PACIFIC JOURNAL OF MATHEMATICS
Vol 107, No. 1, 1983

EXPLICIT PL SELF-KNOTTINGS AND THE STRUCTURE OF PL HOMOTOPY COMPLEX PROJECTIVE SPACES

DOUGLAS MEADOWS

We show that certain piecewise-linear homotopy complex projective spaces may be described as a union of smooth manifolds glued along their common boundaries. These boundaries are sphere bundles and the glueing homeomorphisms are piecewise-linear self-knottings on these bundles. Furthermore, we describe these self-knottings very explicitly and obtain information on the groups of concordance classes of such maps.

A piecewise linear homotopy complex projective space $\widetilde{CP}^n$ is a compact PL manifold M^{2n} homotopy equivalent to CP^n. In [**22**] Sullivan gave a complete enumeration of the set of PL isomorphism classes of these manifolds as a consequence of his Characteristic Variety theorem and his analysis of the homotopy type of G/PL. In [**15**] Madsen and Milgram have shown that these manifolds, the index 8 Milnor manifolds, and the differentiable generators of the oriented smooth bordism ring provide a complete generating set for the torsion-free part of the oriented PL bordism ring. Hence a study of the geometric structure of these exotic projective spaces $\widetilde{CP}^n$, especially with regard to their smooth singularities, may extend our understanding of the PL bordism ring. This paper begins such a study in which we obtain a geometric decomposition of $\widetilde{CP}^n$, into (for many cases) a union of smooth manifolds glued together by PL self-knottings on certain sphere bundles. We also obtain information on groups of concordance classes of PL self-knottings from these decompositions and a number of fairly explicitly constructed examples of self-knottings. I would like to thank by thesis advisor R. J. Milgram for many helpful discussions.

I. Sullivan's classification of PL homotopy $\widetilde{CP}^n$ proceeds as follows: Given a homotopy equivalence $h\colon \widetilde{CP}^n \to CP^n$ make h transverse regular to $CP^j \subset \widetilde{CP}^n$, the standard inclusion. The restriction of h to the transverse inverse image $h^{-1}(CP^j) = N^{2j} \subset \widetilde{CP}^n$ is a degree one normal map with simply connected surgery obstruction

$$\sigma_j \in P_{2j} = \begin{Bmatrix} Z, & j \text{ even} \\ Z/2Z, & j \text{ odd} \end{Bmatrix}.$$

For $j = 2,\ldots,n-1$ these obstruction invariants yield a complete enumeration—i.e. the set of PL isomorphism classes of $\widetilde{CP}^n$ is set-isomorphic to the product $Z \times Z_2 \times Z \times \cdots \times P_{2(n-1)}$ with $n-2$ factors.

We will use the following notation to specify elements with this classification:

$$\widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}) \tag{1}$$

will denote the PL homotopy $\widetilde{CP}^n$ with invariants $\sigma_j \in P_{2j}$ in Sullivan's enumeration.

We recall that a PL homeomorphism f: $M \to M$ is a "self-knotting" and M is said to be "self knotted" if f is homotopic but not PL isotopic to the identity. Also, PL homeomorphisms f, g: $M \to M$ are "PL concordant" (pseudo-isotopic) if we have a PL homeomorphism F: $M \times I \to M \times I$ with $F(m,0) = (f(m),0)$ and $F(m,1) = (g(m),1)$ for $m \in M$. We then define:

(2) $SK(M)$ = "the group (under composition of maps) of PL concordance classes of PL self-knottings of M."

Unless otherwise noted "$CP^j \subset CP^n$" means the standard embedding of CP^j onto the first $(j+1)$ homogeneous coordinates of CP^n or a smooth ambient isotope of this embedding. In this context we define:

(3) $\nu_N(CP^j)$ = "the smooth tubular disc bundle neighborhood of the embedding $CP^j \subset CP^N$."

Our results are as follows:

THEOREM A. *For $n \geq 4$ and $\sigma_2 \equiv 0$ (2) every $\widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1})$ is PL homeomorphic to the identification space*

$$\left[\widehat{CP}^n - \nu_n(CP^1)\right] \cup_{\varphi_{\sigma_{n-1}}} \left[\nu_n(CP^1)\right]$$

where $\widehat{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-2}, 0)$ and the identification is over a PL homeomorphism

$$\varphi_{\sigma_{n-1}}: \partial\nu_n(CP^1) \to \partial\nu_n(CP^1).$$

We prove Theorem A in Part II by a careful description of Sullivan's classification and an easy h-cobordism argument.

An immediate consequence of Theorem A is the decomposition of $\widetilde{CP}^{n+1} \leftrightarrow (0,\ldots,0,\sigma_n)$ into

$$\widetilde{CP}^{n+1} = [CP^{n+1} - \nu(CP^1)] \cup_{\varphi_0} [\nu(CP^1)].$$

THEOREM B. *For every* $n \geq 4$ *and non-zero* $\tau \in P_{2n}$ *there is a PL self-knotting*

$$\varphi_\tau \colon \partial\nu_{n+1}(CP^1) \to \partial\nu_{n+1}(CP^1)$$

which will suffice for the glueing homeomorphism in Theorem A.

We establish this theorem by an explicit construction of φ_τ in Part III.

II. Here we prove Theorem A by beginning with a construction which shows how to obtain $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2,\ldots,\sigma_{n-1},\sigma_n)$ from $\widetilde{CP}^n \leftrightarrow (\sigma_2,\ldots,\sigma_{n-1})$ for $n \geq 4$:

