infty$ *so that* $\chi_E f \neq 0$.

(3) *A nonnegative function w in L^1 is of the form $\chi_{E(w)}|h|$ for some outer h in H^1 if and only if* $\int_E \exp E^{\mathscr{B}}(\log w)dm > 0$ *for any* $\chi_E \in [E^{\mathscr{B}}(A)]_\infty$ *so that* $\chi_E f \neq 0$.

Proof. (1) By Theorem 3, for every nonzero $\chi_E \in [E^{\mathscr{B}}(A)]_\infty$ so that $\chi_E f \neq 0$,

$$\begin{aligned}\inf_{g \in I} \int_X |1 - g|^2 \chi_E |f|\,dm &= \int_X \exp E^{\mathscr{B}}(\log \chi_E |f|)dm \\ &= \int_E \exp E^{\mathscr{B}}(\log |f|)dm > 0\ .\end{aligned}$$

Hence if $M_w = [wA]_2$ and $w = \sqrt{|f|}$, then M_w is an invariant subspace of type I. By Theorem 1, $M_w = \chi_{E(w)}qH^2$ and so $|f| = w^2 = \chi_{E(f)}q^2h^2$ where $|q| = 1$ a.e. and $h \in H^2$. This implies (1). (2) and (3) follows as in the proof of [15, Theorem 2.5.9] and (1).

We can write Theorem 4 in another form.

THEOREM 4′.

(1) *Every f in H^1 with* $\chi_{E(f)}E^{\mathscr{B}}(\log |f|) > -\infty$ a.e. *on* $E(f)$, *is a product of two* H^2 *functions.*

(2) *A function f in H^1 is a product $\chi_{E(f)}qf$ of an inner function q and an outer function F if and only if* $\chi_{E(f)}E^{\mathscr{B}}(\log |f|) > -\infty$ a.e. *on* $E(f)$.

(3) *A nonnegative function w in L^1 is of the form $|h|$ for some outer h in H^1 if and only if* $\chi_{E(w)}E^{\mathscr{B}}(\log w) > -\infty$ a.e. *on* $E(w)$.

If $\mathscr{B} = \{X, \phi\}$, then Theorems 4 and 4′ implies [15, Theorem 2.5.9].

5. Some theorems concerning L^p. We wish to extend some of our theorems in §§3, 4 from L^2 to L^p to general p, i.e., Theorems 1, 3, and 4. However if we use the method of Srinivasan and Wang [15, pp. 242–247], they follow easily. So we omit the proofs. But we shall give two important invariant subspace theorems, known when $\mathscr{B} = \{X, \phi\}$ [12, Lemma 1].

THEOREM 5. *Suppose* $1 \leqq p < q \leqq \infty$. *There is a one-to-one correspondence between invariant subspaces* M_p *of* L^p *and (weak-*closed for* $q = \infty$*) invariant subspaces* M_q *of* L^q, *such that* $M_q = M_p \cap L^q$, *and* M_p *is the closure in* L^p *of* M_q.

Proof. If $w \in L^1$, $w \geqq 0$ and $\log w \in L^1$, then $w = |g|^2$ with outer g in H^2. For then $E^{\mathscr{B}}(\log w) > -\infty$ a.e. and so we can apply Theorem 4′. We shall show that $M_p \cap L^\infty$ is dense in M_p. Let f be in M_p. We shall use the well known method [6, p. 12]. For each n let $k_n = \min(1, n|f|^{-1})$, then $0 \leqq k_n \leqq 1$, $k_n \leqq k_{n+1} \leqq \cdots \to 1$ a.e., and $\log k_n \in L^1$. For each k_n, there exists an outer g_n in H^∞ with $k_n = |g_n|$. Moreover we can assume that $E^{\mathscr{B}}(g_n) > 0$ a.e.. For $|E^{\mathscr{B}}(g_n)| > 0$ a.e., let $q_n = \operatorname{sgn} E^{\mathscr{B}}(g_n)$, then $E^{\mathscr{B}}(\bar{q}_n g_n) = \bar{q}_n E^{\mathscr{B}}(g_n) > 0$ a.e.. Again $\bar{q}_n g_n$ is outer with $k_n = \bar{q}_n g_n$. Write $\bar{q}_n g_n$ as g_n again. We shall show that g_n tends to the constant function in norm, and on a subsequence almost everywhere. Fix n, then for any $\varepsilon > 0$, there exists a h in I such that

