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ANALYTIC SUBGROUPS OF AFFINE ALGEBRAIC GROUPS, II

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Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

Let H be Zariski-dense analytic subgroup of the connected linear complex algebraic group G. It is known that there is a torus T in G with $G = HT$ and $H \cap T$ discrete in H. This paper gives equivalent conditions for $H \cap T$ to be trivial, and considers the connection between these conditions and left algebraic group structures on H induced from the coordinate ring of G.

Let G be a connected linear complex algebraic group, and let H be a Zariski-dense analytic subgroup of G which is integral in the sense of [2, Defn. 1, p. 386]. In [10, Thm. 3] it was shown that there exists an algebraic torus T in G with $G = HT$ such that the Lie algebra of T is a vector space complement to the Lie algebra of H in the Lie algebra of G; T is called a *complementary torus* to H in G. The principal results of this paper deal with conditions under which such a complementary torus meets H trivially. The existence of such a torus is connected, by [10, Prop. 6] and [10, Prop. 7], to left algebraic group structures on H in the sense of [8, Defn. 2.1].

We recall some terminology: let H be an analytic group, let f be an analytic function on H, and let x be in H. Then $x \cdot f$ (respectively, $f \cdot x$) is the function on H whose value at y is $f(yx)$ (respectively, $f(xy)$). f is *representative* if $\{x \cdot f \mid x \in H\}$ spans a finite-dimensional vector space, and $R(H)$ denotes the Hopf algebra of all representative functions on H [5]. A representative function f on H is *semi-simple* if the representation of H on the span of $\{x \cdot f \mid x \in H\}$ is semi-simple, and $R(H)_s$ denotes the subalgebra of all semi-simple representative functions on H [5]. An *analytic left algebraic group structure* on H is a finite-type C-subalgebra A on $R(H)$ such that (1) if $f \in A$ and $x \in H$, $f \cdot x \in A$, and (2) evaluations at element of H correspond bijectively to C-algebra maps from A to C [8, Defn. 2.1]. A *nucleus* of H is a closed, solvable, simply connected normal subgroup K such that H/K is reductive [6, p. 112]. An *additive character* of H is a homomorphism from H to the additive analytic group C and $X^+(H)$ is the free abelian group of additive characters of H. H is an *FR* group if H has a faithful finite-dimensional representation; if V is the space of such a representation then H is a Zariski-dense analytic subgroup of the Zariski-closure of H in $\mathrm{GL}(V)$ which is an algebraic group.

$L(H)$ is the Lie algebra of H. If G is a complex algebraic group, $k[G]$ denotes the affine coordinate ring of G, and we refer to the complex topology on G as the strong topology.

Let H be a connected, commutative FR analytic group. By [10, Lemma 1], $H = V \times T$, where V is a complex vector group and T is a multiplicative torus. Then T is the intersection of the kernels of the additive characters of H, so T is the unique maximal torus of H.

LEMMA 1. *Let H be a connected commutative FR analytic group and let T be the unique maximal torus of H.*

(1) *A closed analytic subgroup U of H is a nucleus if and only if $H = UT$ with $U \cap T = \{e\}$.*

(2) *There is a one-one correspondence between the nuclei of H and the vector space complements to $L(T)$ in $L(H)$.*

Proof. Write $H = V \times T$ with V a complex vector group, and let $p_1 \colon H \to V$ and $p_2 \colon H \to T$ be the projections. Let U be a nucleus of H. Then $p_1(U)$ is a vector subgroup of V. If $p_1(U) \neq V$, there is a nonzero linear functional p on V with $p(p_1 U) = 0$. Let $q = pp_1$. Then q is an additive character of H with $q(U) = 0$, so q induces an onto additive character of H/U. Since H/U is reductive and commutative, H/U is a torus and hence has no surjective additive characters. Thus $p_1(U) = V$ and hence $T \to H/U$ is onto. Then there is a subtorus T_1 of T such that $L(T_1) \to L(H/U)$ is an isomorphism. Since T_1 and H/U are tori, $T_1 \to H/U$ is algebraic, and hence an isogeny. In particular, $T_1 \cap U$ is finite. But U is simply connected, and hence a vector group, so $T_1 \cap U = \{e\}$. Then $H = T_1 \times U$, so T_1 is a maximal torus of H, so $T = T_1$. This establishes half of (1). For the other half, if $H = UT$ with $U \cap T = \{e\}$, then $L(H) = L(U) \oplus L(T)$. Since $\mathrm{Ker}(\exp_H)$ is contained in T, $\exp_H \colon L(U) \to U$ is an isomorphism, so U is simply connected. U is solvable and normal, and H/U is isomorphic to T, and hence reductive, so U is a nucleus of H, and (1) obtains.

