

## ZEROS OF SUCCESSIVE ITERATES OF MULTIPLIER-SEQUENCE OPERATORS

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The authors study the zero free regions of iterates of multiplier-sequence operators for (i) functions analytic in the disc  $|z| < R$  and (ii) functions analytic in  $|z| > T$ . Integral representations for the iterates of each class of functions are given. As a consequence the authors give a generalization of the Post-Widder inversion formula. Other applications include an investigation of the zero free regions for iterates of fractional integrals as well as connections between results obtained here and recent final set results for iterates of operators on balanced sums.

**1. Introduction.** Let  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  be analytic in a neighborhood of 0 and define an operator  $\theta$ , associated with a set of constants  $\{\alpha_k\}_{k=0}^{\infty}$  and an integer  $p \geq 0$ , on the power basis by  $\theta(z^k) = \alpha_k z^{k+p}$ ,  $k = 0, 1, 2, \dots$ . Thus for suitably restricted  $\{\alpha_k\}$  we may write  $(\theta f)(z) = \sum_{k=0}^{\infty} \alpha_k a_k z^{k+p}$ ,  $(\theta^2 f)(z) = \sum_{k=0}^{\infty} \alpha_k \alpha_{k+p} a_k z^{k+2p}$ , and so on. We call an operator of this type a *multiplier-sequence operator*. For example, if  $\alpha_k = k$ , then  $(\theta f)(z) = z^{k+p} f'(z)$ ; that is,

$$(1.1) \quad \theta = z^{1+p} \frac{d}{dz}.$$

In [13] (and in [14] for  $f$  analytic in a neighborhood of  $\infty$ ) the present authors studied the differential operators (1.1) and their *zero-free regions*, that is, regions  $V_n$  in the plane such that  $(\theta^n f)(z) \neq 0$  for  $z \in V_n$  and all  $n$  sufficiently large. Through change-of-variable methods, descriptions of  $V_n$  are transmitted to the classical context of zero-free regions for ordinary derivatives. For instance, when  $p = 1$  and  $F(z) = f(1/z)$ , then  $f(z) = F(1/z)$ ,  $(\theta f)(z) = z^2 f'(z) = -F'(1/z)$ , and  $(\theta^n f)(z) = (-1)^n F^{(n)}(1/z)$  in general. Therefore the regions  $V_n^{-1} = \{z^{-1}: z \in V_n\}$  are free of zeros of  $F^{(n)}$ , for large enough  $n$ . The regions  $V_n$  containing  $z = 0$  are discs which shrink with increasing  $n$  to 0. Their radii have orders of magnitude  $V_n = 1/n$ .

In the present paper we study zero-free regions for rather general multiplier-sequence operators. In particular, we realize all of the results of [13] and a portion of those of [14] as special cases. Additionally, we present a result on the successive iterates of *fractional integrals*, and we indicate a method which can be used for locating regions void of zeros of certain Special Functions (not their derivatives). Included among the applications and examples is a generalized *Post-Widder transform inversion formula* for multiplier-sequence operators. Finally, we connect our

theorems to some results in the literature ([10], [11]) on *final sets* for differential operators of a certain type.

**2. Main results.** We shall work with more general operators  $\theta$  and functions  $f(z)$  than those indicated above. Fix an integer  $M$  (positive, negative or zero) and let  $f(z)$  be given by

$$(2.1) \quad f(z) = \sum_{k=-M}^{\infty} a_k z^k, \quad 0 < |z| < R.$$

If  $M \leq 0$ ,  $z = 0$  is allowed. Let  $p$  be a nonnegative number. Let  $\alpha$  be a function which is defined on all numbers of the form  $k + jp$ , for  $k = -M, -M + 1, -M + 2, \dots$  and  $j = 0, 1, 2, \dots$ . If  $p$  is an integer,  $\alpha$  is of course a *sequence* indexed by the integers  $k \geq -M$ . We will denote the value of  $\alpha$  at  $k + jp$  by  $\alpha_{k+jp}$  even though  $p$  may not be integral. Let an operator  $\theta$  be defined by its action on powers of  $z$  by  $\theta(z^\nu) = \alpha_\nu z^{\nu+p}$ , where  $\nu$  is of the form  $k + jp$  as above. The domain of  $\alpha$  is such that the successive iterates  $\theta^n(z^\nu) = \alpha_\nu \alpha_{\nu+p} \cdots \alpha_{\nu+(n-1)p} z^{\nu+n p}$  are all defined for  $n = 1, 2, 3, \dots$ . Viewed as an analytic function  $z^{\nu=n p}$  may be interpreted as the principal branch of the power function, although this will not be important in what follows.

Applying  $\theta$  to (2.1), we have formally that  $(\theta f)(z) = \sum_{k=-M}^{\infty} \alpha_k a_k z^{k+p}$ , and introducing the notation

$$(2.2) \quad A_{k,n} = \alpha_k \alpha_{k+p} \cdots \alpha_{k+(n-1)p}, \\ k = -M, -M + 1, \dots, n = 1, 2, 3, \dots,$$

we may write, again formally,

$$(2.3) \quad (\theta^n f)(z) = \sum_{k=-M}^{\infty} A_{k,n} a_k z^{k+n p}, \quad n = 1, 2, 3, \dots$$

Note that the factors  $A_{k,n} = A_{k,n}(p)$  depend on the number  $p$ , which we omit to simplify notation.

