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THE SCHWARZIAN DERIVATIVE AND THE POINCARÉ METRIC

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Dedicated to Z. Nehari

Let $\Omega \notin 0_G$ be a plane region and let $\lambda_\Omega(z)$ be its Poincaré metric. Let E_Ω be the complement of $\bar{\Omega}$ and write $\alpha(\zeta)=\alpha(\zeta:\Omega)=\left\{\pi^{-1}\int_{E_\Omega}|z-\zeta|^{-4}\,d\sigma(z)\right\}^{1/2}$, where $d\sigma(z)=dxdy$ and $\zeta\in\Omega$. $\lambda_\Omega(z)=\alpha(z:\Omega)$ for all $z\in\Omega$ only when Ω is a disk less (possibly) a closed subset of inner capacity zero. Let ϕ be holomorphic and univalent in Ω and let $S_\phi(z,\zeta)=-6(\partial^2/\partial z\partial\zeta)\times\log(\phi(z)-\phi(\zeta))/(z-\zeta)$. Here $S_\phi(z,z)$ is the Schwarzian derivative of ϕ. We show

$$|S_\phi(z,\zeta)| \leqq 6\lambda_\Omega(z)\lambda_\Omega(\zeta)\left[1+\left(1-\frac{\alpha^2(\zeta)}{\lambda_\Omega^2(\zeta)}\right)^{1/2}\right]; \qquad z,\zeta\in\Omega\,.$$

1. Introduction. In his paper [4] Gehring was concerned with the problem of extending to an arbitrary simply connected plane region Ω certain results relating the univalence of a function ϕ holomorphic in the unit disk Δ with the magnitude of its Schwarzian derivative

$$S_\phi(z)=\left(\frac{\phi''}{\phi'}\right)'-\frac{1}{2}\left(\frac{\phi''}{\phi'}\right)^2; \qquad \phi=\phi(z),\ z\in\Delta\,.$$

We shall be concerned with generalizing the following two propositions to an arbitrary plane region Ω.

PROPOSITION 1. *If ϕ is holomorphic and univalent in Δ, then*

$$|S_\phi(z)| \leqq 6(1-|z|^2)^{-2}\,, \qquad z\in\Delta\,,$$

and the constant 6 *is sharp.*

PROPOSITION 2. *Let Ω be a simply connected domain and let $\lambda_\Omega(z)$ be its Poincaré metric. If ϕ is holomorphic and univalent in Ω, then*

$$|S_\phi(z)| \leqq 12\lambda_\Omega^2(z)\,, \qquad z\in\Omega\,,$$

and the constant 12 *is sharp.*

Proposition 1 is due to Kraus [5] and Proposition 2 is due to Lehto [6]. In this direction Nehari [7] has shown that if ϕ is holomorphic with $|S_\phi(z)| \leqq 2(1-|z|^2)^{-2}$ in Δ, then ϕ is univalent in Δ with the constant 2 being the best possible.

We shall show that the above two propositions can be immediately read off from one single inequality (Corollary 3) which is valid for any plane region. Our result can be easily extended to open Riemann surfaces too but we shall not pursue this point. Our arguments rely heavily on well known classical results of Bergman and Schiffer [2]. In order to be self contained, however, we will attempt to provide proofs to most crucial statements. The final result obtained in this paper (Theorem 2) involves a string of sharp inequalities amongst the Schwarzian derivative, the span or condenser capacity, the analytic capacity, the capacity and the Poincaré metric. In this string of inequalities, the inequality between the span (condenser) capacity and the analytic capacity is a well known result of Ahlfors and Beurling [1]. Here we provide a different proof of this result which is based on our Theorem 1. The representation formula of Theorem 1 was first mentioned in Schiffer [9] in case $\zeta = \infty \in \Omega$ and Ω has the largest complementary area amongst all regions which are conformally equivalent to Ω.