For (2), if U is a nucleus of H then (1) implies that $L(H) = L(U) \oplus L(T)$. Conversely, if $L(H) = M \oplus L(T)$, let $U = \exp_H(M)$. Then $\exp_H^{-1}(U) = M + \mathrm{Ker}(\exp_H)$ is closed in $L(H)$, so U is a closed analytic subgroup of H. Also, $UT = H$ since $L(H) = M \oplus L(T)$ and $M = L(U)$. If $x \in U \cap T$, there are $m \in M$ and $y \in L(T)$ with $\exp_H(m) = \exp_H(y) = x$. Then $m - y \in \mathrm{Ker}(\exp_H) \subseteq L(T)$ so $m \in L(T) \cap M = 0$ and $x = e$. Thus $U \cap T = \{e\}$ and by (1), U is a nucleus of H.

LEMMA 2. *Let H be a connected analytic group, let R be the radical of H, and let K be a nucleus of H.*

(1) *$(H, R) \subseteqq K \subseteqq R$ and K is a nucleus of R.*

(2) *If L is a closed simply connected normal subgroup of H with $L \subseteqq K$, then K/L is a nucleus of H/L.*

Proof. (1) K is contained in R since K is a connected closed solvable normal subgroup of H. Let $f: H \to H/K$ denote the projection. Then $f(R)$ is the radical of the reductive group H/K, so $e = (H/K, f(R)) = f(H, R)$ and (H, R) is contained in K. Also, K is a closed, simply connected solvable normal subgroup of R, and R/K is the radical of H/K. Since H/K is reductive, its radical is a torus so K is a nucleus of R. (2) K/L is a closed, simply connected solvable normal subgroup of H/L and $H/L/K/L = H/K$ is reductive, so K/L is a nucleus of H/L.

The preceding lemmas combine with [10, Thm. 10] to yield the following characterization of nuclei.

THEOREM 3. *Let H be a connected FR analytic group, let R be the radical of H, let $\bar{R} = R/(H, R)$, let T be the unique maximal torus of $\bar{R}$, and let $f: R \to \bar{R}$ be the canonical map. Then the nuclei of H are the groups $f^{-1}(U)$, where U is a closed analytic subgroup of $\bar{R}$ with $\bar{R} = UT$ and $U \cap T = \{e\}$.*

Proof. [10, Thm. 10] shows that (H, R) is closed in H and that every $f^{-1}(U)$ is a nucleus. Conversely, if K is a nucleus of H then by Lemma 2, part (1), $(H, R) \subseteqq K \subseteqq R$ and K is a nucleus of R. By Lemma 2, part (2), with $L = (H, R) \subseteqq R$, $K/(H, R)$ is a nucleus of $\bar{R}$, and by Lemma 1, $K/(H, R) \times T = H$, so $K = f^{-1}(K/(H, R))$ is of the desired form.

Theorem 3 allows us to improve [10, Thm. 10] somewhat:

COROLLARY 4. *Let G be a connected linear algebraic group, H a Zariski-dense analytic subgroup of G and K a nucleus of H. Then there is a reductive subgroup Q of H Zariski-closed in G and a complementary torus T of H in G such that $H = KQ$ with $K \cap Q = \{e\}$ and $(T, Q) = \{e\}$.*

Proof. [10, Thm. 10] establishes the existence of T and Q when K has the form $f^{-1}(U)$ as in Theorem 3, and Theorem 3 shows that K always has this form.