We now state our first hypotheses and make rigorous the formula (2.3). We will assume throughout this paper that

$$(2.4) \quad \alpha_{-M} \neq 0,$$

and that

$$(2.5) \quad \alpha_k \neq 0 \text{ implies that } \alpha_{k+jp} \neq 0 \text{ for } j = 0, 1, 2, \dots,$$

where  $k = -M, -M + 1, \dots$ . We will assume that there is a first integer  $P \geq 1$  such that

$$(2.6) \quad \alpha_{-M+P} \neq 0.$$

Then it follows that

$$(2.7) \quad A_{-M,n} \neq 0, a_{-M+p,n} \neq 0 \quad n = 1, 2, 3, \dots,$$

but if  $P > 1$  then the definition of  $A_{k,n}$  implies that

$$(2.8) \quad A_{-M+1,n} = A_{-M+2,n} = \dots = A_{-M+p-1,n} = 0, \quad n = 1, 2, 3, \dots$$

Our assumptions on the growth of the function  $\alpha$  come in the form of two alternate hypotheses, the first of which we now state.

*Hypothesis I.* The set  $\{\alpha_\nu\}$  satisfies  $\alpha_{k+jp} \geq 0$  for all  $k$  and  $j$ , and

$$(2.9) \quad \frac{\alpha_{-M+p+j+np}}{\alpha_{-M+np}} \left[ \frac{\alpha_{-M+np}}{\alpha_{-M+p+np}} \right]^{P+j} \leq 1,$$

$$j = 0, 1, 2, \dots; n = 0, 1, 2, \dots$$

Under this condition, the factors  $A_{k,n}$  satisfy

$$(2.10) \quad \frac{A_{-M+p+j,n+1}}{A_{-M,n+1}} \left[ \frac{A_{-M,n+1}}{A_{-M+p,n+1}} \right]^{P+j}$$

$$= \left[ \frac{\alpha_{-M+p+j} \cdots \alpha_{-M+p+j+np}}{\alpha_{-M} \cdots \alpha_{-M+np}} \right] \left[ \frac{\alpha_{-M} \cdots \alpha_{-M+np}}{\alpha_{-M+p} \cdots \alpha_{-M+p+np}} \right]^{P+j}$$

$$\leq 1, \quad \text{for } n, j = 0, 1, 2, \dots$$

Written another way, this becomes

$$(2.11) \quad \frac{A_{-M+p+j,n}}{A_{-M,n}} \leq \left[ \frac{A_{-M+p,n}}{A_{-M,n}} \right]^{P+j}, \quad n = 1, 2, 3, \dots; j = 0, 1, 2, \dots$$

Let  $S_{-M+p+j,n+1}$  denote the term on the left of (2.10).

**LEMMA 2.1.** *For each fixed  $j$ , the sequence  $\{S_{-M+p+j,n+1}\}_{n=0}^\infty$  is convergent.*

*Proof.* First, we have  $0 \leq S_{-M+p+j,n+1} \leq 1$  by (2.10). To obtain  $S_{-M+p+j,n+1}$  from  $S_{-M+p+j,n}$  we multiply it by the factor on the left of (2.9). By this inequality, we see that  $\{S_{-M+p+j,n+1}\}$  is non-increasing for  $n \rightarrow \infty$ . This completes the proof.

We introduce now the notation

$$R_n = \frac{A_{-M,n}}{A_{-M+p,n}}, \quad n = 1, 2, 3, \dots,$$

and the function

$$H_n(w) = 1 + \sum_{j=0}^{\infty} S_{-M+P+j,n} w^{P+j}, \quad n = 1, 2, 3, \dots, |w| < 1.$$

**THEOREM 2.1.** *Assume Hypothesis I and let  $f(z)$  be given by (2.1). Then  $(\theta^n f)(z)$ , as given by (2.3), is defined for  $0 < |z| < (R_n/\rho)$  and is also given by*

(2.12)

$$(\theta^n f)(z) = (2\pi i)^{-1} A_{-M,n} z^{-M+np} \int_{|t|=\rho} (1/t)^{M+1} f(1/t) H_n(zt/R_n) dt$$

$$n = 1, 2, 3, \dots,$$

where  $(1/\rho) < R$ .

*Proof.* We define  $\theta f, \theta^2 f, \dots$  by (2.3). What we have to show is that (2.3) converges for  $0 < |z| < (R_n/\rho)$  and that (2.12) holds there. By (2.7) and (2.8), the series in (2.3) has partial sums

$$\begin{aligned} & \sum_{k=-M}^m A_{k,n} a_k z^{k+np} \\ &= A_{-M,n} a_{-M} z^{-M+np} + \sum_{j=0}^{m+M-P} A_{-M+P+j,n} a_{-M+P+j} z^{-M+P+j+np} \\ &= A_{-M,n} z^{-M+np} \left\{ a_{-M} + \sum_{j=0}^{m+M-P} \frac{A_{-M+P+j,n}}{A_{-M,n}} a_{-M+P+j} z^{P+j} \right\}. \end{aligned}$$

In view of (2.11) we have

$$\frac{A_{-M+P+j,n}}{A_{-M,n}} |z|^{P+j} \leq (|z|/R_n)^{P+j},$$

and this implies convergence of (2.3) in  $0 < |z| < R_n R$ . The convergence is uniform and absolute on compact sets.