2. Capacities and the Poincaré metric. Let Ω be an open region in the extended plane and let $\zeta \in \Omega$. Usually, $\zeta \neq \infty$ but the transition to $\zeta = \infty$ is trivial. $H(\Omega)$ stands for the class of holomorphic functions in Ω and $H_m(\Omega)$ denotes the class of multivalued holomorphic functions f in Ω such that $|f(z)|$, $z \in \Omega$, is single valued. We write

$$||f||_\infty = \sup_{z \in \Omega} |f(z)| , \qquad D[f] = \int_\Omega |f'(z)|^2 d\sigma(z) ,$$

where $d\sigma(z) = dxdy$ is the Lebesgue area measure. Consider the following families:

$$\mathscr{B}_\zeta(\Omega) = \{f \in H(\Omega): ||f||_\infty \leqq 1, f(\zeta) = 0\} ,$$

$$\mathscr{C}_\zeta(\Omega) = \{f \in H_m(\Omega): ||f||_\infty \leqq 1, f(\zeta) = 0\} ,$$

$$\mathscr{D}_\zeta(\Omega) = \{f \in H(\Omega): D[f] \leqq \pi, f(\zeta) = 0\} .$$

We now introduce (cf. [1]) the analytic capacity

$$C_B(\zeta) = C_B(\zeta: \Omega) = \max\{|f'(\zeta)|: f \in \mathscr{B}_\zeta(\Omega)\} ,$$

the capacity

$$C_\beta(\zeta) = C_\beta(\zeta: \Omega) = \max\{|f'(\zeta)|: f \in \mathscr{C}_\zeta(\Omega)\}$$

and the span or condenser capacity

$$C_D(\zeta) = C_D(\zeta: \Omega) = \max\{|f'(\zeta)|: f \in \mathscr{D}_\zeta(\Omega)\} .$$

We note that $C_\beta(\zeta)$ is well defined and that by $f(\zeta) = 0$ in $\mathscr{C}_\zeta(\Omega)$ we mean that at least one branch of $f(z)$ vanishes at ζ.

Assume now that $\Omega \notin 0_G$ (i.e., Ω has a nontrivial Green's function) and thus Ω has the unit disk $\varDelta$ as its universal covering space. The Poincaré metric $\lambda_\Omega(z)$ is defined as follows: For the unit disk $\varDelta$, $\lambda_\varDelta(\omega) = (1 - |\omega|^2)^{-1}$ while for Ω

$$\lambda_\Omega(z) = \lambda_\varDelta(\omega)\,|\pi'(\omega)|^{-1}\,, \qquad z = \pi(\omega) \in \Omega\,,$$

where $\pi\colon \varDelta \to \Omega$ is a universal cover map. We denote by $\delta_\Omega(z)$ the distance from z to the boundary of Ω. $\lambda_\Omega(z)$ is monotonic decreasing with Ω and thus

$$\lambda_\Omega(z)\delta_\Omega(z) \leqq 1\,, \qquad z \in \Omega\,.$$

Moreover, if Ω is simply connected then, in view of Koebe's 1/4 theorem,

$$\lambda_\Omega(z)\delta_\Omega(z) \geqq 1/4\,, \qquad z \in \Omega\,.$$

Clearly, C_B, C_β, C_D and λ_Ω are conformally invariant and therefore

$$C_D(z\colon \Omega) = C_B(z\colon \Omega) = C_\beta(z\colon \Omega) = \lambda_\Omega(z)\,, \qquad z \in \Omega\,, \tag{2.1}$$

whenever Ω is simply connected.

It is also evident that $C_B(z\colon \Omega) \leqq C_\beta(z\colon \Omega)$ and it is a theorem of Ahlfors and Beurling [1] (see also Corollary 2) that $C_D(z\colon \Omega) \leqq C_B(z\colon \Omega)$. Moreover, $C_\beta(z; \Omega) \leqq \lambda_\Omega$. Indeed, let $\pi; \varDelta \to \Omega$ be a universal cover map $z = \pi(\omega)$. Then

$$\begin{aligned} C_\beta(z\colon \Omega) &= \max\{|f'(z)|\colon f \in \mathscr{C}_z(\Omega)\} \\ &= |\pi'(\omega)|^{-1} \max\{|g'(\omega)|\colon g = f \circ \pi \in \mathscr{C}_\omega(\varDelta),\ f \in \mathscr{C}_z(\Omega)\} \\ &\leqq |\pi'(\omega)|^{-1} \max\{|g'(\omega)|\colon g \in \mathscr{C}_\omega(\varDelta)\} \\ &= |\pi'(\omega)|^{-1} C_\beta(\omega\colon \varDelta) \\ &= \lambda_\varDelta(\omega)\,|\pi'(\omega)|^{-1} = \lambda_\Omega(z)\,, \end{aligned}$$