Also, Theorem 3 and Lemma 1 show that, in the notation of Theorem 3, the set of nuclei of H corresponds bijectively to the set of vector space complements to $L(T)$ in $L(\bar{R})$. We now show that this latter set carries the structure of a complex vector space.

To simplify notation, let V be a finite-dimensional complex vector space and let W be a subspace of V. Let S be the set of vector space complements to W in V. Fix M_0 in S, and let M be any element of S. Let $p: V \to M_0$ and $q: V \to W$ be the projections. Since $\mathrm{Ker}(p) = W$ and $W \cap M = 0$, $\mathrm{Ker}(p|M) = 0$. Since $\dim M = \dim V - \dim W = \dim M_0$, $p|M$ is an isomorphism. Let $f_M = q \circ (p|M)^{-1}: M_0 \to W$. Then $M = \{m + f_M(m) \mid m \in M_0\}$, and $M \to f_M$ is a bijection between S and $\mathrm{Hom}_C(M_0, W)$. Thus S carries the structure of a complex vector space.

We relate this calculation to sets of nuclei:

THEOREM 5. *Let H be a connected FR analytic group, and let R be the radical of H. Then the set of nuclei of H is a complex vector space of dimension $rd - r_1 d - d^2$, where $r = \dim(L(R))$, $r_1 = \dim L((H, R))$ and $d = \mathrm{rank}(X^+(H))$.*

Proof. Let $\bar{R} = R/(H, R)$ and let T be the maximal torus of $\bar{R}$. As noted above, the set of nuclei corresponds bijectively to the set of vector space complements to $L(T)$ in $L(\bar{R})$ by Theorem 3 and Lemma 1. Let U be a vector subgroup of $\bar{R}$ with $\bar{R} = U \times T$. By the above considerations, the set of vector space complements to $L(T)$ in $L(\bar{R})$ is in bijection with $\mathrm{Hom}_C(L(U), L(T))$. Let $\bar{H} = H/(H, R)$. Then $\bar{H}$ is also FR (since (H, R) is normal and Zariski-closed in any linear algebraic group in which H is a Zariski-dense analytic subgroup), and $\bar{H} = \bar{R}S$ where S is semi-simple since $\bar{R}$ is the radical of $\bar{H}$. Also, $\bar{R}$ is central in $\bar{H}$, so that $\bar{R} \cap S$ is central in S, and since S is semi-simple and FR, the center of S is finite. Thus $\bar{R} \cap S$ is finite. But every element of finite order of $\bar{R}$ lies in T, so $\bar{R} \cap S \subseteq T$ and $\bar{H} = U \times (TS)$. Now $X^+(H) = X^+(\bar{H})$, and since TS is reductive, $X^+(\bar{H}) = X^+(U)$. Thus $\dim(U) = \dim(L(U)) = \mathrm{rank}(X^+(H)) = d$. Also $\dim(L(T)) = \dim(L(\bar{R})) - \dim(L(U)) = \dim(L(R)) - \dim(L(H, R)) - \dim L(U) = r - r_1 - d$. Thus $\dim(\mathrm{Hom}_C(L(U), L(T)) = (r - r_1 - d)d$.

A similar description of the set of nuclei as a vector space was obtained by other means in [9, Cor. 2.2].

We now consider some further implications of Corollary 4. Let G be a connected linear algebraic group, H a Zariski-dense analytic subgroup, K a nucleus of H, Q a reductive subgroup of G, and T a complementary torus of H in G such that $H = KQ$ with $K \cap Q = \{e\}$, and $(T, Q) = \{e\}$, as in the corollary. Then $P = TQ$ is a reductive subgroup of G with $(P, P) = (Q, Q)$. We show now that P contains a complementary torus T'' of H in G with $T'' \cap Q = \{e\}$.