Now define the function  $K_n(w) = \sum_{k=-M}^{\infty} A_{k,n} w^{k+M}$ ; this is defined for  $|w| < R_n$ , according to (2.11). Writing  $w = zt$ ,

$$K_n(zt) = \sum_{k=-M}^{\infty} A_{k,n} z^{k+M} t^{k+M}, \quad |zt| < R_n,$$

and the Cauchy Integral Formula yields

$$(2.13) \quad A_{k,n} z^{k+M} = (2\pi i)^{-1} \int_{|t|=\rho} \frac{K_n(zt)}{t^{k+M+1}} dt.$$

This is valid when  $|z| < R_n/\rho$ . If we require  $(1/\rho) < R$ , then  $|z| < R_n/\rho$  implies  $|z| < R_n R$  also, and so (2.13) may be substituted into (2.3) when  $0 < |z| < R_n/\rho$ . This gives

$$\begin{aligned}
 (2.14) \quad (\theta^n f)(z) &= z^{-M+np} \sum_{k=-M}^{\infty} A_{k,n} a_k z^{k+M} \\
 &= z^{-M+np} (2\pi i)^{-1} \sum_{k=-M}^{\infty} a_k \int_{|t|=\rho} (K_n(z t)/t^{k+M+1}) dt \\
 &= z^{-M+np} (2\pi i)^{-1} \int_{|t|=\rho} (1/t)^{M+1} f(1/t) K_n(z t) dt,
 \end{aligned}$$

where we have used uniform convergence of the series to integrate termwise. Note that  $f(1/t)$  is defined because  $|1/t| = (1/\rho) < R$ . In (2.14) put  $z = R_n x$ , note (2.7) and (2.8), and expand  $K_n(z t)$  to produce

$$\begin{aligned}
 (\theta^n f)(z) &= A_{-M,n} z^{-M+np} (2\pi i)^{-1} \int_{|t|=\rho} (1/t)^{M+1} f(1/t) \\
 &\quad \times \left\{ 1 + \sum_{j=0}^{\infty} \frac{A_{-M+P+j,n}}{A_{-M,n}} \left[ \frac{A_{-M,n}}{A_{-M+P,n}} x t \right]^{P+j} \right\} dt
 \end{aligned}$$

which is equivalent to (2.12).

**THEOREM 2.2** *Assume Hypothesis I, let  $f(z)$  be given by (2.1), and let  $a_{-M} \neq 0$ . Then there exists a constant  $\gamma, 0 < \gamma < (1/\rho) < R$ , such that for all  $n$  sufficiently large  $(\theta^n f)(z)$  has no zero in  $0 < |z| \leq R_n \gamma$ .*

*Proof.* Writing  $z = R_n x$  in (2.12), we have

$$\begin{aligned}
 (2.15) \quad \frac{z^{M-np} (\theta^n f)(z)}{A_{-M,n}} &= (2\pi i)^{-1} \int_{|t|=\rho} (1/t)^{M+1} f(1/t) H_n(x t) dt \\
 &= I_n(x),
 \end{aligned}$$

with  $I_n(x)$  defined in the obvious way. This is valid for  $0 < |x t| = R_n^{-1} |z t| < R_n^{-1} (R_n/\rho) \rho = 1$ , and consequently  $I_n(x)$  is defined for  $|x| < (1/\rho)$ . By Lemma 2.1 and the definition of  $H_n(w)$ , the sequence  $\{H_n(w)\}_{n=1}^{\infty}$  converges uniformly on compact subsets of  $|w| < 1$  to a function

$$H(w) = 1 + H_p w^P + H_{p+1} w^{P+1} + \dots = \sum_{m=0}^{\infty} H_m w^m.$$

Holding  $x$  to a compact subset of  $|x| < (1/\rho)$  and letting  $n \rightarrow \infty$  in (2.15), we obtain a limit function for  $\{I_n(x)\}_{n=1}^\infty$ ,

$$(2.16) \quad I(x) = (2\pi i)^{-1} \int_{|t|=\rho} (1/t)^{M+1} f(1/t) H(xt) dt, \quad |x| < (1/\rho);$$

that is

$$(2.17) \quad \lim_{n \rightarrow \infty} \left\{ \frac{(R_n x)^{M-np} (\theta^n f)(R_n x)}{A_{-M,n}} \right\} = I(x), \quad |x| < (1/\rho),$$

with uniform convergence on compact sets. An easy calculation gives

$$I(x) = \sum_{m=0}^{\infty} H_m a_{-M+m} x^m = \sum_{m=0}^{\infty} I_m x^m$$

and in particular  $I_0 = a_{-M} \neq 0$ . Thus we can find a neighborhood  $|x| \leq \gamma < (1/\rho)$  in which  $I(x) \neq 0$ . The conclusion of the theorem follows from (2.17).