Let $h\colon \widetilde{CP}^n \to CP^n$ be a homotopy equivalence, and let M^{2n} be the compact $(n-1)$-connected Milnor or Kervaire manifold of Index $8\sigma_n$ or Kervaire-Arf invariant σ_n as the case may be [**4**]. Let $r\colon M^{2n} \to S^{2n}$ be a degree one map. Then $h\#r\colon \widetilde{CP}^n\#M^{2n} \to CP^n\#S^{2n} = CP^n$ is a degree one normal map with 1-connected surgery obstruction σ_n. We define $\hat{H}$ as the D^2 bundle over $\widetilde{CP}^n\#M^{2n}$ induced by $h\#r$ from H, the disc bundle associated to the complex line bundle over CP^n. Let $\hat{h}\colon \hat{H} \to H$ be the bundle mapping. We note that the map $h\#r$ is $(n-1)$-connected with homological kernel $K_n = \pi_n(M_0^{2n})$ where $M_0^{2n} = M^{2n} - D^{2n}$. The bundle $\hat{H}$ is trivial over M_0^{2n} since $M_0^{2n} = (h\#r)^{-1}(\text{point})$. In $M_0^{2n} \times D^2$ we can represent $\pi_n(M_0^{2n})$ by disjointly embedded spheres $S^n \Rightarrow M_0^{2n} \times S^1$ with trivial normal bundles. This follows by general position and the fact that the normal bundles of the generating spheres $S^n \subset M_0^{2n}$ are the stably trivial tangent disc bundles $\tau(S^n)$. We now attach a solid handle $D^{n+1} \times D^{n+1}$ along $S^n \times D^{n+1} \subset M_0^{2n} \times S^1$ for each generator of $\pi_n(M_0^{2n})$ and extend the map $\hat{h}$ across these bundles. This is possible since the embedded spheres are in the homotopy kernel of $\hat{h}$. Call the resulting PL manifold $\tilde{H}$ and the extended map $\tilde{h}\colon \tilde{H} \to H$. In the process of extending $\hat{h}$ across the handles, we may guarantee that $\tilde{h}$ is a map of pairs $(\tilde{H},\partial) \to (H,\partial)$. We observe, then, the:

PROPOSITION. $\tilde{h}\colon (\tilde{H},\partial) \to (H,\partial)$ *is a homotopy equivalence of pairs.*

This follows directly from the construction as $\tilde{H}$ deformation retracts onto $\widetilde{CP}^n \# M^{2n} \cup \{e_\alpha^n\}$ where the n-cells e_α^n are attached so as to kill the entire homology kernel of $(h\#r)$. Hence $\tilde{h}$: $\tilde{H} \to H$ is a homology isomorphism, and as $\tilde{H}$ is 1-connected we have by Whitehead's theorem that it is a homotopy equivalence. The restriction of $\tilde{h}$ to the boundary is likewise a homology isomorphism as the boundaries, $D^{n+1} \times S_\alpha^n$, of the solid handles are precisely the surgeries needed to cobord $\hat{h}$: $\partial\hat{H} \to \partial H$ to a homotopy equivalence.

In particular as $n \geq 3$ we note that the boundary manifold, $\partial\tilde{H}$, is a PL $(2n+1)$-sphere by the Poincaré conjecture. Thus, we attach D^{2n+2} to $\tilde{H}$ as the PL cone on $\partial\tilde{H}$ and define:

$$\widetilde{CP}^{n+1} = \tilde{H} \cup c(\partial\tilde{H}) \quad \text{and} \quad h\colon \widetilde{CP}^{n+1} \to CP^{n+1} = H \cup c(\partial H)$$

by radial extension of $\tilde{h}$ into $c(\partial\tilde{H})$.

Observe that h has 'built-in" transverse inverse image $\widetilde{CP}^n \# M^{2n} = h^{-1}(CP^n)$ with surgery obstruction σ_n. Hence, this $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n)$ is the space we require.

Now, given $\widetilde{CP}^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1})$ let us consider a bit more closely the suspension and generalized suspension constructions described above. First, assume the homotopy equivalence

$$h\colon \widetilde{CP}^n \to CP^n$$

is the identity map on a disc $D^{2n} \subset \widetilde{CP}^n$. Let $\widetilde{CP}_0^n = \widetilde{CP}^n - D^{2n}$, $M_0^{2n} = M^{2n} - D^{2n}$ and observe that $\widetilde{CP}^n \# M^{2n} = \widetilde{CP}_0^n \cup_\partial M_0^{2n}$. Now, let $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, 0)$ be the suspension[1] of $\widetilde{CP}^n$ with homotopy equivalence

$$\tilde{h}\colon \widetilde{CP}^{n+1} \to CP^{n+1}$$

and $\widehat{CP}^n \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n)$ be the general suspension of $\widetilde{CP}^n$ with homotopy equivalence

$$\hat{h}\colon \widehat{CP}^{n+1} \to CP^{n+1}.$$

Let $D^{2n} \subset CP^n$ be the image $h(D^{2n})$ and let $CP^1 = S^2 \subset CP^{n+1}$ be represented as $D_*^2 \cup c(\partial D_*^2)$ in $CP^{n+1} = H \cup c(\partial H)$ with D_*^2 the fiber in H over the center of the disc D^{2n}. Then $\nu_{n+1}(CP^1) \subset CP^{n+1}$ may be represented as the set $D_*^2 \times D^{2n} \cup c(\partial H)$, a D^{2n} bundle over the sphere $S^2 = D_*^2 \cup c(\partial D_*^2)$.

Now let $\tilde{V} = \tilde{h}^{-1}(\nu_{n+1}(CP^1))$ and $\hat{V} = \hat{h}^{-1}(\nu_{n+1}(CP^1))$ in $\widetilde{CP}^{n+1}$ and $\widehat{CP}^{n+1}$ respectively. We observe directly from the constructions that

[1] We say $\widetilde{CP}^{n+1} \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, 0)$ in the "suspension" of $\widetilde{CP}^n \leftrightarrow (\sigma_2, \sigma_3, \ldots, \sigma_{n-1})$ as it is precisely the Thom complex of the line bundle induced over $\widetilde{CP}^n$.

$\widetilde{CP}^{n+1} - \tilde{V}$ and $\widehat{CP}^{n+1} - \hat{V}$ are precisely the same spaces. To prove Theorem A we must show that $\tilde{V}$ and $\hat{V}$ are PL homeomorphic to $\nu_{n+1}(CP^1)$.