$$\int_X E^{\mathscr{B}}(g_n) dm + \varepsilon = \inf_{g \in I} \int_X |1 - g|^2 |g_n| \, dm + \varepsilon > \int_X |1 - h|^2 |g_n| \, dm$$
$$\geqq \exp \int_X \log |1 - h|^2 \, dm \times \exp \int_X \log |g_n| \, dm \, .$$

By Theorem 2 and as $\varepsilon \to 0$, for each n,

$$\int_X E^{\mathscr{B}}(g_n) dm \geqq \exp \int_X \log |g_n| \, dm \, .$$

By Fatou's lemma, it follows that $\exp \int_X \log |g_n| \, dm \to 1$ and hence $\int_X g_n dm = \int_X E^{\mathscr{B}}(g_n) dm \to 1$. Therefore

$$\int_X |g_n - 1|^2 \, dm = \int_X |g_n|^2 \, dm + 1 - 2 \operatorname{Re} \int_X g_n dm$$
$$\leqq 2 - 2 \int g_n dm \longrightarrow 0 \, .$$

There exists a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \to 1$ a.e.. Since $g_{n_k} f \in M_p \cap L^\infty$, f is a limit of bounded functions in M_p. Since $M_p \cap L^\infty$ is dense in M_p, it is clear that $M_p \cap L^q$ is dense in M_p. By the first half of theorem, as in the proof of [6, p. 12], we can show that $[M_q]_p \cap L^q = M_q$.

PROPOSITION 6. *If* M *is an invariant subspace of* $L^p(m)$ $(1 \leqq p$

$\leqq \infty$), then $\chi_{E(M)}q \in M$ for some unimodular q and the support set $E(M)$ of M. Moreover

$$|M| = \chi_{E_0}\cdot\chi_{E(M)}\,|H^p(m)| + (1-\chi_{E_0})\chi_{E(M)}\,|L^p(m)|\ ,$$

where $\chi_{E_0}M$ is the largest subspace that contains no nontrivial reducing subspace of L^∞ and $\chi_{E_0}M \subseteqq M$ and $|M| = \{|f|; f \in M\}$.

Proof. By Theorem 4, if u is a real-valued function in L^∞, then there is $h \in H^\infty$ such that $e^u = |h|$ and $h^{-1} \in H^\infty$. Hence by [14, Theorem] and Theorem 5, the former half of this proposition follows. The latter half can be shown as in the proof of [14, Corollary 5].

6. **Weak-*Dirichlet algebras.** Let A be a weak-*Dirichlet algebra of L^∞, i.e., it is an extended weak-*Dirichlet algebra with respect to $E^{\mathscr{B}}$ which is a conditional expectation for $\mathscr{B}$ with $\mathscr{B} = \{X, \phi\}$. Then m is multiplicative on A. Suppose B^∞ is any weak-*closed subalgebra of L^∞ which contains A. The measure m was called in [12] quasi-multiplicative on B^∞ if $\int_X f^2 dm = 0$ for every f in B^∞ such that $\int_E f dm = 0$ for all χ_E in B^∞. It is a consequence of the definition of a weak-*Dirichlet algebra that if f is in H^∞ and $\int_E f dm = 0$ for all χ_E in H^∞, then $\int_X f^2 dm = 0$. Let

$$B_0^\infty = \left\{f \in B^\infty : \int_X f dm = 0\right\}$$

and let I_B^∞ be a maximum weak-*closed ideal of B^∞ in B_0^∞ [12, Lemma 2]. I_B^∞ is given by $\left\{f \in B^\infty : \int_X fg dm = 0 \text{ for all } g \text{ in } B^\infty\right\}$. Let $\mathscr{L}_B^\infty$ be a self-adjoint part of B^∞. Suppose $E^{\mathscr{B}}$ is a conditional expectation for $\mathscr{L}_B^\infty$.