If  $M \leq 0$  the requirement  $z \neq 0$  can be dropped from the theorem. Also the assumption  $a_{-M} \neq 0$  can be weakened to  $H_m a_{-M+m} \neq 0$  for some  $m$ ; i.e.,  $I(x)$  does not vanish identically. For in this case we can find a neighborhood  $|x| \leq \gamma$  in which  $I(x)$  has no zero other than (possibly)  $x = 0$ . Finally, we remark that the multiple-valued nature of the terms  $z^{np}$  for non-integral  $p$  is immaterial in the theorem, for these can always be factored out as in (2.14).

From one point of view, (2.17) may be regarded as a transform inversion formula, in the sense of the classical Post-Widder formula ([9], [15], [16]). To illustrate, let us assume that  $P = 1$  and that  $H_m \neq 0$  for all  $m$ , under Hypothesis I. Let a measure  $\mu$  of bounded variation on  $[0, \infty)$  solve the Stieltjes moment problem

$$H_k^{-1} = \int_0^\infty \tau^k d\mu(\tau), \quad k = 0, 1, 2, \dots;$$

see [16, p. 139] for details. If  $I(x)$  is an *entire* function, and  $f(z)$  is given by (2.1), then we have

$$\begin{aligned} (2.18) \quad z^M f(z) &= \sum_{k=-M}^{\infty} a_k z^{k+M} = \sum_{k=-M}^{\infty} (I_{k+M}/H_{k+M}) z^{k+M} \\ &= \sum_{k=-M}^{\infty} I_{k+M} \int_0^\infty (z\tau)^{k+M} d\mu(\tau) \\ &= \int_0^\infty I(z\tau) d\mu(\tau), \end{aligned}$$

provided the integral exists and termwise integration can be justified. We may interpret (2.18) as a transform equation:  $\hat{f}(z) = z^M f(z)$  is the “ $\theta$ -transform” of  $I(x)$ . The reciprocal relation is given by (2.16), which can be written as

$$I(x) = (2\pi i)^{-1} \int_{|t|=\rho} (1/t) \hat{f}(1/t) H(xt) dt.$$

Consider the example  $M = -1, \alpha_k = k, p = 1, P = 1$ . Hypothesis I is easily verified, and we calculate that

$$S_{2+j,n} = \frac{(n+1)(n+2) \cdots (n+j+1)}{(n+1)^{j+1}(j+1)!} \rightarrow \frac{1}{(j+1)!}, \quad n \rightarrow \infty,$$

and so  $H_k^{-1} = k!$  The corresponding moment problem is solved by  $k! = \int_0^\infty \tau^k e^{-\tau} d\tau$ . Here (2.18) is easily justified and becomes after a change of variable

$$f(1/z) = \int_0^\infty e^{-zt} I(t) dt,$$

implying that  $F(z) = f(1/z)$  is the Laplace transform of  $I(t)$ . From  $\alpha_k = k$  and  $p = 1$  it follows that  $\theta = z^2 d/dz$ . Now  $(\theta^n f)(z) = (-1)^n F^{(n)}(1/z)$  as noted earlier. Since  $R_n = (n+1)^{-1}$ , (2.17) reduces in this case to

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n!} \left( \frac{n+1}{x} \right)^{n+1} F^{(n)} \left( \frac{n+1}{x} \right) = I(x).$$

This was first derived by E. L. Post in 1930 [9, p. 772]. See [15], [16], [17], [18] for further references.

3. *Alternate hypothesis:* If the terms  $\{\alpha_n\}$  are negative or complex, the previous results do not apply. We will modify them by using an alternate set of growth conditions. We still assume (2.4), (2.5) and (2.6), so that (2.7) and (2.8) hold. The following replaces (2.9).

*Hypothesis II.* There exists a number  $S > 0$  such that

$$(3.1) \quad \left| \frac{\alpha_{-M+P+j+np}}{\alpha_{-M+np}} \right| \left| \frac{\alpha_{-M+np}}{\alpha_{-M+P+np}} (1/S) \right|^{P+j} \leq 1; \quad n, j = 0, 1, 2, \dots$$

Instead of (2.11), but proving it in the same way, we now have

$$(3.2) \quad \left| \frac{A_{-M+P+j,n}}{A_{-M,n}} \right| \leq (S^n/R_n)^{P+j}, \quad n = 1, 2, 3, \dots; j = 0, 1, 2, \dots,$$

where  $R_n = |A_{-M,n}/A_{-M+P,n}|$  as before. This time  $(\theta^n f)(z)$ , still given by (2.3), is defined for  $0 < |z| < R_n R/S^n$ . The function  $K_n(w)$  will be defined as before except that  $|w| < R_n/S^n$ , and for (2.13) we must insist that  $|z| < R_n/(\rho S^n)$  where  $(1/\rho) < R$ . Finally, we see that (2.14) will hold under Hypothesis II when  $0 < |z| < R_n/(\rho S^n)$ . Thus we have proved the following.

