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A NOTE ON TAMELY RAMIFIED EXTENSIONS OF RINGS

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Buhler gave a criterion for a class of finite free extensions of discrete valuation rings to be tamely ramified 1-dimensional regular rings. In this note, we extend this criterion to finite free extensions of general local rings and, in the final section, indicate the extension to schemes.

1. Introduction. To set the notation, let A be a noetherian local ring of Krull dimension n and let $A \to B$ be a finite free extension of rings; denote by $\mathfrak{d}_{B/A}$ the discriminant of this extension, defined as $\det[\operatorname{tr}(b_i b_j)]$ where $b_1, \ldots, b_m$ is a free basis of B over A and tr: $B \to A$ denotes the trace morphism. Let $\mathfrak{m}_A$ be the maximal ideal of A and define a function $\nu_{\mathfrak{m}_A}$ on A by $\nu_{\mathfrak{m}_A}(x) = r$ where r is the largest integer with $x \in \mathfrak{m}_A^r$ and $\nu_{\mathfrak{m}_A}(0) = \infty$. Note that $\nu_{\mathfrak{m}_A}$ is a valuation if $\operatorname{gr}_A(\mathfrak{m}_A)$ has no zero divisors, in particular if A is regular [2].

If $\mathfrak{n}_1, \ldots, \mathfrak{n}_s$ are the maximal ideals of B lying over $\mathfrak{m}_A$ define the *ramification index* $e_{\mathfrak{n}_i/\mathfrak{m}_A}$ to be $l_{B_{\mathfrak{n}_i}}(B_{\mathfrak{n}_i}/\mathfrak{m}_A B_{\mathfrak{n}_i})$ where $l_B(M)$ denotes the length (of a composition series) of the artin B-module M. If A is a discrete valuation ring, the $e_{\mathfrak{n}_i/\mathfrak{m}_A}$ clearly coincide with the usual ramification indices of algebraic number theory. Recall that the embedding dimension $\operatorname{ed}(B)$ of the semi-local ring B is $\max \dim_{\kappa(\mathfrak{n}_i)} \mathfrak{n}_i/\mathfrak{n}_i^2$ where $\mathfrak{n}_i$ runs through all maximal ideals of B. With the above notation the main result of this paper is:

THEOREM 1. *If A is regular* (resp. $\operatorname{gr}_A(\mathfrak{m}_A)$ *has no zero divisors*) *and if $B = A[X]/\langle f(X)\rangle$ where $f(X)$ is a monic polynomial and $\kappa(\mathfrak{m}) \to \kappa(\mathfrak{n}_i)$ is separable for all $i = 1, \ldots, s$, then*

$$\nu_{\mathfrak{m}_A}(\mathfrak{d}_{B/A}) \geq \sum_{i=1}^{s} \left(e_{\mathfrak{n}_i/\mathfrak{m}_A} - 1\right)[\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_A)]$$

with equality if and only if (resp. *only if*) $\operatorname{ed}(B) = \operatorname{ed}(A)$ *and B is tamely ramified over A in that $p \nmid e_{\mathfrak{n}_i/\mathfrak{m}_A}$ for all i, where p is the characteristic of $\kappa(\mathfrak{m}_A)$.*

2. Proof of Theorem 1. We begin by some reductions. Observe that the conditions and conclusions of the theorem remain unchanged on base change by the $\mathfrak{m}_A$-adic completion of A: $A^{\wedge} \to B \otimes_A A^{\wedge} \cong B^{\wedge}$ so we may assume A is complete. Thus B is a product of local rings $\prod B_i$ where each $A \to B_i$ satisfies the conditions of Theorem 1. Since $\nu_{\mathfrak{m}_A}(\mathfrak{d}_{B/A}) = \sum_i \nu_{\mathfrak{m}_A}(\mathfrak{d}_{B_i/A})$, it is easy to see that it is enough to prove Theorem 1 when B is local with maximal ideal $\mathfrak{m}_B$, say.