LEMMA 1. *$\tilde{V} \cong \nu_{n+1}(CP^1)$ if σ_2 is even.*

We observe this from PL block bundle theory as follows: by construction $\tilde{V}$ is the union of two discs $D^2_* \times D^{2n}$ and $c(\partial\tilde{H}) = D^{2n+2}$ along $S^1_* \times D^{2n}$. Hence $\tilde{V}$ is trivially a block bundle regular neighborhood of $CP^1 = D^2_* \cup c(\partial D^2_*)$. Assume the obstruction σ_2 is even. Then as noted by Sullivan ([**23**] p. 43) the splitting obstruction of the homotopy equivalence

$$\tilde{h}\colon \widetilde{CP}^{n+1} \to CP^{n+1}$$

along CP^1 vanishes as it is the mod 2 reduction of σ_2. Hence, by a homotopic deformation we may conclude that the transverse inverse image of CP^1 by $\tilde{h}$ is $CP^1 \subset \widetilde{CP}^{n+1}$. Moreover, as any two homotopic PL embeddings of $CP^1 \subset \widetilde{CP}^{n+1}$ are ambiently PL isotopic (for $n \geq 2$ by Cor. 5.9 p. 65 [**21**]), we see by appeal to the uniqueness of normal block bundles (regular neighborhoods) [**20**] that $\tilde{V}$ is block bundle isomorphic to the bundle induced from $\nu_{n+1}(CP^1)$ by $\tilde{h}$. Conversely, the same argument on the homotopy inverse of $\tilde{h}$ implies $\nu_{n+1}(CP^1)$ is block bundle induced from $\tilde{V}$. As we are in the stable block and vector bundle range and $\pi_2 B_{\mathrm{PL}} = \pi_2 B_0 = Z_2$ we can conclude that $\tilde{C}$ and $\nu(CP^1)$ are block bundle isomorphic; hence PL homeomorphic.

LEMMA 2. *$\hat{V} \simeq S^2$ (homotopy equivalent).*

Proof. By construction $\hat{V} = D^2 \times M_0^{2n} \cup X \cup c(\partial H)$ where X represents the solid handles we attached along $S^1 \times M_0^{2n}$ to kill the homology kernel of $\hat{h}$. The manifold $D^2 \times M_0^{2n} \cup X$ is simply-connected with simply connected boundary and the homology of a point; hence by Smale's theorem (Thm. 1.1 [**22**]) it is a PL disc D^{2n+2}. Thus, $\hat{V} = D^{2n+2} \cup_W D^{2n+2}$ where W is the complement of the embedding

$$D^2 \times S^{2n-1} \subset S^{2n+1} = \partial D^{2n+2}$$

and $S^{2n-1} = \partial M_0^{2n}$. By the Mayer-Vietoris sequence we know that W is a homology circle. Then, by a second application of the Mayer-Vietoris sequence to the union $D^{2n+2} \cup_W D^{2n+2}$ we see that $\hat{V}$ is a homology S^2. Finally, by the Van Kampen theorem $\hat{V}$ is 1-connected and we apply the Whitehead theorem for CW complexes.

LEMMA 3. $\hat{V} \cong \nu_{n+1}(CP^1)$.

Proof. $\partial\hat{V} = \partial[CP^{n+1} - \hat{V}] = \partial[CP^{n+1} - \tilde{V}] = \partial\tilde{V} \cong \partial\nu_{n+1}(CP^1)$ by Lemma 1. Let $S^2 \subset \hat{V}$ be a homotopy equivalence and a PL embedding via Whitney's embedding theorem. Then $S^2 \subset \hat{V} \subset \widehat{CP}^{n+1}$ is homotopic to the standard embedding $CP^1 \subset \widehat{CP}^{n+1}$, and as before, the PL block bundle neighborhoods of these two embeddings must be isomorphic. Let $\nu \subset \hat{V}$ be this block bundle. We note that

$$\partial\nu = \partial\nu_{n+1}(CP^1) \cong \partial\tilde{V} = \partial\hat{V}$$

by the previous lemmas. Hence, if

$$\hat{V} - \nu = Y$$

we have $\partial Y = \partial\hat{V} \cup_{\cdot} \partial\nu$, two copies of the same manifold.

We consider the Mayer-Vietoris sequence for the union $\hat{V} = Y \cup \nu$ over $\partial\nu = Y \cap \nu$:

$$\cdots \to H_1(\partial\nu) \xrightarrow{i_{1*} - i_{2*}} H_1(\nu) \oplus H_q(Y) \xrightarrow{j_{1*} - j_{2*}} H_1(\hat{V}) \to \cdots$$

where

$$i_1 : \partial\nu \hookrightarrow \nu, \quad j_1 : \nu \hookrightarrow \hat{V},$$
$$i_2 : \partial\nu \hookrightarrow Y, \quad j_2 : Y \hookrightarrow \hat{V}.$$

Since ν and V are homotopy 2-spheres and j_1 is a homotopy equivalence, we see that for $q \neq 2$, $i_{2*}: H_q(\partial\nu) \to H_q(Y)$ must be an isomorphism. When $q = 2$ the sequence becomes:

$$Z \xrightarrow{1 - i_{2*}} Z \oplus A \xrightarrow{1 + j_{2*}} Z, \qquad A = H_2(Y)$$

from which we obtain i_{2*} are isomorphisms $Z \xrightarrow{i_{2*}} A \xrightarrow{j_{2*}} Z$. Thus, i_2: $\partial\nu \subset Y$ is a homology isomorphism, and in fact, a homotopy equivalence since $\hat{V} = Y \cup \nu$ and $\hat{V}$, ν, $\partial\nu$ are all 1-connected so that by Van Kampen's theorem Y is 1-connected.

We show next that $\partial\hat{V} \overset{i}{\subset} Y$ is a homology isomorphism so that Y is a h-cobordism from $\partial\nu$ to $\partial\hat{V}$—i.e. $Y \overset{\mathrm{PL}}{\cong} \partial\nu \times I$ and $\hat{V} = Y \cup \nu \cong \nu \cong \tilde{V}\nu_{n+1}(CP^1)$ as required.