where (2.1) has been used. Consequently,

$$C_D(z\colon \Omega) \leqq C_B(z\colon \Omega) \leqq C_\beta(z\colon \Omega) \leqq \lambda_\Omega(z) \leqq \delta_\Omega^{-1}(z)\,, \tag{2.2}$$

and we note that, if $\Omega \notin 0_{AD}$, then $C_D(z\colon \Omega) > 0$ for all $z \in \Omega$. The condition $\Omega \notin 0_{AD}$ means that there exists a nonconstant holomorphic function f in Ω with $D[f] < \infty$.

We conclude this section by recalling the following well-known lemma of Ahlfors and Beurling [1]:

LEMMA 1. *Let E be a measurable set with a finite Lebesgue measure $\sigma(E)$ in $\mathbf{C}$. Then*

$$\sup_{\zeta \in C} \left| \int_E \frac{d\sigma(z)}{z-\zeta} \right| \leqq \sqrt{\pi\sigma(E)}$$

with equality holding, if and only if E is a almost everywhere a disk of radius $\sqrt{\sigma(E)/\pi}$.

3. The slit mappings. Here we assume that the region Ω is bounded by n closed analytic curves $C_1, \cdots, C_n$ and we denote by C the boundary $\partial\Omega = \bigcup_{k=1}^{n} C_k$ of Ω. We assume that C_1 is outer and we let $\zeta \in \Omega$. Let

$$p(z) = p(z: \zeta) = \frac{1}{z-\zeta} + a(z-\zeta) + \cdots$$

and

$$q(z) = q(z: \zeta) = \frac{1}{z-\zeta} + b(z-\zeta) + \cdots$$

be the horizontal and vertical slit mappings, respectively, of Ω. We write

$$\Phi(z) = \Phi(z: \zeta) = \frac{1}{2}(p(z) - q(z))$$

and

$$\Psi(z) = \Psi(z: \zeta) = \frac{1}{2}(p(z) + q(z)) .$$

Then $\Phi(z)$ and $(z-\zeta)\Psi(z)$ are holomorphic on $\bar{\Omega}$ with $\Phi(\zeta) = 0$. Further, $\Psi(z)$ is univalent on Ω with pole at ζ. It maps Ω onto Ω^* with $E = \hat{C} - \bar{\Omega}^*$ being bounded. Clearly, $d\Phi = d\bar{\Psi}$ on C and therefore $\Phi = \bar{\Psi} - \bar{\lambda}_k$ on C_k, where λ_k is a constant depending on the component C_k, $1 \leqq k \leqq n$. Also $\Phi(C_k)$ and $\Psi(C_k)$ are closed analytic and convex curves. One easily shows that

$$D[\phi] = \pi C_D^2(\zeta: \Omega) = \sigma(E) = \frac{\pi}{2}(a-b) . \tag{3.1}$$

Let

$$\psi(z) = \psi(z: \zeta) = \zeta + \frac{1}{\Psi(z: \zeta)} .$$

ψ maps Ω conformally onto $\Omega' = \psi(\Omega)$ with $\phi(\zeta) = \zeta$. We write $\Gamma = \bigcup_{k=1}^{n} \Gamma_k = \partial\Omega'$ with $\Gamma_k = \psi(C_k)$, $1 \leqq k \leqq n$. We now establish an integral formula representing Φ in terms of Ψ (compare also Schiffer [9]).

THEOREM 1. *The integral formula*

$$\Phi(z:\zeta) = \frac{1}{\pi}\int_E \frac{d\sigma(t)}{\Psi(z:\zeta) - t}\,; \qquad E = \hat{C} - \overline{\Psi(\Omega)}\,,$$

holds.

