PACIFIC JOURNAL OF MATHEMATICS
Vol. 84, No. 2, 1979

RIESZ-PRESENTATION OF ADDITIVE AND σ-ADDITIVE SET-VALUED MEASURES

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In this paper we generalize the well known Riesz's representation theorems for additive and σ-additive scalar measures to the case of additive and σ-additive set-valued measures.

1. **Introduction.** Consider a nonvoid set Ω and an algebra $\mathscr{A}$ over Ω. *An additive set-valued measure* Φ on the field $(\Omega, \mathscr{A})$ is a function $\Phi: \mathscr{A} \to \{T \subset \boldsymbol{R}^m: T \neq \emptyset\}$ from $\mathscr{A}$ into the class of all non-empty subsets of $\boldsymbol{R}^m$, which is additive, i.e.,

$$\emptyset \neq \Phi(A) \subset \boldsymbol{R}^m \quad \text{for all} \quad A \in \mathscr{A}$$

and

$$\Phi(A_1 \cup A_2) = \Phi(A_1) + \Phi(A_2)$$

for every pair of disjoint sets $A_1, A_2 \in \mathscr{A}$. If $\mathscr{A}$ is a σ-algebra then Φ is called a *σ-additive set-valued measure*, iff

$$\Phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \Phi(A_n)$$

for every sequence $A_1, A_2, \cdots$ of mutually disjoint elements of $\mathscr{A}$. Here the sum $\sum_{n=1}^{\infty} T_n$ of the subsets $T_1, T_2, \cdots$ of $\boldsymbol{R}^m$ consists of all thevectors: "$x = \sum_{n=1}^{\infty} x_n$ with $x_n \in T_n$ for $n \in \boldsymbol{N}$. In the sequel, "$\Phi | \mathscr{A}$ is an additive [resp. σ-additive] set-valued measure" is an abbreviation for an algebra [resp. a σ-algebra] over Ω and a function $\Phi: \mathscr{A} \to \{T \subset \boldsymbol{R}^m: T \neq \emptyset\}$ which is additive [resp. σ-additive]. The calculus of additive and σ-additive set-valued measures has recently been developed by several authors (see [2], [4], [5], [1] and [6]) and the ideas and techniques have many interesting applications in mathematical economics (see [3], [4] and [10]), in control theory (see [8] and [9]), and other mathematical fields. Additive and σ-additive set-valued measures have also been discussed for their own mathematical interest, because they extend the theory of scalar additive and σ-additive measures in a natural way. This is the background of the present paper. Theorems 1 and 2 extend the known representation th orems of Riesz for bounded, additive [resp. regular, σ-additive] scalar measures to the case of bounded, additive [resp. regular, σ-additive] set-valued measures.

2. **Some properties of additive set-valued measures.** The following Lemma 1 is well known and has appeared in the literature in several forms (see [1], Proposition 3.1, p. 105). We state it here in a form suitable for the sequel, and for completeness we also give the proof.

LEMMA 1. *If $\Phi|\mathscr{A}$ is an additive [resp. σ-additive] set-valued measure, then the function $\mu_{x,\Phi}|\mathscr{A}$ with*

$$\mu_{x,\Phi}(A) := \sup\{\langle x, y\rangle : y \in \Phi(A)\}$$

is an additive [resp. σ-additive] scalar measure for all $x \in \boldsymbol{R}^m$.

Proof. The set function $\mu_{x,\Phi}|\mathscr{A}$ is well defined and with values in $(-\infty, +\infty]$. The additivity of $\mu_{x,\Phi}$ is trivial. Let $A_1, A_2, \cdots$ be a sequence of mutually disjoint sets $A_n \in \mathscr{A}$ and $A = \bigcup_{n=1}^\infty A_n$. If $z \in \Phi(A)$ then $z = \sum_{n=1}^\infty z_n$, where $z_n \in \Phi(A_n)$ for $n \in \boldsymbol{N}$. Then

$$(1) \qquad \langle x, z\rangle = \sum_{n=1}^\infty \langle x, z_n\rangle \leqq \liminf_\kappa \sum_{n=1}^\kappa \mu_{x,}\ (A_n)$$

