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SUPERALGEBRAS

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Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

1. Introduction. The theory of graded Lie algebras, now more widely called Lie superalgebras, underwent a very rapid development starting about 1973, inspired by the interest expressed in the sudject by physicists. I was active in the field for about a year, during 1975 and 1976. Thus far I have published only the announcement [16] (jointly with Peter Freund of Chicago's Physics Department, to whom I am enormously indebted); in addition, the summary [29] is to appear.

The present mature state of the field, and the fact that Hochschild (partly in collaboration with Djoković) made several important contributions, make this an appropriate occasion to publish some further details. Although Victor Kac has brilliantly solved the main problems, there remains the possibility that the different methods I used retain some independent interest.

The large bibliography is intended to be complete on mathematical references not contained in [9]; there is also a selection of physics papers. I hope this bibliography will be useful to some readers.

This article is written so as to keep the overlap with [29] to a minimum.

2. Invariant forms. When I began studying Lie superalgebras I imitated [46] and selected as an initial goal the classification of those simple Lie superalgebras (over an algebraically closed field of characteristic 0) that admit a suitable invariant form.

For basic definitions and facts about Lie superalgebras, I refer to [25]. I shall just recall that if ϕ is a superrepresentation of the Lie superalgebra L then

$$(x, y) = \mathrm{STr}\,(\phi(x)\phi(y))$$

is an invariant form on L, where STr denotes the supertrace. This can be extended to "projective representation", following the model of [28, p. 66], but since the setup will shortly be axiomatic anyway, I shall not pursue the details here.

Assume now that the form ψ on L induced by ϕ is nondegenerate. Write $L = L_0 + L_1$, with L_0 and L_1 the even and odd parts of L. We have that ψ is symmetric on L_0, skew on L_1, and that L_0 and L_1 are orthogonal relative to ψ. It follows that ψ remains nondegenerate when restricted to L_0. Hence L_0 is the direct sum of a semisimple algebra and an abelian algebra.

Since the assumption that the form comes from a representation plays no further role in the investigation it is feasible to weaken the hypothesis by assuming outright that L admits an invariant form and that L_0 is semisimple $\oplus$ abelian. We assume that L is simple.

3. Cartan decomposition. The role of a Cartan subalgebra of L is satisfactorily played by a Cartan subalgebra H of L_0. The decomposition of L_0 relative to H is fully known, for the abelian part of L_0 creates minimal interference. So the even roots and root spaces have standard properties.

The decomposition of L_1 relative to H creates odd roots and root spaces with properties not quite so standard. Odd roots may be isotropic. Also, two-dimensional root spaces are possible; but this happens only in one algebra: the 14-dimensional projective special linear algebra of 4×4 matrices. In this algebra there moreover exist odd roots λ, μ with $(\lambda, \mu) \neq 0$ and $\lambda + \mu$, $\lambda - \mu$ both (even) roots. This is again a unique exception and will be ruled out in the axioms about to be given.

4. Axioms for roots. The system of roots that has arisen can now be treated axiomatically. We postulate a finite-dimensional vector space V over a field of characteristic 0. V is equipped with a nondegenerate symmetric form (,). In V a finite set Γ of non-zero vectors is given; we call the members of Γ "roots". Γ is a disjoint set-theoretic union of two subsets whose members we call "even" and "odd". There are seven axioms.

1. Γ spans V.
2. Along with any vector Γ contains its negative. A root and its negative have the same parity.
3. The even roots in Γ constitute the system of roots of an (ordinary) semisimple Lie algebra. (The form on each simple component is a scalar multiple of the Killing form, the scalar varying with the component.)
4. For any two non-orthogonal odd roots the sum or the difference is a root, but not both.

REMARK. It is probably feasible to classify the larger class of root systems that arise if the phrase "but not both" is deleted; I have not tried, since no application is in sight.

In the final three axioms α is an even root and λ is an odd isotropic root.

5. $2(\alpha, \lambda)/(\alpha, \alpha) = 0, \pm 1$, or ± 2.
6. If $2(\alpha, \lambda)/(\alpha, \alpha) = -1$ then $\lambda + \alpha$ is a root.
7. If $2(\alpha, \lambda)/(\alpha, \alpha) = -2$ then $\lambda + \alpha$ and $\lambda + 2\alpha$ are roots.

$\lambda + \alpha$ is odd.

The roots in the Lie superalgebras of §2 (i.e., with an invariant form, and an even part which is semisimple $\oplus$ abelian) satisfy these axioms, with the solitary 14-dimensional exception mentioned in §3.