We know already that $\partial\hat{V} \simeq Y$ as $\partial\hat{V} \cong \partial\nu \simeq Y$. Moreover, $\partial\nu \cong \partial\nu_{n+1}(CP^1)$ is an S^{2n-1} bundle over S^2. Hence, by the Serre Spectral Sequence we have

$$H_p(Y) = H_p(\partial\hat{V}) = \begin{cases} Z & \text{if } p = 0, 2, 2n-1, 2n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the exact sequence of the pair $(\hat{V}, \partial\hat{V})$ is:

$$\begin{array}{ccccccc} 0 = H_3(\hat{V}, \partial\hat{V}) & \to & H_2(\partial\hat{V}) & \to & H_2(\hat{V}) & \to & H_1(\hat{V}, \partial\hat{V}) = 0 \\ & & \| & & \| & & \\ & & Z & & Z & & \end{array}$$

where the first and last groups are 0 by Poincaré Duality. Thus, the inclusion $\partial\hat{V} \subset Y \subset \hat{V}$ is a homology isomorphism through $p = 2$.

Now, consider the composition f: $\partial\hat{V} \overset{i}{\hookrightarrow} Y \to \partial\hat{V}$ where the second map is a homotopy equivalence. Then f_*: $H_p(\partial\hat{V}) \to H_p(\partial\hat{V})$ is an isomorphism for $p \leq 2$, and by Poincaré Duality so is f^*: $H^I(\partial\hat{V}) \to H^I(\partial\hat{V})$ for $q = 2n-1$, $2n$, $2n+1$. By the Universal Coefficient Theorem f_* is an isomorphism for $p = 2n-1$, $2n$, $2n+1$ and so for all p. Thus, f is a homotopy equivalence, and so is i.

Theorem A is now an immediate consequence of the last lemma as we have:

$$\widehat{CP}^{n+1} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, \sigma_n) = [CP^{n+1} - \tilde{V}] \cup \hat{V},$$
$$\widetilde{CP}^{n} \leftrightarrow (\sigma_2, \ldots, \sigma_{n-1}, 0) = [CP^{n+1} - \nu_{n+1}(CP^1)] \cup_{\alpha_{\sigma_n}} \nu_{n+1}(CP^1)$$

where we have identified $\tilde{V}$ with $\nu_{n+1}(CP^1)$ by Lemma 1, and the PL homeomorphism

$$\varphi_{\sigma_n}: \partial[\widetilde{CP}^{n+1} - \nu(CP^1)] \to \partial\nu(CP^1)$$

comes from the restriction to the boundary of the PL homeomorphism $\hat{V} \to \nu_{n+1}(CP^1)$ of Lemma 3.

III. Construction of the self-knotting φ_σ: Here we construct for $n \geq 4$ a PL self-knotting

$$\varphi_\sigma: \partial\nu_{n+1}(CP^1) \to \partial\nu_{n+1}(CP^1)$$

with the property that it extends to a homotopy equivalence

$$\bar{\varphi}_\sigma: \nu_{n+1}(CP^1) \to \nu_{n+1}(CP^1)$$

which has a transverse-inverse image

$$M_0^{2n} = \bar{\varphi}_\sigma^{-1}(D^{2n})$$

on a fiber D^{2n}. Clearly such a φ_σ will suffice for the map in Theorem A.

We begin the construction by defining

$$\Sigma_\sigma^{2n-1} \subset S^{2n+1}$$

to be the smooth Brieskorn knot represented as the link of the singularity on the hypersurface in C^{n+1} defined by

$$p(Z) = \begin{cases} Z_0^{6\sigma-1} + Z_1^3 + Z_2^2 + \cdots + Z_n^2, & n \text{ even}, \\ Z_0^3 + Z_1^2 + \cdots + Z_n^2, & n \text{ odd}. \end{cases}$$

It is well-known that $S^{2n+1} - \Sigma_\sigma^{2n-1}$ is a smooth fiber bundle over the circle with fiber M_0^{2n}, the smooth Milnor or Kervaire manifold with surgery invariant σ.

Now, let $S^1 \subset S^{2n+1}$ be a fiber on the boundary of the smooth tubular neighborhood $D^2 \times \Sigma_\sigma^{2n-1}$ of the knot (a trivial bundle as $\pi_{2n-1}(\mathrm{SO}(2)) = 0$ for $n > 1$). Since $n > 1$ this circle S^1 is smoothly unknotted in S^{2n+1} so that the complement of a small tube $S^1 \times D^{2n}$ about it is diffeomorphic to $D^2 \times S^{2n-1}$. Hence the knot Σ_σ^{2n-1} lies in this complement with a trivial normal bundle and we can therefore define:

$$\beta\colon D^2 \times \Sigma_\sigma^{2n-1} \hookrightarrow D^2 \times S^{2n-1}$$

as this embedding. Let W^{2n+1} be the complement of this smooth embedding. Then we observe:

(a) $\partial W = S^1 \times S^{2n-1} \mathbin{\dot\cup} S^1 \times \Sigma_\sigma^{2n-1}$.

(b) W is a smooth fiber bundle over the circle S^1 with fiber $F^{2n} = M_0^{2n} - D^2$ and $\partial F = S^{2n-1} \mathbin{\dot\cup} \Sigma_\sigma^{2n-1}$.

(c) the bundle projection is trivial on $\partial W \to S^1$.

Now, using the smooth embedding β we define a piecewise-linear embedding

$$\gamma_\sigma\colon D^2 \times S^{2n-1} \hookrightarrow D^2 \times S^{2n-1}$$

as the composite map

$$D^2 \times S^{2n-1} \xrightarrow{\mathrm{id}\times\alpha_\sigma} D^2 \times \Sigma_\sigma^{2n-1} \xrightarrow{\beta} D^2 \times S^{2n-1}$$

where $\alpha_\sigma\colon S^{2n-1} \to \Sigma_\sigma^{2n-1}$ is a specific PL homeomorphism.

We now describe the normal bundle $\nu_{n+1}(CP^1)$ in CP^{n+1} as:

$$\nu_{n+1}(CP^1) = D^2_- \times S^{2n-1} \cup_\rho D^2_+ \times S^{2n-1}$$

(*) where $\rho\colon S^1 \times S^{2n-1} \to S^1 \times S^{2n-1}$ is a smooth bundle automorphism representing an element in $\pi_1(\mathrm{SO}(2n)) = Z/2Z$ $(n > 1)$. [We note in fact that $\gamma_{n+1}(CP^1)$ is trivial for n even and non-trivial for n odd as it is the Whitney sum of n copies of the canonical line bundle over $CP^1 = S^2$.]