and therefore $\mu_{x,\Phi}(A) \leqq \liminf_\kappa \sum_{n=1}^\kappa \mu_{x,\Phi}(A_n)$. If $\mu_{x,\Phi}(A) = \infty$ there is nothing else to show. If $\mu_{x,\Phi}(A) < \infty$, the additivity implies $\mu_{x,\Phi}(A_n) < \infty$ for every n. Given $\varepsilon > 0$, choose for each n an element $y_n \in \Phi(A_n)$ such that $\mu_{x,\Phi}(A_n) \leqq \langle x, y_n\rangle + \varepsilon \cdot 2^{-n}$. Denote $\tilde{y}_\kappa = \sum_{n=1}^\kappa y_n + \sum_{n>\kappa} z_n$. Then $\tilde{y}_\kappa \in \Phi(A)$ and

$$(2) \qquad \limsup_\kappa \sum_{n=1}^\kappa \mu_{x,\Phi}(A_n) - \varepsilon \leqq \limsup_\kappa \langle x, \tilde{y}_k\rangle \leqq \mu_{x,\Phi}(A)\ .$$

Since ε is arbitrarily small, (1) and (2) imply $\mu_{x,\Phi}(A) = \sum_{n=1}^\infty \mu_{x,\Phi}(A_n)$.

We call an additive set-valued measure $\Phi|\mathscr{A}$ *bounded*, iff $\bigcup_{A\in\mathscr{A}} \Phi(A)$ is a bounded subset of $\boldsymbol{R}^m$. In the case that Φ is σ-additive the following Lemma 2 is a result of Z. Artstein (see [1], p. 105). If Φ is only additive, the proof is given in [12], Korollar 2a. $|\nu|$ denotes the total variation of an additive scalar measure $\nu|\mathscr{A}$ and $e_1, \cdots, e_{2m}$ the $2m$ vectors of the form $(0, \cdots, \pm 1, \cdots, 0)$.

LEMMA 2. *Let $\Phi|\mathscr{A}$ be a bounded, additive set-valued measure [resp. a σ-additive set-valued measure with bounded $\Phi(\Omega)$] and $\hat{\mu} := \sum_{i=1}^{2m} |\mu_{e_i,\Phi}|$. Then $\hat{\mu}|\mathscr{A}$ is a nonnegative, finite additive [resp. σ-additive] scalar measure with*

$$\sup\{|y| : y \in \Phi(A)\} \leqq \hat{\mu}(A)$$

for all $A \in \mathscr{A}$.

Let $B(\Omega, \mathscr{A})$ denote the set of all uniform limits of finite linear combinations characteristic functions of sets in $\mathscr{A}$ and $B_+(\Omega, \mathscr{A})$ the subset of all nonnegative functions of $B(\Omega, \mathscr{A})$. $B(\Omega, \mathscr{A})$ is a Banach space. The norm on $B(\Omega, \mathscr{A})$ is denoted by $\|\ \|$.

LEMMA 3. *If $\Phi|\mathscr{A}$ is a bounded, additive set-valued measure, then:*

(a) *Every $f \in B(\Omega, \mathscr{A})$ is $\mu_{x,\Phi}$-integrable for all $x \in \boldsymbol{R}^m$.*

(b) *If $f \in B_+(\Omega, \mathscr{A})$ then $\int f d\Phi$ with $\left(\int f d\Phi\right)(x) := \int f d\mu_{x,\Phi}$ is a sublinear functional on $\boldsymbol{R}^m$.*

Proof. (a) Choose $x \in \boldsymbol{R}^m$ and $A \in \mathscr{A}$. By Lemma 1 $\mu_{x,\Phi}$ is an additive scalar measure and by Lemma 2

$$|\mu_{x,\Phi}(A)| \leqq |x|\hat{\mu}(A) .$$

Therefore

$$|\mu_{x,\Phi}|(A) \leqq |x|\hat{\mu}(A)$$

and hence

$$\left|\int f d\mu_{x,\Phi}\right| \leqq \int |f| d|\mu_{x,\Phi}| \leqq \|f\| |\mu_{x,\Phi}|(\Omega) < \infty \quad \text{for all} \quad f \in B(\Omega, \mathscr{A}) .$$

(b) The function $\mu_{\cdot,\Phi}(A)|\boldsymbol{R}^m$ with $(\mu_{\cdot,\Phi}(A))(x) := \mu_{x,\Phi}(A)$ is sublinear for every $A \in \mathscr{A}$. Therefore $\int t d\Phi$ is sublinear for every simple function $t \in B_+(\Omega, \mathscr{A})$ and hence $\int f d\Phi$ for every $f \in B_+(\Omega, \mathscr{A})$.