Proof. We first note that if f is holomorphic on $\bar{\Omega}$ then, using the residue theorem,

$$f(z) = \frac{1}{2\pi i}\int_C f(\tau)\frac{\psi'(\tau)}{\psi(\tau) - \psi(z)}d\tau\,.$$

Specializing this formula for $f = \Phi\Psi$ we obtain

$$\Phi(z)\Psi(z) = \frac{1}{2\pi i}\int_C |\Psi(\tau)|^2 \frac{\psi'(\tau)}{\psi(\tau)-\psi(z)}d\tau - \frac{1}{2\pi i}\sum_{k=1}^{n}\bar{\lambda}_k\int_{C_k}\psi(\tau)\frac{\psi'(\tau)}{\psi(\tau)-\psi(z)}d\tau\,.$$

Writing $w = \psi(z)$ and $\omega = \psi(\tau)$, $\tau = C$, and recalling the definition of $\psi(z) = \psi(z:\zeta)$ we have

$$\int_{C_k}\psi(\tau)\frac{\psi'(\tau)}{\psi(\tau) - \psi(z)}d\tau = \int_{\Gamma_k}\frac{1}{\omega - \zeta}\,\frac{1}{\omega - w}d\omega$$

which is zero for each $k = 1, \cdots, n$. Therefore, writing $f = \Phi\Psi$,

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{|\omega - \zeta|^2}\,\frac{d\omega}{\omega - w}\,.$$

Let $E' = C - \bar{\Omega}'$, then according to Green's formula,

$$f(z) = \frac{1}{\pi}\int_{E'}\frac{1}{\overline{(\omega - \zeta)^2}}\,\frac{d\sigma(\omega)}{(\omega - \zeta)(\omega - w)}\,.$$

Hence

$$\begin{aligned}\Phi(z) &= \frac{w - \zeta}{\pi}\int_{E'}\frac{d\sigma(\omega)}{\overline{(\omega - \zeta)^2}(\omega - \zeta)(\omega - w)} \\ &= \frac{w - \zeta}{\pi}\int_{E'}\frac{d\sigma(\omega)}{\overline{(\omega - \zeta)^2}(\omega - \zeta)\left[1 - \dfrac{w - \zeta}{\omega - \zeta}\right]} \\ &= \frac{w - \zeta}{\pi}\int_E \frac{d\sigma(t)}{1 - t(w - \zeta)} = \frac{1}{\pi}\int_E \frac{d\sigma(t)}{\Psi(z) - t}\,.\end{aligned}$$

This concludes the proof.

REMARK. The theorem remains valid for the general case that $\Omega \notin O_{AD}$. This can be accomplished via a canonical exhaustion of Ω by regular regions $\{\Omega_m\}$.

The following corollary was also obtained in Burbea [3] and Sakai [8]. The methods in [8] are different from ours.

COROLLARY 1. *Let $\Omega \notin 0_{AD}$. Then $\|\Phi\|_\infty \leqq C_D(\zeta)$ with equality holding if and only if Ω is conformally equivalent to the unit disk Δ less (possibly) a closed null C_D-set.*

Proof. According to Lemma 1 and (3.1) we have

$$\|\Phi\|_\infty \leqq \frac{1}{\pi}\sqrt{\pi\sigma(E)} = C_D(\zeta) .$$

The statement about equality follows from Lemma 1 too.

The theorem of Ahlfors and Beurling [1] is also a consequence of the theorem as the following corollary shows.

COROLLARY 2. $C_D(\zeta) \leqq C_B(\zeta)$.

Proof. We may assume that $C_D(\zeta) > 0$. Let $f(z) = \Phi(z)/C_D(\zeta)$. Since $\Phi(\zeta) = 0$ it follows from Corollary 1 that $f \in \mathscr{B}_\zeta(\Omega)$. Thus $|\Phi'(\zeta)|/C_D(\zeta) \leqq C_B(\zeta)$. However, $\Phi'(\zeta) = (1/2)(a - b)$ and the assertion follows by appealing to (3.1).

REMARK. One can show (see [8]) that equality in the last corollary occurs if and only if either (i) $C_B(\zeta) = 0$, or (ii) Ω is conformally equivalent to the unit disk Δ less (possibly) a closed null C_B-set.