Let e be the ramification index of B over A and $a_1, \ldots, a_n$ form a basis of the cotangent space $\mathfrak{m}_A/\mathfrak{m}_A^2$ over $\kappa(\mathfrak{m}_A)$. There is a monic polynomial $g \in A[X]$ with $f = g^e + \sum_i a_i h_i$ where $h_i \in A[X]$ for all i. Letting $R(p(X), q(X))$ denote the resultant of the polynomials p and q (see **[3]** or **[5]** for the properties of resultants we will use), then

$$\mathfrak{d}_{B/A} = R(f, f') = R\Big(f, eg^{e-1}g' + \sum_i a_i h_i'\Big).$$

If $p \mid e$ where $p = \operatorname{char} \kappa(\mathfrak{m}_A)$, then $e \in \mathfrak{m}_A$ and so

$$\begin{aligned}\nu_{\mathfrak{m}_A}(\mathfrak{d}_{B/A}) &= \nu_{\mathfrak{m}_A}\Big(R\Big(f, eg^{e-1}g' + \sum_i a_i h_i'\Big)\Big)\\ &\geq \operatorname{degree} f = e[\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)].\end{aligned}$$

This completes the proof for the case of wild ramification.

Assume from now on that $e \notin \mathfrak{m}_A$. Since $f' \equiv eg'g^{e-1} \bmod \mathfrak{m}_A$ and $\kappa(\mathfrak{m}_A) \to \kappa(\mathfrak{m}_B)$ is separable, eg' and g^{e-1} are relatively prime in $\kappa(\mathfrak{m}_A)[X]$. Thus by Hensel's lemma

$$f' = \Big(eg' + \sum_i a_i p_i\Big)\Big(g^{e-1} + \sum_i a_i q_i\Big)$$

where $p_i q_i \in A[X]$ with $\deg(p_i) < \deg(g')$, $\deg(q_i) < \deg g^{e-1}$ for all i.

Since $\nu_{\mathfrak{m}_A}(R(g^e, eg')) = e\nu_{\mathfrak{m}_A}(R(g, g')) = 0$ we have

$$\nu_{\mathfrak{m}_A}\Big(R\Big(f, eg' + \sum a_i p_i\Big)\Big) = 0.$$

Thus

$$\begin{aligned}\nu_{\mathfrak{m}_A}(\mathfrak{d}_{B/A}) &= \nu_{\mathfrak{m}_A}\Big(R\Big(f, eg' + \sum_i a_i p_i\Big)\Big) + \nu_{\mathfrak{m}_A}\Big(R\Big(f, g^{e-1} + \sum_i a_i q_i\Big)\Big)\\ &= \nu_{\mathfrak{m}_A}\Big(R\Big(f, g^{e-1} + \sum_i a_i q_i\Big)\Big);\end{aligned}$$

we conclude that if $e = 1$ then $\nu_{\mathfrak{m}_A}(\mathfrak{d}_{B/A}) = 0$ and $\mathfrak{m}_A B = \mathfrak{m}_B$ proving the theorem for the unramified case $e = 1$.

Assume from now on that $e \geq 2$ and put $r = g^{e-1} + \sum_i a_i q_i$. Then

$$\nu_{\mathfrak{m}_A}(\mathfrak{d}_{B/A}) = \nu_{\mathfrak{m}_A}(R(f - gr, r)) = \nu_{\mathfrak{m}_A}\left(R\left(\sum_i a_i(h_i - gq_i), r\right)\right)$$
$$\geq \deg r = (e-1)[\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$$

and equality holds if and only if (resp. only if)

$$I = \langle h_1 - gq_1, \ldots, h_n - gq_n, r \rangle \kappa(\mathfrak{m}_A)[X] = \kappa(\mathfrak{m}_A)[X]$$

by Lemma 1 below. This completes the proof, for $I = \kappa(\mathfrak{m}_A)[X]$ if and only if some h_i is invertible in $\kappa(\mathfrak{m}_B)$ and so if and only if $\sum a_i h_i = f - g^e \equiv 0 \bmod \mathfrak{m}_B^2$ gives a non-trivial linear relation between the a_i's and g in $\mathfrak{m}_B/\mathfrak{m}_B^2$.