Consider the system $(\mathscr{K}, \delta)$ of all nonvoid, compact subsets of $\boldsymbol{R}^m$ with the Hausdorff distance δ and $\mathscr{L}_m := \{K \in \mathscr{K}: K \text{ convex}\}$. $(\mathscr{K}, \delta)$ is a metric space and

(1.1) $$(\mathscr{L}_m, \delta) \quad \text{is complete}$$

(see [4], (5.6), p. 362). Let Λ_m be the closed unit ball in $\boldsymbol{R}^m$ and $s: \mathscr{L}_m \to \mathscr{C}(\Lambda_m)$ with $s(T) := s(\cdot, T)$ and $s(x, T) := \sup\{\langle x, y\rangle : y \in T\}$ for $x \in \Lambda_m$, $T \in \mathscr{L}_m$. By [11]

(1.2) $$s \quad \text{is an isometric function} .$$

LEMMA 4. *If $\Phi|\mathscr{A}$ is an additive set-valued measure such that $\Phi(A)$ is compact for all $A \in \mathscr{A}$, then Φ is σ-additive iff $\delta(\Phi(A_n), \{0\}) \to 0$ for every sequence $A_1, A_2, \cdots,$ in $\mathscr{A}$ with $A_n \downarrow \varnothing$.*

Proof. See [12], Satz 1 or [6], Prop. 3.4.

3. Representation theorems. Our aim is to identify certain additive [resp. σ-additive] set-valued measures as linear mappings between suitable linear topological spaces. Let $BA(\Omega, \mathscr{A}, m)$ be the set of all bounded, additive set-valued measures $\Phi | \mathscr{A}$ with $\Phi(A) \in \mathscr{L}_m$ for all $A \in \mathscr{A}$ and E_m the set of all functions $s(\cdot, T): \Lambda_m \to \boldsymbol{R}$ with $T \in \mathscr{L}_m \cdot E_m$ is a convex cone in the Banach space $\mathscr{C}(\Lambda_m)$ of all real-valued continuous functions on Λ_m. Therefore $V_m := E_m - E_m$ is a linear subspace of $\mathscr{C}(\Lambda_m)$. The norm on $\mathscr{C}(\Lambda_m)$ is denoted by $\| \ \|_1$. Finally $\mathscr{L}_+(B(\Omega, \mathscr{A}); V_m)$ denotes the set of all continuous, linear mappings $\varphi: B(\Omega, \mathscr{A}) \to V_m$, where $\varphi(f) \in E_m$ for all $f \in B_+(\Omega, \mathscr{A})$.

THEOREM 1. *The mapping* $\pi: BA(\Omega, \mathscr{A}, m) \to \mathscr{L}_+(B(\Omega, \mathscr{A}); V_m)$ *defined by* $(\pi(\Phi))(f) := \int f d\Phi$ *is one-to-one and onto for all* $m \in \boldsymbol{N}$.

Proof. (1) First we show that π is well defined. Choose $\Phi \in BA(\Omega, \mathscr{A}, m)$ and $f \in B(\Omega, \mathscr{A})$. By Lemma 3(a) the function $\int f d\Phi$ is well defined and by Lemma 3(b) $\int f^+ d\Phi$ and $\int f^- d\Phi$ are sublinear functionals on $\boldsymbol{R}^m$. With the Hahn-Banach theorem it follows that

$$\left(\int f^+ d\Phi\right)(x) = \sup\left\{\langle x, y\rangle : \langle \cdot, y\rangle \leqq \left(\int f^+ d\Phi\right)(\cdot)\right\}$$

and

$$\left(\int f^- d\Phi\right)(x) = \sup\left\{\langle x, y\rangle : \langle \cdot, y\rangle \leqq \left(\int f^- d\Phi\right)(\cdot)\right\}$$