**THEOREM 3.1.** *Assume Hypothesis II and let  $f(z)$  be given by (2.1). Then  $(\theta^n f)(z)$  is defined (by (2.3)) for  $0 < |z| < R_n/(\rho S^n)$  and is also given by (2.12), where  $(1/\rho) < R$ .*

**THEOREM 3.2.** *Assume Hypothesis II and let  $f(z)$  be given by (2.1) with  $a_{-M} \neq 0$ . Let  $\tau$  and  $\gamma$  be constants with  $0 < \tau < 1$  and  $0 < \gamma < (1/\rho)$ . Then for all  $n$  sufficiently large  $(\theta^n f)(z)$  has no zero in  $0 < |z| \leq R_n(\tau^n/S^n)\gamma$ .*

*Proof.* In (2.14) put  $z = R_n(\tau^n/S^n)x$ , and write the result as

$$\begin{aligned} (\theta^n f)(z) &= A_{-M,n} z^{-M+np} (2\pi i)^{-1} \int_{|t|=\rho} (1/t)^{M+1} f(1/t) \\ &\quad \times \left\{ 1 + \sum_{j=0}^{\infty} \frac{A_{-M+P+j,n}}{A_{-M,n}} \left[ \frac{R_n \tau^n x t}{S^n} \right]^{P+j} \right\} dt \\ &= A_{-M,n} z^{-M+np} L_n(x), \end{aligned}$$

with  $L_n(x)$  defined as indicated. This is valid for  $0 < |z| < R_n/(\rho S^n)$ ; i.e.,  $0 < \tau^n |xt| = \tau^n |x| \rho < 1$ . In fact, take  $0 < |x| \leq \gamma < (1/\rho)$ , which implies  $\tau |x| \rho < 1$ . For  $x$  so restricted, and  $0 < \tau < 1$ , (3.2) implies that the sequence  $\{L_n(x)\}_{n=1}^{\infty}$  converges uniformly to

$$(2\pi i)^{-1} \int_{|t|=\rho} (1/t)^{M+1} f(1/t) dt = a_{-M} \neq 0.$$

This produces a revised version of (2.17), namely

$$\lim_{n \rightarrow \infty} \left\{ \frac{z^{M-np} (\theta^n f)(z)}{A_{-M,n} a_{-M}} \right\} = 1, \quad 0 < |z| = |R_n \tau^n x/S^n| \leq R_n \tau^n \gamma/S^n,$$

and the desired conclusion follows.

**4. Examples.** We calculate here some examples to illustrate the wide variety of situations covered by Theorems 2.2 and 3.2.

(i)  $M = -1, p = 0, \alpha_k = k, P = 1, \theta = z d/dz$ . We can use Hypothesis I since (2.9) reduces to

$$\frac{\alpha_{2+j}}{\alpha_1} \left[ \frac{\alpha_1}{\alpha_2} \right]^{1+j} \leq 1,$$

which is equivalent to  $2 + j \leq 2^{1+j}$ , which holds for all  $j \geq 0$ . Now  $R_n = A_{1,n}/A_{2,n} = (1/2)^n$ , and therefore the discs  $|z| \leq \gamma 2^{-n}$  are free of zeros of  $(\theta^n f)(z)$  for  $n$  large enough. For this operator, the change of variable  $z = e^w$ ,  $F(w) = f(z) = f(e^w)$  leads to  $(\theta^n f)(z) = (D^n F)(w)$  where  $D$  stands for ordinary differentiation. The half planes  $\text{Re}(w) \leq \log \gamma - n \log 2$  are free of zeros of  $D^n F$  for such  $F$ . This example was given in [13].

(ii)  $M = -1, p = 1, \alpha_k = k, P = 1, \theta = z^2 d/dz$ . Hypothesis I applies again, for (2.9) becomes

$$\frac{n + j + 2}{n + 1} \leq \left( \frac{n + 2}{n + 1} \right)^{j+1};$$

but this is the same as

$$1 + \frac{j + 1}{n + 1} \leq \left( 1 + \frac{1}{n + 1} \right)^{j+1},$$

which is true by the binomial theorem. We calculate that  $R_n = A_{1,n}/A_{2,n} = n!/(n + 1)! = (n + 1)^{-1}$ , and so the discs  $|z| \leq \gamma/(n + 1)$  have no zeros of  $(\theta^n f)(z)$ . Putting  $F(z) = f(1/z)$ , we obtain the zero-free regions for  $F^{(n)}(z)$  recently described by R. P. Boas [2].

(iii)  $M = 1, p = 0, \alpha_k = k, \theta = z d/dz$ . Since  $\alpha_{-1} = -1$  and  $\alpha_0 = 0$  we have  $P = 2$ . Condition (3.1) specializes to  $1 + j \leq S^{2+j}$ , or  $S \geq (j + 1)^{1/(j+2)}$ . The sequence  $(j + 1)^{1/(j+2)}$  attains a maximum at  $j = 3$  and therefore Hypothesis II is satisfied for  $S = 4^{1/5}$ . Since  $R_n = 1$ , the regions  $0 < |z| \leq \gamma r^n / 4^{n/5}$  are zero-free for  $\theta$ . This bound is asymptotically best possible because of the example  $f(z) = z^{-1} + z^4$ , for which we have  $(\theta^n f)(z) = (-1)^n z^{-1} + 4^n z^4$ .