PROPOSITION 7. *Suppose B^∞ is any weak-*closed subalgebra of L^∞ which contains A. Then the following are equivalent.*

(1) *m is quasi-multiplicative on B^∞.*

(2) $[B^\infty \cap \bar{B}^\infty]_2 = [B^\infty]_2 \cap [\bar{B}^\infty]_2$.

(3) *$E^{\mathscr{B}}$ is multiplicative on B^∞.*

(4) *B^∞ is an extended weak-*Dirichlet algebra with respect to $E^{\mathscr{B}}$.*

Proof. (1) $\Leftrightarrow$ (2) is known in [12, Theorem 4]. Since $B^\infty + \bar{B}^\infty$ is weak-*dense in L^∞, (3) $\Leftrightarrow$ (4) is clear.

(2) $\Leftrightarrow$ (3). Let $K = L^2 \ominus [B^\infty]_2$, then $[I_B^\infty]_2 = \bar{K}$ by [12, Lemma 2] and so $K \subset [B^\infty]_2$. Proposition 2 implies this equivalence.

By Proposition 7, [12, Theorem 2] is a corollary of Theorem 1. For w in L^1 with $w \geqq 0$, $\log w \in L^1$ if and only if $w = |g|$ for some outer function g in H^1 [15, Theorem 2.5.9]. Since g is outer, $\exp \int_X \log |g| \, dm = \left| \int_X g dm \right| > 0$. We want to know when $\log w \notin L^1$. Suppose B^∞ is any weak-*closed subalgebra of L^∞ which contains H^∞ properly and on which $E^{\mathscr{B}}$ is multiplicative. Even if $\log w \notin L^1$, it can happen that $E^{\mathscr{B}}(\log w) > -\infty$ a.e.. Then by Theorem 4′, $w = |g|$ for some g in $[B]_1$ with $[gI_B^\infty]_1 = [I_B^\infty]_1 \subset H^1$. If $g \in H^1$,

$$0 = \left| \int_X g dm \right| = \exp \int_X \log |g| \, dm = \exp \int_X E^{\mathscr{B}}(\log |g|) dm$$
$$\lneqq \int_X \exp E^{\mathscr{B}}(\log |g|) dm = \int_X |E^{\mathscr{B}}(g)| \, dm ,$$

and $[gA]_1 \subsetneqq H^1$. In general, $[gA]_1 = q[B^\infty]_1$ for some unimodular q in H^∞ or $H^\infty = \{h \in L^\infty : h[gA]_1 \subseteqq [gA]_1\}$ and $[gA]_1$ is type III for H^∞.

Set $A_0 = \left\{ f \in A : \int f dm = 0 \right\}$, then Szegö's theorem implies

$$\begin{aligned} \inf_{g \in A_0} \int_X |1 - g|^2 w dm &= \inf_{g \in H_0^\infty} \int_X |1 - g|^2 w dm \\ &= \exp \int_X \log w dm . \end{aligned} \tag{1}$$

When $B^\infty \supsetneqq H^\infty$ and $E^{\mathscr{B}}$ is multiplicative on B^∞, $H_0^\infty \supsetneqq I_B^\infty$. By Theorem 3

$$\inf_{g \in I_B^\infty} \int_X |1 - g|^2 w dm = \int_X \exp E^{\mathscr{B}}(\log w) dm . \tag{2}$$

If $f \in \mathscr{L}_B^\infty \cap H^\infty$ and $g \in I_B^\infty$, then by Theorem 2,

$$\int_X \log |f + g| \, dm \geqq \int_X \log |f| \, dm \geqq \log \left| \int_X f dm \right| .$$