for every $x \in \boldsymbol{R}^m$. The set $T_\pm := \left\{y \in \boldsymbol{R}^m : \langle \cdot, y\rangle \leqq \left(\int f^\pm d\Phi\right)(\cdot)\right\}$ is an element of $\mathscr{L}_m$ and therefore $\int f^\pm d\Phi \in E_m$. Since $\int f d\Phi = \int f^+ d\Phi - \int f^- d\Phi$, $\int f d\Phi \in V_m$. Obviously the equality

$$(\pi(\Phi))(\alpha f + \beta g) = \alpha(\pi(\Phi))(f) + \beta(\pi(\Phi))(g)$$

holds and

$$\left\|\int f d\Phi - \int g d\Phi\right\|_1 \leqq \|f - g\| \sup_{x \in \ \ m} |\mu_{x,\Phi}|(\Omega)$$

for all $f, g \in B(\Omega, \mathscr{A})$ and $\alpha, \beta \in \boldsymbol{R}$. So π is well defined.

(2) Second we show that $\pi(\Phi) = \pi(\Phi')$ implies $\Phi = \Phi'$ for all $\Phi, \Phi' \in BA(\Omega, \mathscr{A}, m)$. Let $\Phi, \Phi' \in BA(\Omega, \mathscr{A}, m)$ and $\pi(\Phi) = \pi(\Phi')$. Then $\mu_{x,\Phi}(A) = \mu_{x,\Phi'}(A)$ for every $x \in \Lambda_m$ and $A \in \mathscr{A}$. The Hahn-Banach theorem and $\Phi(A), \Phi'(A) \in \mathscr{L}_m$ for every $A \in \mathscr{A}$ imply $\Phi = \Phi'$.

(3) Third we have to show that for an arbitrarily chosen $\varphi \in \mathscr{L}_+(B(\Omega, \mathscr{A}); V_m)$ there is a $\Phi \in BA(\Omega, \mathscr{A}, m)$ with $\pi(\Phi) = \varphi$. Choose $\varphi \in \mathscr{L}_+(B(\Omega, \mathscr{A}); V_m)$. For every $f \in B_+(\Omega, \mathscr{A})$ there exists

only one $T(f) \in \mathscr{L}_m$ with $\varphi(f) = s(\cdot, T(f))$. Define $\Phi | \mathscr{A}$ by $\Phi(A) := T(\chi_A)$, where χ_A is the characteristic function of A. Since φ is linear the equation

$$T(\chi_{A_1} + \chi_{A_2}) = T(\chi_{A_1}) + T(\chi_{A_2})$$

holds for disjoint sets $A_1, A_2 \in \mathscr{A}$, i.e., $\Phi | \mathscr{A}$ is an additive set-valued measure with $\Phi(A) \in \mathscr{L}_m$ for all $A \in \mathscr{A}$. Moreover, by (1.2) and the continuity of φ, it follows

$$\begin{aligned} \delta(\Phi(A), \{0\}) &= \|s(\cdot, T(\chi_A))\|_1 \\ &= \|\varphi(\chi_A)\|_1 \\ &\leqq \sup\{\|\varphi(g)\|_1 : g \in B(\Omega, \mathscr{A}), \|g\| \leqq 1\} < \infty \end{aligned}$$

for all $A \in \mathscr{A}$. Therefore Φ is bounded. Let $x \in \Lambda_m$. Then $\varphi_x : B(\Omega, \mathscr{A}) \to \boldsymbol{R}$ with $\varphi_x(f) := (\varphi(f))(x)$ is a continuous linear functional and by the Riesz representation theorem ([7], Theorem 1, p. 258) there is a bounded, additive scalar measure $\lambda_x | \mathscr{A}$ with $\varphi_x(f) = \int f d\lambda_x$ for $f \in B(\Omega, \mathscr{A})$. So

$$\mu_{x,} (A) = s(x, T(\chi_A)) = \varphi_x(\chi_A) = \lambda_x(A)$$

holds for all $A \in \mathscr{A}$. That means $\pi(\Phi) = \varphi$.

$B(\Omega, \mathscr{A})'$ denotes the topological dual of $B(\Omega, \mathscr{A})$ and $ba(\Omega, \mathscr{A})$ the set of all bounded, additive scalar measures ν on $\mathscr{A}$. So we get the following corollary of Theorem 1.