PROPOSITION 6. *Let P be a reductive algebraic group and let Q be a reductive algebraic subgroup with $(P, P) = (Q, Q)$. Then there is an algebraic torus T'' in P such that $P = QT''$ with $Q \cap T'' = \{e\}$.*

Proof. Let $S = (Q, Q) = (P, P)$. We note that S and Q are normal in P. Let R_0 be the radical of P and R_1 be the radical of Q, so R_1 is contained in R_0. Let T_1 be a torus in S containing the center of S [1, Cor. 11.1, p. 270]. Then $T_0 = R_0T_1$ is a torus in P and $T_1 \subseteq T_0 \cap S$. We claim that $T_0 \cap S = T_1$: for if $rt \in T_0 \cap S$ with $r \in R_0$ and $t \in T_1$, then $r \in R_0 \cap S$ which is central in S so $r \in T_1$ and $rt \in T_1$. Let $T_0' = R_1T_1$. Then $T_1 \subseteq T_0' \cap S \subseteq T_0 \cap S = T_1$, so $T_0' \cap S = T_1$. Also, T_1 is a subtorus of T_0', so by [1, Cor., p. 206] there is a torus T' in T_0' so that $T_0' = T_1T'$ and $T' \cap T_1 = \{e\}$. Since $T' \cap S \subseteq T_0' \cap S = T_1$, $T' \cap S = \{e\}$. Since T_0' is a subtorus of T_0, by [1, Cor., p. 206] again there is a torus T'' in T_0 so that $T_0 = T_0'T''$ and $T_0' \cap T'' = \{e\}$. Since $T'' \cap S \subseteq T_0 \cap S = T_1$ and $T_1 \subseteq T_0'$ so $T'' \cap T_1 = \{e\}$, then $T'' \cap S = \{e\}$. Let $T = T'T''$. Then $T_0 = T_1T$ and $T_1 \cap T = \{e\}$. Let $x = t_1t_2$ be in $T \cap S$ with $t_1 \in T'$ and $t_2 \in T''$. Then $x \in T_0 \cap S = T_1$ so $x \in T \cap T_1 = \{e\}$ and $t_1 = t_2^{-1}$. Since $T' \cap T'' \subseteq T_0' \cap T'' = \{e\}$, $t_1 = e$. Then $T \cap S = \{e\}$. Now $Q = R_1S$ so $Q = T_0'S$, and $T_0' = T'T_1$ with $T_1 \subseteq S$, so $Q = T'S$. Similarly, since $P = R_0S$, $P = T_0S$ and $T_0 = T_0^1T'' = T_1T'T'' = T_1T$ with $T_1 \subseteq S$, $P = TS$. Since $T = T''T'$, $P = T''(T'S) = T''Q$. Now let $x \in T'' \cap Q$. Since $x \in Q$ and $Q = T'S$, $x = ts$ with $t \in T'$ and $s \in S$. Then $s = t^{-1}x$ is in $T'T'' = T$ and in S, and weshowed above that $T \cap S = \{e\}$. Thus $s = e$ and $x = t$ is in T'. But x is also in T'' and $T'' \cap T' = \{e\}$ so $x = \{e\}$. Thus $Q \cap T'' = \{e\}$, and the proposition follows.

COROLLARY 7. *Let G be a connected linear algebraic group, H a Zariski-dense analytic subgroup of G and K a nucleus of H. Then there is a reductive subgroup Q of H Zariski-closed in G and a complementary torus T'' to H in G such that $H = KQ$ with $K \cap Q = \{e\}$, $T'' \cap Q = \{e\}$, and T'' normalizes Q.*

Proof. Let Q and T be as in Corollary 4. Let $P = TQ$. Then P is reductive and $(P, P) = (Q, Q)$. Let T'' be as in Proposition 6. Then $G = HT = KQT = KP = KQT'' = HT''$, and $L(G) = L(H) \oplus L(T) = L(K) \oplus L(Q) \oplus L(T) = L(K) \oplus L(P) = L(K) \oplus L(Q) \oplus L(T'') = L(H) \oplus L(T'')$, so T'' is a complementary torus to H in G, and Q and T'' possess the desired properties.

The examples following [10, Thm. 3] show that, in the notation of Corollary 7, it is not always possible to find a T'' with $T'' \cap H = \{e\}$. Complementary tori with this property are connected to left algebraic group structures on H [10, Prop. 6], and we now examine when such exist.