Now we shall show other versions of Szegö's theorem.

$$\inf_{u \in H_0^\infty \cap \mathscr{L}_B^\infty} \int_X |1 - u|^2 w dm = \exp \int_X \log |E^{\mathscr{B}}(w)| \, dm . \tag{3}$$

For since $H^\infty = H^\infty \cap \mathscr{L}_B^\infty + I_B^\infty$ [12], it follows that $H^\infty \cap \mathscr{L}_B^\infty$ is a weak-*Dirichlet algebra of $\mathscr{L}_B^\infty$. Thus

$$\begin{aligned} &\inf_{u \in H_0^\infty \cap \mathscr{L}_B^\infty} \int_X |1 - u|^2 w dm \\ &\quad = \inf \int_X E^{\mathscr{B}}(|1 - u|^2 w) dm = \inf \int_X |1 - u|^2 E^{\mathscr{B}}(w) dm \\ &\quad = \exp \int_X \log E^{\mathscr{B}}(w) dm . \end{aligned}$$

Fix $v \in \mathscr{L}_B^\infty$ with v^{-1} in $\mathscr{L}_B^\infty$.

$$(2)' \qquad \inf_{g \in I_B^\infty} \int_X |v - g|^2 w dm = \int_X \exp E^{\mathscr{B}}(\log w) |v|^2 dm .$$

For the $L^2(|v|^2 wdm)$-closure of $v^{-1}I_B^\infty$ contains I_B^∞ and so by (2)

$$\begin{aligned}
&\inf_{g \in I_B^\infty} \int_X |1 - v^{-1}g|^2 |v|^2 wdm \\
&\quad = \inf_{g \in I_B^\infty} \int_X |1 - g|^2 |v|^2 wdm = \int_X \exp E^{\mathscr{B}}(\log |v|^2 w) dm \\
&\quad = \int_X \exp E^{\mathscr{B}}(\log w) |v|^2 dm .
\end{aligned}$$

The following is Szegö's theorem by Arveson [1, pp. 611-615]. We shall give another proof to connect (4) with (2) and (2)$'$.

$$(4) \qquad \begin{aligned}
&\inf \left\{ \int_X |v - g|^2 wdm;\ g \in I_B^\infty,\ v \in \mathscr{L}_B^\infty \text{ and } \int_X \log |v| dm \geqq 0 \right\} \\
&= \inf \left\{ \int_X \exp E^{\mathscr{B}}(\log w) |v|^2 dm;\ v \in \mathscr{L}_B^\infty \text{ and } \int_X \log |v| dm \geqq 0 \right\} \\
&= \exp \int_X \log wdm .
\end{aligned}$$

For

$$\exp \int_X \log wdm = \inf \left\{ \int_X e^u wdm;\ u \in L_R^\infty \text{ and } \int udm = 0 \right\} .$$

By Lemma 7 and Theorem 2, there exists f in $(H^\infty)^{-1}$ such that $E^{\mathscr{B}}(\log |f|) = p^{-1}E^{\mathscr{B}}(u) = \log |E^{\mathscr{B}}(f)|$ and so $\int_X \log |E^{\mathscr{B}}(f)| dm = 0$. So

$$\begin{aligned}
&\exp \int_X \log wdm \\
&= \inf \left\{ \int_X |f|^2 wdm;\ f \in (H^\infty)^{-1} \text{ and } \int_X \log |E^{\mathscr{B}}(f)| dm = 0 \right\} \\
&\geqq \inf \left\{ \int_X |v - g|^2 wdm;\ g \in I_B^\infty,\ v \in \mathscr{L}_B^\infty \text{ and } \int \log |v| dm \geqq 0 \right\} \\
&= \inf \left\{ \int_X \exp E^{\mathscr{B}}(\log |v - g|^2) \exp E^{\mathscr{B}}(\log w) dm;\ g \in I_B^\infty, \right. \\
&\qquad \left. v \in \mathscr{L}_B^\infty \text{ and } \int \log |v| dm \geqq 0 \right\}
\end{aligned}$$