COROLLARY 1. *There is an isometric isomorphism between $B(\Omega, \mathscr{A})'$ and $ba(\Omega, \mathscr{A})$ such that the corresponding elements η and ν satisfy the identity $\eta(f) = \int f d\nu$ for all $f \in B(\Omega, \mathscr{A})$.*

Proof. We have to show only that each $\eta \in B(\Omega, \mathscr{A})'$ determines a $\nu \in ba(\Omega, \mathscr{A})$ such that $\int f d\nu = \eta(f)$ for $f \in B(\Omega, \mathscr{A})$. Let $\eta \in B(\Omega, \mathscr{A})'$ and $(\varphi(f))(x) := x\eta(f)$ for $f \in B(\Omega, \mathscr{A})$ and $x \in [-1, 1]$. φ is an element of $\mathscr{L}_+(B(\Omega, \mathscr{A}); V_1)$ and by Theorem 1 there exists a $\Phi \in BA(\Omega, \mathscr{A}, 1)$ with $\pi(\Phi) = \varphi$, i.e., $\int f d\mu_{x,\Phi} = x\eta(f)$ for $f \in B(\Omega, \mathscr{A})$ and $x \in [-1, 1]$. Therefore

$$\eta(\chi_A) = \sup\{y : y \in \Phi(A)\}$$

and

$$-\eta(\chi_A) = -\inf\{y : y \in \Phi(A)\}$$

for $A \in \mathscr{A}$. This means that $\Phi(A)$ consists only of one point $\nu(A)$ and ν is an element of $ba(\Omega, \mathscr{A})$. Furthermore

$$\int f d\nu = \left(\int f d\Phi\right)(1) = \eta(f) \quad \text{for} \quad f \in B(\Omega, \mathscr{A}) .$$

Now let Ω be a topological space. A σ-additive set-valued measure $\Phi | \mathscr{B}(\Omega)$ on the Borel sets $\mathscr{B}(\Omega)$ of Ω is called *regular*, iff $\mu_{x,\Phi} | \mathscr{B}(\Omega)$ is regular for every $x \in \Lambda_m$. $RCA(\Omega, \mathscr{B}(\Omega), m)$ denotes the set of all regular, σ-additive set-valued measures $\Phi | \mathscr{B}(\Omega)$ such that $\Phi(B) \in \mathscr{L}_m$ for $B \in \mathscr{B}(\Omega)$. If Ω is a compact Hausdorff space, $\mathscr{C} := \mathscr{C}(\Omega)$ and $\mathscr{C}'$ the topological dual of $\mathscr{C}$ then $\mathscr{L}_+^b(\mathscr{C}, V_m)$ denotes the set of all $\varphi \in \mathscr{L}_+(\mathscr{C}, V_m)$ such that: there is a $\eta \in \mathscr{C}'$ with $\|\varphi(f)\|_1 \leqq \eta(|f|)$ for $f \in \mathscr{C}$.

THEOREM 2. *If Ω is a compact Hausdorff space then the mapping π: $RCA(\Omega, \mathscr{B}(\Omega), m) \rightarrow \mathscr{L}_+^b(\mathscr{C}, V_m)$ defined by $(\pi(\Phi))(f) := \int f d\Phi$ is one-to-one and onto for all $m \in N$.*

Proof. By Lemma 2 each $\Phi \in RCA(\Omega, \mathscr{B}(\Omega), m)$ is bounded and hence $RCA(\Omega, \mathscr{B}(\Omega), m) \subset BA(\Omega, \mathscr{B}(\Omega), m)$. Analogous to (1) of Theorem 1 one shows $\pi(RCA(\Omega, \mathscr{B}(\Omega), m)) \subset \mathscr{L}_+(\mathscr{C}, V_m)$. Let $\Phi \in RCA(\Omega, \mathscr{B}(\Omega), m)$. By Lemma 2 the σ-additive scalar measure $\hat{\mu} = \sum_{i=1}^{2m} |\mu_{e_i,\Phi}|$ is finite and

$$\begin{aligned} \|(\pi(\Phi))(f)\|_1 &\leqq \sup_{x \in \Lambda_m} \int |f| \, d|\mu_{x,\Phi}| \\ &\leqq \int |f| \, d\hat{\mu} , \end{aligned}$$