We will need to use some facts about representation theory in this examination. We fix the following terminology: if H is an analytic group, an H-module V is a finite-dimensional complex vector space with an analytic left H-action; we let $r_V: H \to \mathrm{GL}(V)$ be the corresponding representation. The associated semi-simple module to V is the direct sum V' of the H-module composition factors of H; we let $r'_V = r_{V'}$ and call r'_V the associated semi-simple representation. If H is an analytic subgroup of the analytic group G, then every G-module is, by restriction, an H-module. In this case, if V is an H-module, we say that r_V extends to G if there is a G-module W containing V as an H-submodule. In [3], a criterion is given for determining when a representation of H extends to G in the case H is a normal semi-direct factor of G.

THEOREM 8. *Let G be a connected linear algebraic group and H a Zariski-dense analytic subgroup of G. Then the following are equivalent*:

(1) *Every additive character of H is the restriction of an additive character of G.*

(2) *There is a complementary torus T to H in G with $T \cap H$ finite.*

(3) *There is a complementary torus T'' to H in G with $T'' \cap H = \{e\}$.*

(4) *Every nucleus of H is a nucleus of G.*

(5) *Every H-module is an H-submodule of a G-module.*

Proof. (1) and (2) are equivalent by [10, Thm. 3] and (3) implies (2) trivially. We next show that (2) implies (4). Let K be a nucleus of H and let T, Q be as in Corollary 4. Let $P = TQ$. Then P is a reductive subgroup of G and $G = KP$. We assume $T \cap H$ is finite. By [10, Thm. 3], H is strongly closed in G, and hence K is a strongly closed simply connected analytic subgroup of G. K is normal in G since H is Zariski-dense in G and K is normal in H. $K \cap P$ is solvable and normal in P, so $K \cap P$ is contained in the center Z of the reductive group P. $Z = TZ'$, where Z' is the center of Q. Let $x \in K \cap P$. Then $x = tq$ where $t \in T$ and $q \in Z'$. Since $x \in H$ and $q \in H$, $t \in T \cap H$. Let n be the order of $T \cap H$. Then $x^n = q^n$, so $q^n \in K \cap Q = \{e\}$. Let ${}_nZ'$ denote the n-torsion in Z'. Then ${}_nZ'$ is finite since Q is reductive, and $K \cap P \subseteq (T \cap H)({}_nZ')$ so $K \cap P$ is finite. Since K is simply connected, $K \cap P = \{e\}$. Thus $G/K = P$ is reductive, so K is a nucleus of G and (2) implies (4). We now show that (4) implies (3). Let K, T, Q, P be as above. Since K is then a nucleus of G, $G/K = P/K \cap P$ is reductive. The analytic map $f: P \to P/K \cap P$ is then a morphism of algebraic groups by [10, Lemma A1]. Since f induces an isomorphism on Lie algebras, it follows that f has finite kernel, i.e., $P \cap K$ is finite, and since K is simply connected, $P \cap K = \{e\}$. Thus $G = KP$ with $P \cap K = \{e\}$. By Proposition 6, $P = QT''$ with $T'' \cap Q = \{e\}$. It follows that $G = KQT'' = HT''$ and $T'' \cap H = \{e\}$, so (4) implies (3). We next show that (3) implies (5): Assume condition (3) holds; i.e., $G = HT$ with T a torus in G with $T \cap H = \{e\}$. Let V be an H-module and let $r = r_V$ be the corresponding representation. Let R be the radical of H. By [3, Thm. 2.2, p. 215], V is an H-submodule of a G-module if and only if $r'((G, R)) = 1$. We claim that $(G, R) = (H, R)$. First, (H, R) is contained in the unipotent radical of G, so (H, R) is Zariski-closed in G. Let $(\ )_c$ denote Zariski-closure. Then $(H, R)_c = (H_c, R_c)$ by [1, Prop., p. 108]. Thus $(G, R) = (H_c, R) \subseteq (H_c, R_c) = (H, R)_c = (H, R)$ and it follows that $(G, R) = (H, R)$. Since (H, R) acts trivially on simple H-modules, $r'((G, R)) = r'((H, R)) = 1$, so every H-module is an H-submodule of a G-module. Finally, we show that (5) implies (1): Let $f \in X^+(H)$, $f \neq 0$ and let V be the two dimension complex space with basis e_1, e_2 and let H act on V by $he_1 = e_1$ and $he_2 = f(h)e_1 + e_2$ for $h \in H$. Then V is an H-module. Let W_0 be a G-module containing V as an H-submodule. Let K be the kernel of f. Since $(G, G) = (H, H)$ is contained in K, K is normal in G. Let $W = \{x \in W_0 \mid kx = x \text{ for all } k \text{ in } K\}$. Since K is normal in G, W is a G-submodule of W_0 and W contains V. Let $\bar{H} = r_W(H)$ and $\bar{G} = r_W(G)$. W is a $\bar{G}$- and $\bar{H}$-module, and since $K \subseteq \operatorname{Ker}(r_W)$ and $(G, G) \subseteq K$, $\bar{G}$ is abelian. Let T be the unique maximal torus of $\bar{G}$. If every additive character of $\bar{G}$ vanishes on $\bar{H}$, then $\bar{H}$ is contained in T. W is semi-simple as a T-module, so W is semi-simple as an $\bar{H}$-module, if $\bar{H} \subseteq T$. But then V is also semi-simple as an $\bar{H}$-module, hence as an H-module, so $f = 0$, contrary to assumption. Thus there is an additive character of $\bar{G}$ which is not trivial on $\bar{H}$. This character defines an additive character g on G whose kernel contains K but whose restriction to H is not trivial. Let g_1 be the restriction of g to H. Both g_1 and f induce isomorphisms $H/K \to \boldsymbol{C}$, so there is a nonzero scalar α such that $\alpha g_1 = f$. It follows that f is the restriction of ag to H, and $ag \in X^+(G)$. Thus (5) implies (1), and Theorem 8 is complete.