In the above description we are expressing CP^1 as $S^2 = D^2_- \cup D^2_+$. Using this representation we will define the self-knotting φ_σ by showing that the PL embedding

$$\gamma_\sigma\colon D^2_+ \times S^{2n-1} \hookrightarrow D^2_+ \times S^{2n-1}$$

may be extended to a PL homeomorphism on all of $V_{n+1}(CP^1)$. We will show this using the very agreeable bundle structure on the complement W of the embedding γ_σ.

The map

$$\varphi_\sigma\colon D^2_- \times S^{2n-1} \cup_\rho D^2_+ \times S^{2n-1} \to D^2_- \times S^{2n-1} \cup_\rho D^2_+ \times S^{2n-1}$$

will in fact be defined as the union of three maps—

$$(1) \qquad \gamma_\sigma\colon D^2_+ \times S^{2n-1} \hookrightarrow D^2_+ \times S^{2n-1},$$

$$(2) \qquad \eta\colon \tilde{W}^{2n+1} \to W^{2n+1},$$

$$(3) \qquad \mathrm{id} \times \mu\colon D^2 \times \Sigma^{2n-1}_{-\sigma} \to D^2_- \times S^{2n-1}$$

where η is a bundle homeomorphism of bundles over S^1 and $\mu\colon \Sigma^{2n-1}_{-\sigma} \to S^{2n-1}$ is a PL homeomorphism and

$$D^2 \times \Sigma^{2n-1}_{-\sigma} \cup \tilde{W}^{2n+1} = D^2_- \times S^{2n+1}.$$

Essentially what we are producing in this construction is a map with the symmetric property that φ_σ embeds a fiber (the core of $D^2_+ \times S^{2n-1}$) piecewise linearly onto the smooth fibered knot $\Sigma^{2n-1}_{-\sigma} \subset D^2_- \times S^{2n-1}$ while φ_σ^{-1} embeds a fiber (the core of $D^2_- \times S^{2n-1}$) piecewise linearly onto the smooth fibered knot $\Sigma^{2n-1}_\sigma \subset D^2_- \times S^{2n-1}$.

The construction will be completed by (a) defining the bundle $\tilde{W}$ and the bundle map η in (2), (b) showing that $D^2 \times \Sigma^{2n-1}_{-\sigma} \cup \tilde{W}$ is in fact $D^2 \times S^{2n-1}$ by a PL homeomorphism which is the identity on the boundary, (c) showing that the maps (1), (2), (3) agree on boundaries after taking the defining automorphism ρ into account, and finally by (d) showing that φ_σ is homotopic to the identity.

We define the bundle $\tilde{W}$ over S^1 by defining its fiber $\tilde{F}$ and its monodromy map $\tilde{h}\colon \tilde{F} \to \tilde{F}$.

Recall that the $2n$-manifold F (fiber of W) is $(n-1)$ connected and that $\partial F = S^{2n-1} \dot{\cup}\, \Sigma^{2n-}_{-\sigma}$ where the smooth exotic sphere is defined as $\Sigma^{2n-1}_{\sigma} = D^{2n-1}_{-} \cup_{\sigma} D^{2n+1}_{+}$ and $\sigma\colon S^{2n-2} \to S^{2n-2}$ is an exotic diffeomorphism.

Let $I \subset F$ be a path connecting the centers of the discs D^{2n-1}_{+} and D^{2n-1}_{+} of Σ^{2n-1}_{σ} and S^{2n-1}. Then a tubular neighborhood of I is $I \times D^{2n-1}_{+}$. We define $\tilde{F}$ as the smooth manifold

$$\tilde{F} = \left[F - I \times D^{2n-1}_{+}\right] \cup \left[I \times D^{2n-1}_{+}\right]$$

where the union is taken over the diffeomorphism

$$\mathrm{id}_I \times \sigma^{-1}\colon I \times S^{2n-2} \to I \times S^{2n-2}.$$

Then $\partial \tilde{F} = \Sigma^{2n-}_{-\sigma} \dot{\cup}\, S^{2n-1}$ as a smooth manifold and we can define a PL homeomorphism

$$\hat{\eta}\colon \tilde{F} \to F$$

where $\hat{\eta}$ is the identity on $F - I \times D^{2n-1}_{+}$ and is $\mathrm{id}_I \times$ (cone extension of σ) on $I \times D^{2n-1}_{+}$.

Then we define the monodromy $\tilde{h}\colon \tilde{F} \to \tilde{F}$ as the composite map

$$\tilde{h} = \hat{\eta}^{-1} \circ h \circ \hat{\eta}$$

where $h\colon F \to F$ is the monodromy map defining the bundle W. Since ∂W is a trivial bundle we know that h is the identiy map on ∂F. Hence, $\tilde{h}$ is the identity on $\partial \tilde{F}$ and the bundle $\tilde{W}$ has the trivial boundary

$$\partial \tilde{W} = S^1 \times \Sigma^{2n-}_{-\sigma} \dot{\cup}\, S^1 \times S^{2n-1}.$$

Since $\hat{\eta} \circ \tilde{h} = h \circ \hat{\eta}$ the PL homeomorphism $\hat{\eta}\colon \tilde{F} \to F$ induces a well-defined bundle homeomorphism

$$\eta\colon \tilde{W}^{2n+1} \to W^{2n+1}.$$

Restricted to the boundary η is a pair of bundle maps

$$\mathrm{id}_{S^1} \times \alpha^{-1}_{-\sigma}\colon S^1 \times \Sigma^{2n-}_{-\sigma} \to S^1 \times S^{2n-1},$$

$$\mathrm{id}_{S^1} \times \alpha_{\sigma}\colon S^1 \times S^{2n-1} \to S^1 \times \Sigma^{2n-1}_{\sigma}$$

where the PL homeomorphism $\alpha_{-\sigma}$ and α_{σ} are the identity on D^{2n-1}_{-} and the cone extension of σ^{-1} and σ respectively on D^{2n-1}_{+}.