therefore $\pi(\Phi) \in \mathscr{L}_+^b(\mathscr{C}, V_m)$. If Φ' is also an element of $RCA(\Omega, \mathscr{B}(\Omega), m)$, then $\pi(\Phi) = \pi(\Phi')$ implies $\int f d\mu_{x,\Phi} = \int f d\mu_{x,\Phi'}$ for $x \in \Lambda_m$, $f \in \mathscr{C}$, and by the regularity of $\mu_{x,\Phi}$ and $\mu_{x,\Phi'}$ we have $\Phi = \Phi'$. Now we show that for each $\varphi \in \mathscr{L}_+^b(\mathscr{C}, V_m)$ there is a $\Phi \in RCA(\Omega, \mathscr{B}(\Omega), m)$ such that $\pi(\Phi) = \varphi$. Let $\varphi \in \mathscr{L}_+^b(\mathscr{C}, V_m)$. By the Riesz representation theorem ([7], Theorem 3, p. 265) there is a nonnegative, regular, σ-additive scalar measure $\lambda_\varphi | \mathscr{B}(\Omega)$ with $\|\varphi(f)\|_1 \leqq \int |f| d\lambda_\varphi$ for $f \in \mathscr{C}$. Furthermore for each $f \in \mathscr{C}$, $f \geqq 0$, there is only one $T(f) \in \mathscr{L}_m$ such that $\varphi(f) = s(\cdot, T(f))$. Let $B \in \mathscr{B}(\Omega)$. Since λ_φ is regular there exists a sequence $f_1, f_2, \cdots$, in $\mathscr{C}$ such that $0 \leqq f_n \leqq 1$ and $\int |\chi_B - f_n| d\lambda_\varphi \rightarrow 0$. (1.2) implies

$$\begin{aligned} \delta(T(f_n), T(f_\kappa)) &= \|\varphi(f_n - f_\kappa)\|_1 \\ &\leqq \int |f_n - f_\kappa| \, d\lambda_\varphi \xrightarrow{n,\, \kappa \rightarrow \infty} 0 \end{aligned}$$

and by (1.1) there is a $\tilde{T}(B) \in \mathscr{L}_m$ with $\delta(T(f_n), \tilde{T}(B)) \to 0$. Define $\Phi \mid \mathscr{B}(\Omega)$ by $\Omega(B) := \tilde{T}(B)$. The definition is independent of the choice of the sequence $f_1, f_2, \cdots$, and, since φ is linear and $\delta(T_1 + T_2, T_1' + T_2') \leqq \delta(T_1, T_1') + \delta(T_2, T_2')$ for $T_i, T_i' \in \mathscr{L}_m (i = 1, 2)$, we have $\tilde{T}(B_1 \cup B_2) = \tilde{T}(B_1) + \tilde{T}(B_2)$ for disjoint sets $B_1, B_2 \in \mathscr{B}(\Omega)$, i.e., $\Phi \mid \mathscr{B}(\Omega)$ is an additive set-valued measure with $\Phi(B) \in \mathscr{L}_m$ for $B \in \mathscr{B}(\Omega)$. Furthermore, Φ is σ-additive, since by (1.2) and Lemma 4

$$\delta(\Phi(B_n), \{0\}) \leqq \lambda_\varphi(B_n) \longrightarrow 0$$

for every sequence $B_1, B_2, \cdots$ in $\mathscr{B}(\Omega)$ such that $B_n \downarrow \varnothing$. Let $x \in \Lambda_m$ and $\varphi_x(f) := (\varphi(f))(x)$ for $f \in \mathscr{C}$. φ_x is a continuous linear functional on $\mathscr{C}$ and by the Riesz representation theorem ([7], Theorem 3, p. 265) there is a regular, σ-additive scalar measure ν_x on $\mathscr{B}(\Omega)$ such that $\int f d\nu_x = \varphi_x(f)$ for $f \in \mathscr{C}$. If we can show the equality $\nu_x = \mu_{x,\Phi}$, then the regularity of Φ and $\pi(\Phi) = \varphi$ follows. Since $\left|\int f d\nu_x\right| \leqq \int |f| d\lambda_\varphi$ for $f \in \mathscr{C}$ and because of the regularity of ν_x and λ_φ the inequality