4. The Schwarzian derivative.

We again assume that Ω is a regular analytic region as mentioned before. The more general case can be always obtained by a canonical exhaustion. Let $H_S(\Omega)$ be the Hilbert space of all holomorphic functions f in Ω, having single valued integrals and so that

$$\|f\|^2 = \int_\Omega |f(z)|^2 d\sigma(z) < \infty .$$

This space possesses the (reduced) Bergman kernel function $K_\Omega(z, \bar{\zeta})$. We have the obvious identity

$$K_\Omega(z, \bar{\zeta}) = \frac{1}{\pi}\Phi'(z\colon \zeta)$$

and therefore

$$C_D(\zeta\colon \Omega) = \sqrt{\pi K_\Omega(\zeta, \bar{\zeta})} . \tag{4.1}$$

The "adjoint" kernel [2] is given by

$$L_\Omega(z, \zeta) = -\frac{1}{\pi}\Psi'(z: \zeta) .$$

This kernel is symmetric in z and ζ. Since $d\Phi = d\bar{\Psi}$ on the boundary, we have $\overline{K_\Omega(z, \zeta)dz} = -L_\Omega(z, \zeta)dz$ for $z \in \partial\Omega$ and $\zeta \in \Omega$. Also,

$$L_\Omega(z, \zeta) = \frac{1}{\pi}\frac{1}{(z-\zeta)^2} - l_\Omega(z, \zeta) ,$$

where $l_\Omega(z, \zeta)$ is symmetric and holomorphic in $(z, \zeta) \in \bar{\Omega} \times \bar{\Omega}$. If $\zeta \in \Omega$ is fixed then $l_\Omega(\ , \zeta) \in H_S(\Omega)$. We have (see also [2, p. 243])

$$|| l_\Omega(\ , \zeta) ||^2 = \int_\Omega | l_\Omega(z, \zeta) |^2 d\sigma(z) = K_\Omega(\zeta, \bar{\zeta}) - \Gamma_\Omega(\zeta, \bar{\zeta})$$

where

$$\Gamma_\Omega(\zeta, \bar{\zeta}) = \frac{1}{\pi^2}\int_{E_\Omega}\frac{d\sigma(z)}{|z-\zeta|^4} .$$

Here $E_\Omega = \hat{\boldsymbol{C}} - \bar{\Omega}$. We also write

$$\alpha(\zeta) = \alpha(\zeta: \Omega) = \sqrt{\pi\Gamma_\Omega(\zeta, \bar{\zeta})} \tag{4.2}$$

and thus $\pi\alpha^2(\zeta)$ represents the image area of E_Ω under the linear mapping $(z-\zeta)^{-1}$. Therefore

$$\alpha(\zeta) \leqq \sqrt{\pi K_\Omega(\zeta, \bar{\zeta})} = C_D(\zeta: \Omega) ,$$

equality holding, for each $\zeta \in \Omega$, only if $\partial\Omega$ is a circle (including circles passing through ∞). Further, we have

$$l_\Omega(z, \zeta) = (l_\Omega(\ , \zeta), K_\Omega(\ , \bar{z}))$$

and therefore

$$| l_\Omega(z, \zeta) |^2 \leqq || l_\Omega(\ , \zeta) ||^2 || K_\Omega(\ , \bar{z}) ||^2$$

or

$$| l_\Omega(z, \zeta) |^2 \leqq K_\Omega(z, \bar{z})[K_\Omega(\zeta, \bar{\zeta}) - \Gamma_\Omega(\zeta, \bar{\zeta})] . \tag{4.3}$$

A fortiori,

$$| l_\Omega(z, \zeta) |^2 \leqq K_\Omega(z, \bar{z})K_\Omega(\zeta, \bar{\zeta}) . \tag{4.4}$$

Let $\omega = \phi(z)$ be a conformal mapping of Ω onto Ω^*. Then, for $\tau = \phi(\zeta)$,

$$K_\Omega(z, \bar{\zeta}) = K_{\Omega^*}(\omega, \bar{\tau})\phi'(z)\overline{\phi'(\zeta)} \tag{4.5}$$

and

$$L_\Omega(z,\zeta) = L_{\Omega^*}(\omega,\tau)\phi'(z)\phi'(\zeta)\ .$$

From the last formula it follows that

$$l_{\Omega^*}(\omega,\tau)\phi'(z)\phi'(\zeta) = l_\Omega(z,\zeta) - \frac{1}{6\pi}S_\phi(z,\zeta)\ , \tag{4.6}$$

where

$$S_\phi(z,\zeta) = -6\frac{\partial^2}{\partial z\partial\zeta}\log\frac{\phi(z)-\phi(\zeta)}{z-\zeta}\ .$$

we note that

$$S_\phi(z,z) = S_\phi(z) = \left(\frac{\phi''}{\phi'}\right)' - \frac{1}{2}\left(\frac{\phi''}{\phi'}\right)^2; \qquad \phi = \phi(z),\ z\in\Omega\ ,$$

is the Schwarzian derivative of $\phi(z)$.