We next embed $\tilde{W}$ in $D^2 \times S^{2n-1}$ as a knot complement which will act as an inverse to W:

Recall the bundle isomorphism

$$(*) \qquad \rho: S^1 \times S^{2n-1} \to S^1 \times S^{2n-1}$$

which defines $\partial\nu_{n+1}(CP^1)$. We define a PL bundle map

$$\hat{\rho}: S^1 \times \Sigma_{-\sigma}^{2n-1} \to S^1 \times \Sigma_{-\sigma}^{2n-1}$$

as the composite: $\hat{\rho} = (\mathrm{id}_{S^1} \times \alpha_{-\sigma}) \cdot \rho \cdot (\mathrm{id}_{S^1} \times \alpha_{-\sigma})^{-1}$. We consider the PL manifold

$$D^2 \times \Sigma_{-\sigma}^{2n-1} \cup_{\hat{\rho}} \tilde{W}^{2n+1}$$

where the union is over the appropriate component of $\partial\tilde{W}$ and show:

PROPOSITION. *The PL manifold $D^2 \times \Sigma_{-\sigma}^{2n-1} \cup_{\hat{\rho}} \tilde{W}^{2n+1}$ is isomorphic to $D^2 \times S^{2n-1}$ by a PL homeomorphism Λ which restricted to the boundary $S^1 \times S^{2n-1}$ is an S^{2n-1} bundle isomorphism λ.*

Proof. We recall from the definition of W^{2n+1} that $S^1 \times D^{2n} \cup W^{2n+1}$ is the knot complement of our original Brieskorn knot and so has the homology of S^1. A simple exercise with the Mayer-Vietoris sequence implies then that the manifold $\tilde{W}^{2n+1} \cup S^1 \times D^{2n}$ likewise is a homology circle, and a second application of the sequence implies that the PL manifold.

$$P^{2n+1} = D^2 \times \Sigma_{-\sigma}^{2n-1} \cup_{\hat{\rho}} \tilde{W} \cup S^1 \times D^{2n}$$

has the homology of S^{2n+1}. Moreover, P^{2n+1} is simply connected since $\tilde{W} \cup S^1 \times D^{2n}$ fibers over S^1 with fiber $\tilde{F}^{2n} \cup D^{2n}$ which is $(n-1)$-connected. Hence $\pi_1(\tilde{W} \cup S^1 \times D^{2n}) = Z$ and by the Van Kampen theorem on the union

$$[D^2 \times \Sigma_{-\sigma}^{2n-1}] \cup_{S^1 \times \Sigma_{-\sigma}} [\tilde{W} \cup S^1 \times D^{2n}]$$

we have $\pi_1(P^{2n+1}) = 0$. By the Hurewicz and Whitehead theorems any simply-connected homology sphere is a homotopy sphere, and by the generalized Poincaré conjecture $(2n + 1 \geq 9) P^{2n+1}$ is a PL sphere.

The identification $D^2 \times \Sigma_{-\sigma}^{2n-1} \cup \tilde{W} S^1 \times D^{2n} \cong S^{2n+1}$ provides a PL embedding $S^1 \overset{i}{\subset} S^{2n+1}$ and exhibits $i(S^1 \times D^{2n}) \subset S^{2n+1}$ as a representative for the PL normal microbundle to this embedding. We apply a theorem due to Lashof and Rothenberg (Thm. 7.3 in [13]) to obtain a piecewise differentiable homeomorphism $g\colon S^{2n+1} \to S^{2n+1}$ so that $g \circ i\colon S^1 \times D^{2n} \to S^{2n+1}$ is the smooth vector bundle to the smooth embedding $g \circ i\colon S^1 \to S^{2n+1}$. By smoothly unknotting this circle and applying the smooth tubular neighborhood theorem we obtain a diffeomorphism $h\colon S^{2n+1} \to S^{2n+1}$ so that

$$\begin{tikzcd} h \circ g \circ i\colon S^1 \times D^{2n} \arrow[r] \arrow[dr, "\bar{\lambda}"'] & S^{2n+1} \\ & S^1 \times D^{2n} \arrow[u, "j"'] \end{tikzcd}$$

commutes where j is the standard embedding and $\bar{\lambda}$ is a vector bundle isomorphism. Hence, the restriction map

$$h \circ g \big| : S^{2n+1} - i(S^1 \times D^{2n}) \to S^{2n+1} - j(S^1 \times D^{2n}) \overset{\displaystyle \|}{\underset{\displaystyle D^2 \times S^{2n-1}}{}}$$

defines a piecewise differentiable homeomorphism

$$\Lambda\colon \left[D^2 \times \Sigma^{2n-}_{-\sigma} \cup_{\hat{\rho}} \hat{W}\right] \to D^2 \times S^{2n-1}$$

which restricts as $\lambda = \bar{\lambda}$ on the boundary. Finally, we observe that (cf. Cor. 10.13 in [19]) we may choose a smooth triangulation of $D^2 \times S^{2n-1}$ so that Λ is PL. Now, using the homeomorphisms Λ and η we define a PL homeomorphism:

$$\varphi_\sigma\colon \xi \to \partial \nu_{n+1}(CP^1) \tag{1}$$

where ξ is the S^{2n-1} bundle over $CP^1 = S^2$ defined by λ^{-1}:

$$\begin{aligned} \xi &= D^2_- \times S^{2n-1} \cup_{\lambda^{-1}} D^2_+ \times S^{2n-1} \\ &\xrightarrow{\Lambda^{-1} \cup \mathrm{id}} D^2 \times \Sigma^{2n-1}_{-\sigma} \cup_{\hat{\rho}} \tilde{W}^{2n+1} \cup_{\mathrm{id}} D^2_+ \times S^{2n-1} \\ &\xrightarrow{(\mathrm{id} \times \alpha_{-\sigma}) \cup \eta \cup (\mathrm{id} \times \alpha_\sigma)} D^2_- \times S^{2n-1} \cup_\rho W \cup D^2 \times \Sigma^{2n-1}_{-\sigma} \\ &= D^2_- \times S^{2n-1} \cup_\rho D^2_+ \times S^{2n-1} = \partial \nu_{n+1}(CP^1). \end{aligned}$$

From the next lemma to the effect that two non-isomorphic sphere bundles over S^2 cannot be PL homeomorphic it follows that the existence of the map φ_σ itself guarantees that ξ and $\partial \nu_{n+1}(CP^1)$ are the same bundle.