5. The structure theorem. Indecomposable systems satisfying these axioms were classified in a piece of work I completed in August 1975. The result can today be stated briefly. One gets precisely the systems attached to the following simple Lie superalgebras: special linear, orthosymplectic, and the exceptional algebras of dimensions 17, 31, and 40. The proof was elementary but long.

It is a routine matter to exhibit these root systems, so two samples will suffice.

Special linear. Take an orthogonal direct sum $X \oplus Y$ where X has an orthonormal basis $e_1, \cdots, e_m$ and Y has a negative orthonormal basis $f_1, \cdots, f_n$ (this means that the f's are orthogonal and each $(f_j, f_j) = -1$). The even roots consist of all $e_i - e_r$ and $f_j - f_s$ $(i \neq r, j \neq s)$. The odd roots are the $2mn$ vectors $\pm(e_i + f_j)$.

$G(3)$, the 31-dimensional algebra. Let p, q, r be vectors satisfying $(p, p) = (q, q) = (r, r) = -2$, $(q, r) = (r, p) = (p, q) = 1$. Let f be a vector perpendicular to p, q, r satisfying $(f, f) = 2$. The roots are as follows (the negatives are to be inserted as well).

Even: $p, q, r, q - r, r - p, p - q, 2f$.

Odd isotropic: $f \pm p, f \pm q, f \pm r$.

Odd non-isotropic: f.

6. A model of $G(3)$. I present a model of $G(3)$ which may be useful for some purposes. Take the even part L_0 to be $G_2 \oplus A_1$ and the odd part L_1 as $C \otimes V$, where C denotes the 7-dimensional space of elements of trace 0 in a Cayley matrix algebra and V is a 2-dimensional space carrying a nonsingular alternate form (,). Let G_2 act on C in the standard way and A_1 on V as linear transformations skew relative to (,). It remains to define the multiplication $L_1 \times L_1 \to L_0$. This is done via two auxiliary maps ϕ and ψ.

$\phi: C \times C \to G_2$. This is the map which appears on page 143 of [21]:

$$\phi(c, d) = [L_c L_d] + [L_c R_d] + [R_c R_d] \ ,$$

where L and R denote left and right multiplication.

$\psi: V \times V \to A_1$. For v, w in V define $\psi(v, w)$ to be the linear transformation on V that sends x into $(x, v)w + (x, w)v$. The product from $L_1 \times L_1$ to L_0 is now defined by

$$(c \otimes v)(d \otimes w) = (v, w)\phi(c, d) + 4 \operatorname{tr}(c, d)\psi(v, w)$$

where tr denotes the trace on the Cayley matrix algebra, normalized so that $\operatorname{tr}(1) = 1$. One must of course verify the Jacobi identity.

7. **Jordan superalgebras.** For the basic facts on Jordan superalgebras, I refer to [27]. In my version of the theory, completed in June, 1976, I used the classical method of idempotents and Peirce decompositions, rather than Kac's Lie method. The key hurdle that had to be surmounted was to exclude the case where the even part is unit element plus radical (called the "nodal" case in the literature on nonassociative algebras). Here is the proof.

PROPOSITION. *Let $J = J_0 + J_1$ be a Jordan superalgebra over a field of characteristic* 0. *Let N be the radical of J_0. Assume that J has a unit element* 1 *and that every element of J_0 is an element of N plus a scalar multiple of* 1. *Then $N + NJ_1$ is an ideal in J.*

Proof. It is easy to see that $NJ_1 \cdot J_1 \subset N$ is the only nontrivial inclusion that needs verification. Thus, for $a, b \in J_1$ and $n \in N$ we need to show that $z = na \cdot b$ lies in N. Assume not. Let c be another element in J_1. We have that $R_bR_c + R_cR_b$ is a derivation of J (this is a special case of a general principle for converting algebra identities into superalgebra identities). Likewise, R_b^2 is a derivation. These derivations restrict to derivations on J_0, and by ordinary Jordan theory carry N into N (this is where characteristic 0 is used). Thus $zb = (na \cdot b)b \in NJ_1$. It follows that $b \in NJ_1$ and then that $(na \cdot c)b \in NJ_1$. Next

$$(na \cdot b)c + (na \cdot c)b \in NJ_1 \,.$$

Hence $zc \in NJ_1$. c is arbitrary in J_1 and so $zJ_1 \subset NJ_1$, $J_1 = NJ_1$, and $J_1 = 0$ by a Nakayama lemma argument. Everything is trivial if $J_1 = 0$. The proof is complete.

Added in proof (May 28, 1980). I missed some references, and many additional ones have now appeared. I have compiled a supplementary bibliography.

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