Condition (3) of Theorem 8 is related to the existence of analytic left algebraic group structures on H by [10, Prop. 6] and [10, Prop. 7]. Thus the other conditions, especially condition (1), are also so related, as the following corollary makes precise.

COROLLARY 9. *Let H be an FR analytic group, and B a Hopf-*

subalgebra of $R(H)$ a finite type over C. Then the following are equivalent:

(1) *B contains an analytic left algebraic group structure on H.*

(2) *B separates the points of H and contains $X^{+}(H)$.*

Proof. Assume (1) and let A be the left algebraic group structure. Let B' be the smallest sub-Hopf-algebra of $R(H)$ containing A, and let G' be the algebraic group with $k[G'] = B'$. By [10, Prop. 7], H is a Zariski-dense and strongly closed analytic subgroup of G' and there is a complementary torus T' to H in G' with $T' \cap H = \{e\}$. By Theorem 8, every additive character of H extends to G'. Since additive analytic characters of algebraic groups are algebraic, the additive characters are in $k[G'] = B'$. Thus $X^{+}(H) \subseteqq B' \subseteqq B$. By definition, A separates points of H, hence so does B, so (1) implies (2). Conversely, assume (2). Let G be the algebraic group with $k[G] = B$. Then H becomes a Zariski-dense analytic subgroup of G. Let $f \in X^{+}(H)$. Then f is a primitive element of $R(H)$: i.e., the comultiplication sends f to $f \otimes 1 + 1 \otimes f$, so f is primitive in B and hence defines an additive character of G. By Theorem 8, there is a complementary torus T to H in G with $T'' \cap H = \{e\}$. By [10, Prop. 6], $A = B^{T}$ is an analytic left algebraic group structure on H and A is contained in B so (2) implies (1).

Let H be an analytic group and A a subgroup of $R(H)$. We recall that $A_s = A \cap R(H)_s$ denotes the semi-simple representative functions in A. If A is a left algebraic group structure on H, A is said to be *normal basic* if for every f in A_s and x in H, $x \cdot f$ and $f \cdot x$ are in A_s [6, p. 116], and a sub-Hopf-algebra of $R(H)$ of finite type over C is *regular* if it contains a normal basic left algebraic group structure on H [7, p. 873]. We will now interpret this concept in terms of complementary tori. The following lemma determines the semi-simple part of the coordinate ring of an algebraic group.