LEMMA. *For $m \geq 3$ the unique non-trivial orthogonal S^m bundle over S^2, ξ, is not PL homeomorphic to $S^2 \times S^m$.*

Proof. Suppose $t\colon \xi \to S^2 \times S^m$ is a PL homeomorphism. Let E be the non-trivial D^{m+1} bundle over S^2 with $\partial E = \xi$ and define the PL manifold

$$M^{m+3} = E \cup_t D^3 \times S^m$$

M is the union of simply connected spaces over a path connected intersection. Hence, $\pi_1(M) = \{1\}$. For $m \geq 3$ the homotopy exact sequence of the fibration $S^m \to \partial E \xrightarrow{p} S^2$ implies that $p_*\colon \pi_2(\partial E) \to \pi_2(S^2)$ is an isomorphism, and by the Whitehead theorem so is the inclusion $H_2(\partial E) \to H_2(E)$. Hence, in the Mayer-Vietoris sequence

$$\cdots \to H_j(S^2 \times S^m) \xrightarrow{\psi_j} H_j(E) \oplus H_j(D^3 \times S^m) \to H_j(M)$$
$$\to H_{j-1}(S^2 \times S^m) \to \cdots$$

ψ_j is an isomorphism for $j \leq m + 1$. Trivally, $H_{m+2}(M) = 0$, and again we have an $(m + 2)$-connected $(m + 3)$-dimensional PL manifold which is consequently a PL sphere.

Then, $E \cup_t D^3 \times S^m \cong S^{m+3}$ defines the vector bundle E as a PL normal micro-bundle to the embedding of its zero section $S^2 \hookrightarrow S^{m+3}$. By Zeeman's PL unknotting theorem and the uniqueness [7] of stable PL normal microbundles, we see that E and $S^2 \times D^{m+1}$ must be micro-bundle isomorphic. Let $S^2 \xrightarrow{b} \mathrm{BO}$ classify E as a vector bundle. Then $S^2 \xrightarrow[h]{} \mathrm{BO} \to \mathrm{BPL}$ is trivial, and as by smoothing theory the fiber PL/0 is 6-connected we see that b is homotopically trivial. As E was assumed non-trivial as a vector bundle the PL homeomorphism t cannot exist.

Thus, we define

$$\varphi_\sigma\colon \partial\nu_{n+1}(CP^1) = \zeta \to \partial\nu_{n+1}(CP^1) \quad \text{from (1) as required.}$$

Next we show that the φ_σ just constructed is indeed a self-knotting and that it will suffice for Theorem A.

Recalling from bundle theory that every S^N bundle over S^2 for $N \geq 2$ has a section, we show

PROPOSITION. *Any orientation preserving PL homeomorphism $\varphi\colon \nu \to \nu$, ν an orthogonal S^N bundle over S^2, which embeds a section $S^2 \overset{j}{\hookrightarrow} \nu$ homotopically to itself is homotopic to the identity.*

Proof. A tubular neighborhood of the section $j(S^2)$ is a D^N bundle U in the same stable bundle class as ν. $\varphi(U)$ PL embeds this bundle in ν with an inherited smooth structure. By the main theorem of smoothing theory ([8] or [13], Thm. 7.3) and the uniqueness of smoothings on S^2 we can piecewise differentially isotope this embedding to a smooth embedding of $U \to \nu$. We may easily make the isotopy ambient. Next, we smoothly unknot the core sphere of U and apply the smooth tubular neighborhood theorem. We have, therefore, P.D. isotoped φ so that restricted to U it is a D^N bundle isomorphism. Since $\pi_2(\mathrm{SO}(N)) = 0$ we can isotope this bundle mapping to the identity through bundle isomorphisms on U all of which extend to ν as U is a sub-bundle. Thus, we have isotoped φ so that it is the identity on U. Now, $\nu - U \cong U$ as each fiber of U is a hemisphere of a fiber in ν. We isotope $\varphi\,\mathrm{rel}(U)$ so that it is the identity on the zero section of the bundle $\nu - U$. Finally, we homotope φ to the identity by collapsing the fibers of $\nu - U$ to the zero-section.

We observe that the φ_σ constructed above satisfies the hypothesis of this last proposition as follows: φ_σ is orientation preserving by construction. Also, as the original Brieskorn knot embedded a fiber S^{2n+1} homotopically to the usual embedding, we know that φ_σ does also. That is $(\varphi_\sigma)_*[\partial\nu] = [\partial\nu]$ and $(\varphi_\sigma)^*(e^{2n-1}) = e^{2n-1}$, where $e^{2n-1} \in H^{2n-1}(\partial\nu)$ is the class represented by inclusion of a fiber. By Poincaré Duality, then, $(\varphi_\sigma)_*(e_2) = e_2$ for $e_2 \in H_2(\partial\nu)$ the class dual to e^{2n-1}. This implies by the Hurewicz Theorem that φ_σ induces the identity homomorphism on $\pi_2(\partial\nu)$, which is generated by the inclusion of a section.

The map φ_σ constructed in section C embeds a fiber S^{2n-1} onto the image of the Brieskorn knot. Hence, in the decomposition

$$\widetilde{CP}^{n+1} = \left[CP^{n+1} - \nu_{n+1}(CP^1)\right] \cup_{\varphi_\sigma} \left[\nu_{n+1}(CP^1)\right]$$

the identification is in the order:

$$\varphi_\sigma\colon \partial\left[CP^{n+1} - \nu\right] \to \partial\nu.$$

To show, therefore, that $\widetilde{CP}^{n+1} \leftrightarrow (0,\ldots,0,\sigma)$ we must extend φ_σ^{-1} to a homotopy equivalence $\bar{\varphi}_\sigma^{-1}\colon \nu \to \nu$ with transverse-inverse image of a fiber being the Milnor or Kervaire manifold M_0^{2n}. Note that any extension will be a homotopy equivalence as $\nu \simeq S^2$ and φ_σ^{-1} induces the identity on $\pi_2(\partial\nu) = \pi_2(\nu)$.