$$\geqq \inf \left\{ \int_X \exp E^{\mathscr{B}}(\log w) \cdot |v|^2 dm; \; v \in \mathscr{L}_B^\infty \text{ and } \int_X \log |v| dm \geqq 0 \right\}$$

$$\geqq \inf \left\{ \exp \int_X \log w dm \exp \int_X \log |v|^2 dm; \; v \in \mathscr{L}_B^\infty \text{ and } \int_X \log |v| dm \geqq 0 \right\}$$

$$\geqq \exp \int_X \log w dm \; .$$

7. Applications.

(I) Let G be a compact abelian group dual to a discrete group Γ. The Haar measure m on G is finite, and normalized so that $m(G) = 1$. Suppose a semigroup P is given in Γ such that $\Gamma = P \cup (-P)$, i.e., P orderes Γ. Let A be the set of all trigonometric polynomials f on G the form $f = \Sigma a_\lambda \chi_\lambda$ $(\lambda \in P)$. Let $\mathscr{L}^\infty$ be the weak-*closed linear span of $\Sigma a_\lambda \chi_\lambda$ $(\lambda \in P \cap (-P))$ and let $E^{\mathscr{B}}$ be the conditional expectation for $\mathscr{L}^\infty$. Then A is an extended weak-*Dirichlet algebra with respect to $E^{\mathscr{B}}$.

In particular, when $P \cap (-P) = \{0\}$, it is called that P orders Γ totally. Then A is a weak-*Dirichlet algebra. Let P_α be a semigroup of Γ which contains P properly. Let H_α be the weak-*closed linear span of all trigonometric polynomials f on G of the form $f = \Sigma a_\lambda \chi_\lambda$ $(\lambda \in P_\alpha)$. Define $\mathscr{L}^\infty = \mathscr{L}_\alpha^\infty$ and $E^{\mathscr{B}} = E^{\mathscr{B}(\alpha)}$ as the above. Then H_α is not a weak-*Dirichlet algebras, but it is an extended one with respect to $E^{\mathscr{B}}$. Let I_α be the weak-*closed linear span of all trigonometric polynomials f on G of the form $f = \Sigma a_\lambda \chi_\lambda$ $(\lambda \notin -P_\alpha)$. Then $I_\alpha = \ker E^{\mathscr{B}} \mid H_\alpha = I_{H_\alpha}^\infty$.

(II) Let $(X, \mathscr{A}, m)$ be a probability measure space and $\{T_t; t \in R\}$ be a flow. Suppose m is invariant under T_t. The action of R on X induces a weak-*continuous, one-parameter group $\{T_t\}_{t \in R}$ of automorphism of $L^\infty = L^\infty(m)$. They are defined by

$$\int_X T_t f(x) g(x) dm(x) = \int_X f(T_{-t}x) g(x) dm(x)$$

for f in L^∞ and g in L^1. For each element f in L^∞ and a function ϕ in $L^1(R)$, we define the convolution $f * \phi$ in M by

$$f * \phi = \int_{-\infty}^{\infty} \phi(t) T_t f dm \; .$$

The above integral exists in the sense that

$$\int_X f*\phi g dm = \langle f*\phi, g\rangle = \int_{-\infty}^{\infty}\phi(t)\left(\int T_t f g dm\right)dt$$
$$= \int_{-\infty}^{\infty}\phi(t)\langle T_t f, g\rangle dt$$

for g in L^1 [2, Proposition 1.6]. Define the ideals of $L^1(R)$ $J(f)$ by

$$J(f) = \{\phi \in L^1(R): f*\phi = 0\}\ .$$

The hull of the ideal $J(f)$ is said to be the spectrum of f and is denoted by $\mathrm{sp} f$. A is defined to be the set of all f in L^∞ with $\mathrm{sp} f \subseteqq [0, \infty)$.