(iv)  $M = 0, p = 0, \alpha_k = 2^{-k}, P = 1$ . Here the operator  $\theta$  is given by  $(\theta^n f)(z) = f(z/2^n)$ ,  $n = 0, 1, 2, \dots$ , which is defined for  $|z| < 2^n R$ . The quantity on the left of (2.9) is

$$\frac{\alpha_{1+j}}{\alpha_0} \left[ \frac{\alpha_0}{\alpha_1} \right]^{1+j} = 2^{-(1+j)} [2]^{1+j} = 1,$$

so that both sets of hypotheses are satisfied, the second for  $S = 1$ . Since  $R_n = 2^n$ , Theorem 2.2 says that the regions  $0 < |z| \leq \gamma 2^n$  ( $\gamma < R$ ) are zero-free. Of course, this is equivalent to the fact that  $f(z)$  itself has isolated zeros. This illustrates also the fact that the constant  $\gamma$  depends on  $f(z)$ .

(v)  $M = 0, p = 1, \alpha_k = 1/(k + 1), P = 1$ . In this case  $\theta$  is the anti-derivative operator,  $\theta(z^k) = z^{k+1}/(k + 1)$ ,  $(\theta f)(z) = \int_0^z f(s) ds$ . Condition (3.1), dropping absolute values, becomes

$$\frac{\alpha_{1+j+n}}{\alpha_n} \left[ \frac{\alpha_n}{\alpha_{1+n}} \right]^{1+j} \leq S^{1+j},$$

that is,

$$\frac{n + 1}{n + j + 2} \left[ \frac{n + 2}{n + 1} \right]^{j+1} \leq S^{j+1}.$$

This inequality is false if  $S = 1$  (in fact the reverse inequality holds) which means that Hypothesis I fails. However Hypothesis II holds with  $S = 2$ , which we get by setting  $n = 0$  and  $j \rightarrow \infty$ . Here  $R_n = n + 1$ , giving zero-free regions of the form  $|z| \leq (n + 1)\gamma\tau^n/2^n$ .

(vi)  $M = 0, p \geq 0, \alpha_k = \Gamma(k + 1)/\Gamma(k + p + 1), P = 1$ . Applying  $\theta$  termwise gives  $(\theta f)(z) = \sum_{k=0}^{\infty} a_k [\Gamma(k + 1)/\Gamma(k + p + 1)] z^{k+p}$ , and it follows that  $\theta$  corresponds to the  $p$ th order fractional integral. We use the definition of [8, p. 67]. If  $p = 1$  this example reduces to the previous one. The left side of (3.1) is asymptotically

$$(np + j)^{-p} (np)^p [(np)^{-p} S^{-1} (np)^p]^{1+j}$$

for large  $n$  and  $j$ , where we have used the asymptotic formula  $[\Gamma(n + a)/\Gamma(n + b)] \sim n^{a-b}$  ([1, p. 257]). Consequently, Hypothesis II will hold for some  $S \geq 1$ , whose value we do not compute. After a calculation,  $R_n = np + 1$ . The zero-free regions for fractional integration are then  $|z| \leq (np + 1)\gamma\tau^n/S^n$ .

(vii) Let  $G(w)$  be an entire function of finite exponential type  $b$ , so that  $|G(w)| = O(e^{(b+\epsilon)w})$ , for  $\epsilon > 0$  and all large  $w$ . Fix the operator  $\delta = zd/dz$  and consider, as a differential operator, the expression  $G(\delta)$ , which is defined by

$$(4.1) \quad G(\delta)f = \lim_{n \rightarrow \infty} \sum_{j=0}^n \frac{G^{(j)}(0)}{j!} \delta^j(f).$$

This operator is considered by Hille [7, p. 48] in connection with questions on analytic continuation, and also by the first author [11]. One can show that, for  $f(z)$  given by (2.1),

$$(4.2) \quad G(\delta)f(z) = \sum_{k=-M}^{\infty} a_k G(k) z^k,$$

with uniform convergences on compact subsets of  $0 < |z| < \text{Re}^{-b}$ . Thus  $G(\delta)$  is a multiplier-sequence operator,  $\theta = G(\delta)$ , with  $\alpha_k = G(k)$  and  $p = 0$ . If we assume  $G(-M) \neq 0$  and  $G(-M + P) \neq 0$ , then Hypothesis II will be satisfied for some  $S$  because of the growth condition  $G(k) = O(e^{(b+\epsilon)k})$ . Since  $R_n = [G(-M)/G(-M + P)]^n$ , then the zero-free regions for  $\theta = G(\delta)$  are of the form  $0 < |z| \leq \gamma |\tau G(-M)/SG(-M + P)|^n$ .