PROPOSITION. *The PL homeomorphism $\varphi_\sigma^{-1}\colon \partial\nu_{N+1}(CP^1) \to \partial\nu_{n+1}(CP^1)$ constructed above extends to $\bar{\varphi}_\sigma^{-1}\colon \nu_{n+1}(CP^1) \to \nu_{n+1}(CP^1)$ with transverse-inverse image*

$$\left(\bar{\varphi}_\sigma^{-1}\right)^{-1}(D^{2n}) = M_0^{2n}$$

Proof. $(\varphi_\sigma^{-1})^{-1}(S^{2n-1}) = \varphi_\sigma(S^{2n-1}) = \Sigma_\sigma^{2n-} \subset \partial\nu$ by the construction of φ_σ. Moreover, the restriction $\varphi_\sigma^{-1}| : D^2 \times \Sigma_\sigma^{2n-1} \to D_+^2 \times S^{2n-1}$ is a product map. Now, Σ_σ^{2n-1} bounds a fiber $F^{2n} \subset W^{2n+1}$ whose other boundary component is a fiber S^{2n-1} of $\partial\nu$. Let $D^{2n} \subset \nu$ be the fiber whose boundary is this same sphere. Then, $F^{2n} \cup D^{2n} = M_0^{2n}$ by the definition of F^{2n}. By pushing F^{2n} into ν along a vector field normal to $\partial\nu$ and smoothing the corner at S^{2n-1} between F^{2n} and D^{2n} we obtain a smooth embedding $M_0^{2n} \hookrightarrow \nu$ extending

$$\partial M_0^{2n} = \Sigma_\sigma^{2n-1} \subset \partial\nu.$$

Moreover, this embedding will have trivial normal D^2 bundle as $H^1(M_0^{2n}, Z) = 0$. Hence, we can extend the product map

$$\varphi_\sigma^{-1}: D^2 \times \Sigma_\sigma^{2n-1} \to D_+^2 \times S^{2n-1}$$

to a bundle map $\hat{\varphi}_\sigma^{-1}: D^2 \times M_0^{2n} \to D_+^2 \times D^{2n}$ covering a degree one extension $M_0^{2n} \to D^{2n}$. Since $[\nu - D_+^2] \times D_-^2 \times D^{2n} = D^{2n-2}$ there are no cohomology obstructions to extending

$$\varphi_\sigma^{-1} \cup \hat{\varphi}_\sigma^{-1} \text{ to } \overline{\varphi_\sigma^{-1}}: \nu \to \nu$$

with the required transverse-inverse image built in.

References

[1] Y. S. Akbulut, *Algebraic Equations for a Class of P. L. Manifolds*, Ph.D. Thesis, Berkeley, 1974.

[2] Y. S. Akbulut, and H. C. King, *Real algebraic variety structures on PL manifolds*, Bull. Amer. Math. Soc., **83** (1977), 281–282.

[3] W. Browder, *Manifolds with $\pi_1 = Z$*, Bull. Amer. Math. Soc., **72** (1966), 238–244.

[4] ———, *Surgery on Simply-Connected Manifolds*, Springer-Verlag, New York, 1972.

[5] G. Brumfiel, I. Madsen, and R. J. Milgram, *PL characteristic classes and cobordism*, Annals of Math., **97** (1973), 82–159.

[6] A. Haefliger, *Plongemonts differentiables de varietés dans varietés*, Comm. Math. Helv., **36** (1961), 47–82.

[7] A. Haefliger and C. T. C. Wall, *Piecewise linear bundles in the stable range*, Topology, **4** (1965), 209–214.

[8] M. W. Hirsch and B. Mazur, *Smoothings of Piecewise Linear Manifolds*, Princeton, 1974.

[9] F. Hirzebruch, *Singularities and Exotic Spheres*, Seminar Bourbaki No. 314, 1966.

[10] M. Kato, *A concordance classification of PL homeomorphisms of $S^p \times S^q$*, Topology, **8** (1969), 371–383.

[11] M. Kervaire and J. Milnor, *Groups of homotopy spheres I*, Annals of Math., **77** (1963), 504–537.

[12] S. Lang, *Introduction to Differentiable Manifolds*, Interscience, New York, 1962.

[13] R. Lashof and M. Rothenberg, *Microbundles and Smoothing*, Topology, **3** (1965), 357–388.

[14] I. Madsen and R. J. Milgram, *The oriented topological and PL cobordism rings*, Bull. Amer. Math. Soc., **80** (1974), 855–860.
[15] _____, *The Classifying Spaces for Surgery and Cobordism of Manifolds*, Princeton University Press, Princton New Jersey, 1979.
[16] D. Meadows, *Some groups of PL self-knotting*, Houston J. Math., (to appear).
[17] J. Milnor, *Singular Points of Complex Hypersurfaces*, Princeton University Press, Princeton, New Jersey, 1968.
[18] J. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton, New Jersey, 1974.
[19] J. Munkres, *Elementary Differential Topology*, Princeton, 1966.
[20] C. P. Rourke and B. J. Sanderson, *Block bundles I, II, and III*, Ann. of Math., **87** (1968), 1–28, 256–278, 431–483.
[21] _____, *Introduction to Piecewise-Linear Topology*, Springer-Verlag, New York, 1972.
[22] S. Smale, *Differentiable and combinatorial structures on manifolds*, Ann. of Math., **74** (1961), 498–502.
[23] D. Sullivan, *Triangulating and smoothing homotopy equivalences*, mimeographed notes, Princeton University.
[24] C. T. C. Wall, *Surgery on Compact Manifolds*, Academic Press, New York, 1970.
[25] G. W. Whitehead, *Homotopy Theory*, M.I.T. Press, Cambridge, Massachusetts, 1966.
[26] R. E. Williamson Jr., *Cobordism of combinatorial manifolds*, Ann. of Math., **83** (1966), 1–133.
[27] H. Winkelnkemper, *Manifolds as open books*, Bull. Amer. Math. Soc., **79** (1973), 45–51.
[28] E.-C. Zeeman, *Seminar on combinatorial topology*, I.H.E.S., 1963.

Received March 24, 1980.

University of Rochester
Rochester, NY 14627