Let $d\nu = dt/\pi(1+t^2)$ and $L^\infty(R \times X) = L^\infty(\nu \times m)$, where $\nu \times m$ is a completion of the product measure of ν and m. Set $F(t, x) = T_t f(x)$ for f in L^∞, then $F(t, x) \in L^\infty(R \times X)$. Set $q = (1-it)(1+it)^{-1}$, then $q \in H^\infty(R)$ and there exists $\sum_{-N}^{+N} f_n^N q^n$ such that

$$\iint\left|F(t, x) - \sum_{-N}^{N} f_n^{(N)} q^n\right|^2 d\nu dm \longrightarrow 0\ ,$$

where $f_n^N \in L^\infty(m)$ and $H^\infty(R)$ is the class of all functions ϕ in $L^\infty(R)$ such that $\mathrm{sp}\,\phi \subseteqq [0, \infty)$. If $\mathrm{sp} f \subseteqq [0, \infty)$, then it is easy to show that $\int_X T_t f g dm \in H^\infty(R)$ for every g in L^1 and hence it follows that

$$\iint\left|F(t, x) - \sum_{0}^{N} f_n^{(N)} q^n\right|^2 d\nu dm \longrightarrow 0\ .$$

Thus $T_t f(x) = F(t, x) \in H^\infty(R)$ a.e. $x(m)$. If $T_t f(x) = F(t, x) \in H^\infty(R)$ a.e. $x(m)$, then it is clear that $\int_X T_t f g dm \in H^\infty(R)$ for every g in L^1 and hence $\mathrm{sp} f \subseteqq [0, \infty)$. This implies that A is a weak-*closed subalgebra of L^∞ which contains the constants. Let $\mathscr{L}^p = \{f \in L^p: T_t f = f\}$ for $1 \leqq p \leqq \infty$ and $E^{\mathscr{D}}$ be a conditional expectation for $\mathscr{L}^\infty$.

THEOREM 6. [11] [8]. *A is an extended weak-*Dirichlet algebra with respect to $E^{\mathscr{D}}$. If the flow is ergodic, then A is a weak-*Dirichlet algebra.*

We shall give the proof in which spectral condition (cf. [2] [8]) is not used but Proposition 2 is used.

LEMMA 8 [11]. *Suppose $1 \leqq p \leqq \infty$. Then*

$$\{f \in L^p: \mathrm{sp} f \subseteqq \{0\}\} = \{f \in L^p: T_t f = f \text{ a.e.}\}\ .$$

Proof. If $T_t f = f$, since $\langle f*\phi, g\rangle = \langle f, g\rangle\hat{\phi}(0)$ for every g in L^q, then $\mathrm{sp} f \subseteqq \{0\}$. If $\mathrm{sp} f \subseteqq \{0\}$, set $F(t) = \int_{-\infty}^{\infty} T_t f g dm$. Then we can show as in the proof of [4, p. 50] that $\operatorname{sp} F \subseteqq -\operatorname{sp} f$. Hence F is a constant a.e. on R and $T_t f = f$ a.e..

LEMMA 9 [4, Proposition 2]. *Suppose* $1 \leqq p \leqq \infty$. *Then if* $f \in L^p$

$$\int_X fg\,dm = 0 \qquad \textit{for all } g \textit{ in } A,$$

then $\operatorname{sp} f \subseteqq [0, \infty)$.

Proof. For any h in L^∞ and any ϕ in $L^1(R)$, $\langle f * \phi, h\rangle = \langle f, h * \tilde{\phi}\rangle$ where $\tilde{\phi}(t) = \phi(-t)$. Hence if $\hat{\phi}(s) = 1$ for $s < 0$ with $\operatorname{supp}\hat{\phi} \subseteqq (-\infty, 0)$, it follows that $f * \phi = 0$. This implies $\operatorname{sp} f \subseteqq [0, \infty)$.