LEMMA 10. *Let G be a connected linear complex algebraic group and let U be its unipotent radical. Then $k[G]_s = k[G]^{U}$.*

Proof. $k[G]^{U} = k[G/U]$ and since G/U is reductive, $k[G/U]_s = k[G/U]$. Thus $k[G]^{U}$ is contained in $k[G]_s$. Conversely, let $f \in k[G]_s$, let $V = \langle x \cdot f \mid x \in G \rangle$ and let $r = r_V$ be the associated representation. Since V is semi-simple, U is in the kernel of r. Since $f \in V$, $x \cdot f = r(x)f = f$ for all x in U, so f is in $k[G]^{U}$. Thus $k[G]_s$ is contained in $k[G]^{U}$ and the result follows.

THEOREM 11. *Let G be a connected linear algebraic group and H a Zariski-dense analytic subgroup of G. Let Q be a maximal reductive subgroup of H. Then the following conditions are equivalent:*

(1) *Every additive character of H is the restriction of an additive character of G, and there is a normal algebraic subgroup L of G such that $LQ = G$ and $L \cap Q = \{e\}$.*

(2) *There is a complementary torus T to H in G with $T \cap H = \{e\}$ and $(T, Q) = \{e\}$.*

(3) *$k[G]^T$ is a normal basic left algebraic group structure on H for some complementary torus T to H in G.*

(4) *$k[G]$ is a regular sub-Hopf-algebra of $R(H)$.*

Proof. Assume condition (1) and let $g: G \to G$ be the algebraic endomorphism with $\mathrm{Ker}(g) = L$ and $g(x) = x$ for all x in Q. Let $K = L \cap H$. Then K is the kernel of the restriction of g to H, and $H = KQ$ with $K \cap Q = \{e\}$, so K is a connected closed normal subgroup of H. By [10, Thm. 10], $K = K_0Q_0$ where K_0 is a nucleus of K and Q_0 is a reductive subgroup of K with $Q_0 \cap K_0 = \{e\}$. Since Q is maximal reductive in H, some conjugate of Q_0 is contained in Q: then there is an $x \in H$ with $xQ_0x^{-1} \subseteqq Q$. But $xQ_0x^{-1} \subseteqq K$ so $Q_0 = \{e\}$ and $K = K_0$. Thus K is simply connected and hence a nucleus of H. Let $\bar{K}$ be the Zariski-closure of K in G. Then $\bar{K} \subseteqq L$, and $\bar{K}Q$ is Zariski-closed in G. Since $H \subseteqq \bar{K}Q$, and H is Zariski-dense in $\bar{K}Q = G$, it follows that $\bar{K} = L$. In particular, L is solvable. Since every additive character of H extends to G, Theorem 8 implies that K is a nucleus of G. Let P be a (necessarily maximal) reductive subgroup of G such that $G = KP$ with $K \cap P = \{e\}$. If necessary, we replace P by a conjugate so that $Q \subseteqq P$. Let $T = L \cap P$. Then $P = TQ$ with $T \cap Q = \{e\}$, and T is a closed connected normal algebraic subgroup of P which is solvable since L is solvable. It follows that T is a torus with $(T, Q) = \{e\}$, and $G = KP = KTQ = HT$ with $T \cap H \subseteqq K \cap P = \{e\}$. Thus condition (2) obtains.

Now assume T is as in condition (2). By [10, Prop. 6], $A = k[G]^T$ is an analytic left algebraic group structure on H. We need to show if $f \in A_s$ and $x \in H$, then $x \cdot f$ and $f \cdot x$ are in A_s. Let U be the unipotent radical of G and let $L = UT$. By Lemma 10, $A_s = k[G]^L$. Let K be a nucleus of H. Then $G = HT = KQT$ and it follows that QT is a maximal reductive subgroup of G. By [4, Thm. 14.2, p. 96], $G = UQT = LQ$, and Q normalizes U and T so L is normal in G. Thus if $f \in k[G]^L$ and $x \in G$, $x \cdot f$ and $f \cdot x$ are in $k[G]^L$. So condition (3) obtains.

Condition (3) implies condition (4) by definition, and condition (4) implies condition (1) by [7, Thm. 2.1, p. 875].

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