(viii) Now take  $\theta = \delta = z d/dz$ . The functions  $P_n(z) = \delta^n(e^z)e^{-z}$  are known as Stirling polynomials ([11],[12]). By example (i) (the constant

term disappears after one iteration, so we may as well take  $M = -1$ ) the  $n$ th Stirling polynomial, for  $n$  large enough, will have no zero inside  $|z| \leq \gamma 2^{-n}$ . The function  $H(w)$  is the constant 1, so that  $I(x) = a_1 = 1$ . Then (2.17) reduces to the asymptotic formula

$$\lim_{n \rightarrow \infty} \frac{P_n(x/2^n)}{(x/2^n)\exp(x/2^n)} = 1, \quad |x| \leq \gamma < (1/\rho).$$

It is shown in [11] that  $\theta^n[\exp(z^r)] = P_n(z^r)\exp(z^r)$  and that the final set of  $\exp(z^r)$  for  $\theta$  (the set of all  $\xi$  such that  $\xi = \lim_{k \rightarrow \infty} \xi_k$  where  $\theta^{n_k}[\exp(\xi_k^r)] = 0$  for some sequence  $n_k \rightarrow \infty$ ) consists of the rays from 0 going through the  $r$ th roots of  $-1$ . The fact, observed above, that  $P_n(z^r) \neq 0$  for  $|z| < (\gamma/2^n)^{1/r}$  indicates the rate at which these rays are "filled in" with zeros of  $P_n(z^r)$ . Closely related results on the final set for derivatives of  $\exp[-e^z]$  are found in [6].

**5. Reciprocal powers.** A completely dual theory to that of §3 exists for multiplier-sequence operators on power series

$$(5.1) \quad f(z) = \sum_{k=-M}^{\infty} a_k z^{-k} = a_{-M} z^M + a_{-M+1} z^{M-1} + \dots, \quad |z| > T.$$

We give a brief account of it in this section.

Let  $p \geq 0$  and let  $\beta$  be a function defined on all numbers of the form  $-k + np$ ,  $k = -M, -M + 1, -M + 2, \dots$  and  $n = 0, 1, 2, \dots$ , where  $M$  is an integer. We continue to write  $\beta_{-k+np} = \beta(-k + np)$ . We assume that

$$(5.2) \quad \beta_{-M} \neq 0, \quad \text{and}$$

$$(5.3) \quad \beta_{-k} \neq 0 \text{ implies } \beta_{-k+np} \neq 0, \quad n = 0, 1, 2, \dots,$$

where  $k = -M, -M + 1, \dots$ . Assume that there is a first integer  $P \geq 1$  such that

$$(5.4) \quad \beta_{M-P} \neq 0.$$

We define the operator  $\theta$  on the power basis by  $\theta(z^\nu) = \beta_\nu z^{\nu+p}$ , for  $\nu$  in the domain of  $\beta$ , so that  $\theta^n(z^\nu) = \beta_\nu \beta_{\nu+p} \cdots \beta_{\nu+(n-1)p} z^{\nu+np}$ . Letting  $B_{\nu,n} = \beta_\nu \beta_{\nu+p} \cdots \beta_{\nu+(n-1)p}$  and applying  $\theta^n$  termwise to (5.1), we arrive at the definition for  $\theta^n f$ , namely

$$(5.5) \quad (\theta^n f)(z) = \sum_{k=-M}^{\infty} B_{-k,n} a_k z^{-k+np}, \quad n = 1, 2, 3, \dots,$$

whenever this series converges. Conditions (5.2), (5.3) (5.4) imply that

$$B_{M,n} \neq 0, B_{M-P,n} \neq 0, \quad n = 1, 2, 3, \dots, \quad \text{and} \\ B_{M-1,n} = \cdots = B_{M-P+1,n} = 0, \quad \text{if } P > 1.$$

We now state the condition corresponding to Hypothesis II. Hypothesis I has no practical correspondent since the standard operators ( $\theta = z^{1+p} d/dz$ ) involve negative  $\beta_v$ .

*Hypothesis III.* There exists a number  $S > 0$  such that

$$(5.6) \quad \left| \frac{\beta_{M-P-j+np}}{\beta_{M+np}} \right| \left| \frac{S\beta_{M+np}}{\beta_{M-P+np}} \right|^{P+j} \leq 1, \quad n, j = 0, 1, 2, \dots$$

It follows that

$$(5.7) \quad \left| \frac{B_{M-P-j,n}}{B_{M,n}} \right| \left| \frac{B_{M,n}S^n}{B_{M-P,n}} \right|^{P+j} \leq 1,$$

and introducing  $T_n = |B_{M-P,n}/B_{M,n}|$ , (5.7) can be written

$$\left| \frac{B_{M-P-j,n}}{B_{M,n}} \right| \leq (T_n/S^n)^{P+j}; \quad n = 1, 2, 3, \dots; j = 0, 1, 2, \dots$$

This condition implies that (5.5) converges and defines  $\theta^n f$  for  $|z| > TT_n/S^n$ ,  $n = 1, 2, 3, \dots$

The rest of the program parallels that of § 3, and we give no further details except statements of pertinent theorems, which generalize corresponding results for differential operators in [14].