The proof of Theorem 6. If $f \in L^1$, $\int_X f(k + \bar{h})dm = 0$ for all h, k in A, then $\operatorname{sp} f \subseteqq \{0\}$ by Lemma 9. By Lemma 8, $T_t f = f \in \mathscr{L}^1$ and f annihilates $A \cap \bar{A} = \mathscr{L}^\infty$. Since $\mathscr{L}^\infty$ is dense in $\mathscr{L}^1$, $f = 0$ a.e.. Thus $A + \bar{A}$ is weak-*dense in L^∞. In order to prove that $E^{\mathscr{B}}$ is multiplicative, by Proposition 2, it is sufficient to show that $K = L^2 \ominus H^2 \subset \bar{H}^2$ and $[A \cap \bar{A}]_2 = H^2 \cap \bar{H}^2$. Set $\mathscr{H}^2 = \{f \in L^2 \colon \operatorname{sp} f \subseteqq [0, \infty)\}$, then $\mathscr{H}^2 \supseteqq H^2$. Since $\mathscr{H}^2 \cap \bar{\mathscr{H}}^2 = \mathscr{L}^2$ and $A \cap \bar{A} = \mathscr{L}^\infty$, it is clear that $[A \cap \bar{A}]_2 = H^2 \cap \bar{H}^2$. By Lemma 9, $K \subset \bar{\mathscr{H}}^2$. So if $H^2 = \mathscr{H}^2$, the proof is complete. If $f \in \mathscr{H}^2 \ominus H^2$, then $\operatorname{sp} f \subseteqq \{0\}$ and hence $f \in \mathscr{L}^2$. While $\mathscr{L}^2 \subset H^2$, this implies $f = 0$ a.e..

(III) Let $C(X_1)$ be the set of all continuous complex-valued functions on a compact Hausdorff space X_1 and let A_2 be a function algebra on a compact Hausdorff space X_2. Moreover let A_2 be a Dirichlet algebra of $C(X_2)$, i.e., $A_2 + \bar{A}_2$ is uniformly dense in $C(X_2)$. Suppose A is the set of all functions of the form; for $u, v \in C(X_1)$ and $f \in A_2$, $u + vf$. Then A is an subalgebra of $C(X_1 \times X_2)$.

Let m_1 be any probability measure on X_1 and m_2 be a nontrivial representing measure of any complex homomorphism of A_2. Let $\mathscr{A}$ be the σ-algebra of all Borel sets of $X_1 \times X_2$ and m be the completion of $m_1 \times m_2$. Let $\mathscr{B}$ be the σ-subalgebra of $\mathscr{A}$ consisting of all Borel sets of the form $E_1 \times X_2$ where E_1 is a Borel set of X_1. Let $E^{\mathscr{B}}$ denote the conditional expectation for $\mathscr{B}$. Then A is an extended weak-*Dirichlet algebra of $L^\infty(m)$ with respect to $E^{\mathscr{B}}$. For it is clear that (i) the constant functions lie in A; (ii) $A + \bar{A}$ is weak-*dense in L^∞; (iv) $E^{\mathscr{B}}(A) \subseteqq A \cap \bar{A}$. For $u, u', v, v' \in C(X_1)$ and $g, g' \in A_2$,

$$\begin{aligned} &E^{\mathscr{B}}(\{u + vg\}\{u' + v'g'\}) \\ &\quad = uu' + u'v\int_{X_2} g\,dm_2 + uv'\int_{X_2} g'dm_2 + vv'\int_{X_2} g\,dm_2 \times \int_{X_2} g'dm_2 \\ &\quad = E^{\mathscr{B}}(u + vg)E^{\mathscr{B}}(u' + v'g)\,. \end{aligned}$$

This implies that (iii) for all f and g in A, $E^{\mathscr{B}}(fg) = E^{\mathscr{B}}(f)E^{\mathscr{B}}(g)$. Then $I = \{f \in A: E^{\mathscr{B}}(f) = 0\} = \left\{u + vg: \int_{X_2} gdm_2 = 0 \text{ and } v \in C(X_1),\ g \in A_2\right\}$.

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