From (4.4) we have

$$|l_{\Omega^*}(\omega,\tau)|^2 \leqq K_{\Omega^*}(\omega,\bar\omega)K_{\Omega^*}(\tau,\bar\tau)\ ,$$

and therefore, using (4.5) and (4.6),

$$\left|l_\Omega(z,\zeta) - \frac{1}{6\pi}S_\phi(z,\zeta)\right|^2 \leqq K_\Omega(z,\bar z)K_\Omega(\zeta,\bar\zeta)\ .$$

Consequently,

$$|S_\phi(z,\zeta)| \leqq 6\pi\{[K_\Omega(z,\bar z)K_\Omega(\zeta,\bar\zeta)]^{1/2} + |l_\Omega(z,\zeta)|\}\ .$$

In view of (4.1), (4.2) and (4.3) we therefore have

$$|S_\phi(z,\zeta)| \leqq 6C_D(z)C_D(\zeta)\left[1+\left(1-\frac{\alpha^2(\zeta)}{C_D^2(\zeta)}\right)^{1/2}\right].$$

This is the desired result. If now we use (2.2), we arrive at our main theorem:

THEOREM 2. *Let $\Omega \notin 0_{AD}$. If ϕ is holomorphic and univalent in Ω we have the following sharp string of inequalities*

$$\begin{aligned} |S_\phi(z,\zeta)| &\leqq 6C_D(z)C_D(\zeta)\left[1+\left(1-\frac{\alpha^2(\zeta)}{C_D^2(\zeta)}\right)^{1/2}\right] \\ &\leqq 6C_B(z)C_B(\zeta)\left[1+\left(1-\frac{\alpha^2(\zeta)}{C_B^2(\zeta)}\right)^{1/2}\right] \\ &\leqq 6C_\beta(z)C_\beta(\zeta)\left[1+\left(1-\frac{\alpha^2(\zeta)}{C_\beta^2(\zeta)}\right)^{1/2}\right] \end{aligned}$$

$$\leqq 6\lambda_\Omega(z)\lambda_\Omega(\zeta)\left[1+\left(1-\frac{\alpha^2(\zeta)}{\lambda_\Omega^2(\zeta)}\right)^{1/2}\right],$$

Proof. The above holds for regular regions. The general case is obtained by a canonical exhaustion.

COROLLARY 3. *Let $\Omega \notin 0_G$ and let ϕ be holomorphic and univalent in Ω. Then*

$$|S_\phi(z,\zeta)| \leqq 6\lambda_\Omega(z)\lambda_\Omega(\zeta)\left[1+\left(1-\frac{\alpha^2(\zeta)}{\lambda_\Omega^2(\zeta)}\right)^{1/2}\right]; \qquad z,\zeta\in\Omega,$$

and in particular

$$|S_\phi(z)| \leqq 6\lambda_\Omega^2(z)\left[1+\left(1-\frac{\alpha^2(z)}{\lambda_\Omega^2(z)}\right)^{1/2}\right]; \qquad z\in\Omega.$$

The inequalities are sharp. The inequality

$$|S_\phi(z)| \leqq 6\lambda_\Omega^2(z)$$

is sharp only when Ω is a disk less (possibly) a closed subset of innear capacity zero. Otherwise, we have the sharp inequality

$$|S_\varphi(z)| \leqq 12\lambda_\Omega^2(z).$$

Proof. This follows from the fact that $\alpha(z)=\alpha(z\colon\Omega)\leqq\lambda_\Omega(z)$ and equality at all points $z\in\Omega$ holds if and only if Ω is a disk less (possibly) a closed subsets of inner capacity zero.

This generalizes the contents of Propositions 1 and 2.

Added in proof. The author has learned A. F. Beardon and F. W. Gehring have recently also generalized the contents of the present Proposition 2.

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