**THEOREM 5.1.** *Assume Hypothesis III and let  $f(z)$  be given by (5.1). Then  $(\theta^n f)(z)$  is defined by (5.5) for  $|z| > T_n/(\rho S^n)$  and is also given by*

$$\begin{aligned} (\theta^n f)(z) &= B_{M,n} z^{M+np} (2\pi i)^{-1} \\ &\times \int_{|t|=\rho} t^{M-1} f(1/t) \left\{ 1 + \sum_{j=0}^{\infty} \frac{B_{M-P-j,n}}{B_{M,n}} \left( \frac{1}{zt} \right)^{P+j} \right\} dt, \end{aligned}$$

where  $(1/\rho) > T$ .

**THEOREM 5.2.** *Assume Hypothesis III and let  $f(z)$  be given by (5.1), with  $a_{-M} \neq 0$ . Let  $\tau$  and  $\gamma$  be constants with  $0 < \tau < 1$  and  $(1/\gamma) > (1/\rho) > T$ . Then for all  $n$  sufficiently large,  $(\theta^n f)(z)$  has no zero which satisfies  $|z| \geq T_n/(S^n \tau^n \gamma)$ .*

**EXAMPLE.** If we take  $\theta = G(\delta)$  as in example (vii), where  $G(w)$  is entire and of finite exponential type  $b$ , and  $f(z)$  a ‘‘balanced’’ Laurent series

$$f(z) = \sum_{k=-M}^M C_k z^k, \quad M > 0, \quad |C_M| = |C_{-M}| = 1,$$

then the results both §3 and §5 apply. We assume  $G(-M)G(M) \neq 0$  and, for simplicity,  $G(-M + 1)G(M - 1) \neq 0$  so that  $P = 1$  for both sets of hypotheses. We have  $\alpha_{-k} = \beta_k = G(k)$ ,  $-M \leq k \leq M$ . From the growth conditions on  $G(w)$  it is possible to choose a number  $\mu > b$  so that (3.1) holds for an  $S$  such that  $|G(-M + 1)S/G(-M)| = e^\mu$  and (5.6) holds for an  $S'$  such that  $|G(M - 1)/G(M)S'| = e^\mu$ . Since

$$R_n = |G(-M)/G(-M + 1)|^n \quad \text{and} \quad T_n = |G(M - 1)/G(M)|^n,$$

the conclusions of Theorems 3.2 and 5.2 say that  $(\theta^n f)(z) = (G(\delta))^n f(z)$  is zero-free when either

$$0 < |z| \leq \gamma(\tau/e^\mu)^n \quad \text{or} \quad |z|^{-1} \leq \gamma(\tau/e^\mu)^n.$$

These bounds become rather crude when one specializes  $G$ . Prather [11] has shown that if  $G(w) = e^{i\lambda w} \prod_{k=1}^\infty (1 - iw/w_k)$ , where  $\lambda$  is real and the  $w_k$  are real and satisfy  $\sum w_k^{-1} < \infty$ , then the final set of a balanced  $f(z)$  taken with respect to  $\theta = G(\delta)$  lies on the unit circle. Consequently, the sets  $0 < |z| \leq r_1 < 1$ , and  $|z| \geq r_2 > 1$  are zero-free for all large  $n$ .

In two very interesting related papers, Craven and Csordas [4], [5] have results related to but not overlapping the results done here. In particular, they consider multiplier sequences  $\{\gamma_k\}_{k=0}^\infty$  which do not increase the number of nonreal zeros of real polynomials. An example of multipliers which they take is  $\gamma_k = Q(k)$ , where  $Q$  is a polynomial having real negative zeros, or more generally an entire function of exponential type having real negative zeros of the form

$$Q(z) = \exp(bz) \prod_{n=1}^\infty (1 + z/a_n),$$

say, where  $b > 0$ ,  $\sum_{n=1}^\infty |a_n|^{-1} < \infty$ . We allow the same multiplier in our example (vii) and the example after Theorem 5.2. When  $M \leq 0$  in example (vii) so that  $z = 0$  is allowed and  $\gamma_k = Q(k)$  as above, the asymptotic size of the zero free region shrinks to 0, as  $n \rightarrow \infty$ . This is consistent with their results.

Moreover, if in example (vii), we take  $G(w)$  to be an entire function of exponential type  $b < \pi$ , and  $G$  vanishes at a set of integers from  $\{-M, -M + 1, \dots\}$  having density 1, yet not vanishing at all the integers (in light of Carlson's Theorem), then the boundary of convergence  $|z| = R \exp(-b)$  of  $G(\delta)f(z)$ , given in (4.2), is a natural boundary, by the Fabry gap theorem (see e.g. Bieberbach [3], Satz 2.21 or Hille [7], Theorem 11.7.2). Our hypotheses ( $G(-M) \neq 0$ ,  $G(-M + P) \neq 0$ ) do not allow  $G$  to vanish at all the integers. Consequently, the boundary  $|z| = R \exp(-bm)$  of the domain of convergence for  $(G(\delta))^m f(z)$  is a natural boundary, for all  $m = 2, 3, \dots$

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