$$|\nu_x|(U) \leqq \lambda_\varphi(U)$$

is true for every open subset U of Ω and therefore

$$(*) \qquad |\nu_x|(B) \leqq \lambda_\varphi(B)$$

for $B \in \mathscr{B}(\Omega)$. If $B \in \mathscr{B}(\Omega)$ then there is a sequence $f_1, f_2, \cdots$ in $\mathscr{C}$ such that $0 \leqq f_n \leqq 1$ and $\int |\chi_B - f_n| d\lambda_\varphi \to 0$. By (*)

$$\int |\chi_B - f_n| d|\nu_x| \longrightarrow 0$$

and therefore

$$\mu_{x,\Phi}(B) = \lim_{n\to\infty} s(x, T(f_n)) = \lim_{n\to\infty} \int f_n d\nu_x = \nu_x(B) .$$

$rca(\Omega, \mathscr{B}(\Omega))$ denotes the set of all regular, σ-additive scalar measures ν on $\mathscr{B}(\Omega)$. From Theorem 2 we get the following corollary.

COROLLARY 2. *If Ω is a compact Hausdorff space, then there is an isometric isomorphism between $\mathscr{C}'$ and $rca(\Omega, \mathscr{B}(\Omega))$ such that the corresponding elements η and ν satisfy the identity $\eta(f) = \int f d\nu$ for all $f \in \mathscr{C}$.*

Proof. We have to show only that each $\eta \in \mathscr{C}'$ determines a $\nu \in rca(\Omega, \mathscr{B}(\Omega))$ such that $\int f d\nu = \eta(f)$ for $f \in \mathscr{C}$.

Let $\eta \in \mathscr{C}'$. Then there are positive linear functionals $\eta_1, \eta_2 \in \mathscr{C}'$ with $\eta = \eta_1 - \eta_2$. For each $i = 1, 2$ we define $(\varphi_i(f))(x) := x \cdot \eta_i(f)$ for $f \in \mathscr{C}$ and $x \in [-1, 1]$. φ_i is an element of $\mathscr{L}_+(\mathscr{C}, V_1)$ and since

$$\|\varphi_i(f)\|_1 \leqq |\eta_i(f)| \leqq \eta_i(|f|)$$

for $f \in \mathscr{C}$, we conclude $\varphi_i \in \mathscr{L}^b_+(\mathscr{C}, V_1)$ for $i = 1, 2$. By Theorem 2 there is a $\Phi_i \in RCA(\Omega, \mathscr{B}(\Omega), 1)$ such that $\int f d\mu_{x,\Phi_i} = x \cdot \eta_i(f)$ for $x \in [-1, 1]$, $f \in \mathscr{C}$ and $i = 1, 2$. Therefore $\int f d(\mu_{1,\Phi_i} + \mu_{-1,\Phi_i}) = 0$ for every $f \in \mathscr{C}$ and the regularity of μ_{x,Φ_i} implies $\mu_{1,\Phi_i} = -\mu_{-1,\Phi_i}$ for $i = 1, 2$. Since

$$\mu_{1,\Phi_i}(B) = \sup\{y \colon y \in \Phi_i(B)\}$$

and

$$\mu_{-1,\Phi}(B) = -\inf\{y \colon y \in \Phi_i(B)\}\ ,$$

the set $\Phi_i(B)$ consists of only one point $\nu_i(B)$ for every $B \in \mathscr{B}(\Omega)$ and ν_i is an element of $rca(\Omega, \mathscr{B}(\Omega))$ for $i = 1, 2$. The σ-additive measure $\nu := \nu_1 - \nu_2$ is also an element of $rca(\Omega, \mathscr{B}(\Omega))$ and

$$\begin{aligned}\int f d\nu &= \int f d\nu_1 - \int f d\nu_2 \\ &= \left(\int f d\Phi_1\right)(1) - \left(\int f d\Phi_2\right)(1) \\ &= \eta_1(f) - \eta_2(f) \\ &= \eta(f)\end{aligned}$$

for every $f \in \mathscr{C}$.

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Received October 9, 1977.

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