LEMMA 1. *If A is a regular local ring* (resp. *a local ring*) *and $a_1, \ldots, a_n$ a basis of $\mathfrak{m}_A$ and $p_0, \ldots, p_n \in A[X]$ with p_0 monic, then*

$$R\left(\sum_{i=1}^{n} a_i p_i, p_0\right) \notin \mathfrak{m}_A^{1+\deg p_0}$$

if and only if (resp. *only if*) *$p_0, \ldots, p_n$ are coprime in $\kappa(\mathfrak{m}_A)[X]$.*

Proof. If A is an arbitrary local ring, let $m \in A[X]$ be a monic polynomial with residue in $\kappa(\mathfrak{m}_A)[X]$ the highest common factor of $p_0, \ldots, p_n$. Then for some $q_i \in A[X]$ with q_0 monic and $\deg mq_0 = \deg p_0$, $R(\sum_i a_i p_i, p_0) \equiv R(m\sum a_i q_i, mq_0) \bmod \mathfrak{m}_A^{\deg p_0 + 1}$ so if $\deg m \geq 1$, $R(\sum a_i p_i, p_0) \in \mathfrak{m}_A^{\deg p_0 + 1}$ as required.

Conversely, if A is regular $\mathrm{gr}_A(\mathfrak{m}_A)$ is a polynomial ring $\kappa(\mathfrak{m}_A)[X_1, \ldots, X_n]$ **[2]**, with the usual grading, so that monomials of total degree d in $a_1, \ldots, a_n$ are linearly independent in $A/\mathfrak{m}_A^{d+1}$. Since $R(\sum_i a_i p_i, p_0)$ is a homogeneous polynomial of degree $\deg p_0$ in the a_i in A, $\nu_{\mathfrak{m}_A}(R(\sum_i a_i p_i, p_0)) = 1 + \deg p_0$ if and only if $R(\sum Z_i p_i, p_0)$ is the zero polynomial in the ring $\kappa(\mathfrak{m}_A)[Z_1, \ldots, Z_n]$ where the Z_i's are indeterminates.

Now if p_i are coprime in $\kappa(\mathfrak{m}_A)[X]$ then $\sum_{i=0}^{n} c_i p_i \equiv 1 \bmod \mathfrak{m}_A$ for some $c_i \in A[X]$. Thus

$$\nu_{\mathfrak{m}_A}\left(R\left(\sum_{i=1}^{n} c_i p_i, p_0\right)\right) = \nu_{\mathfrak{m}_A}\left(R\left(\sum_{i=0}^{n} c_i p_i, p_0\right)\right) = 0$$

so $R(\sum Z_i p_i, p_0)$ is not the zero polynomial in $\kappa(\mathfrak{m}_A)[Z]$ proving the lemma. $\square$

3. The obstruction for non-regular rings. Throughout this section the local ring A is assumed to have no zero divisors in $\mathrm{gr}_A(\mathfrak{m}_A)$.

For regular rings, Theorem 1 gives a necessary and sufficient numerical criterion for $A \to A[X]/\langle f(X)\rangle$ to be tamely ramified with $\mathrm{ed}(f) = \mathrm{ed}(A)$. The failure of this criterion to be necessary for non-regular rings is examined in this section; we will see that the obstruction lies in the equations defining the tangent cone $\mathrm{gr}_A(\mathfrak{m}_A)$. Indeed, we construct a cohomology group $H^2(C_g^{\cdot})$ so that the numerical criterion is necessary and sufficient for all polynomials with a fixed reduction g mod $\mathfrak{m}_A$, say, if and only if $H^2(C_g^{\cdot})$ is isomorphic to the vector space of homogeneous equations defining the tangent cone of degree equal to that of $g(X)$.

In the sense of Hilbert schemes classifying polynomials over A, this failure is not exceptional: "almost all" tamely ramified polynomials $f(X)$ over a non-regular ring A with $\nu(\mathrm{discr}\, f) > \Sigma_i(e_i - 1)[\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m}_A)]$ have $\mathrm{ed}(f(X)) = \mathrm{ed}(A)$;

Nevertheless, for polynomials which are unramified or totally ramified or have degree ≤ 3, the numerical criterion is necessary and sufficient over arbitrary local rings.

Fix a monic polynomial $g(X) = X^m + \Sigma_{i=0}^{m-1} \bar{b}_{i+1} X^i$ in $\kappa(\mathfrak{m}_A)[X]$ and let $b_i \in A$ be elements with residue $\bar{b}_i$ for all i. Then $g(X)$ factorizes as $\Pi_i\, \bar{g}_i(X)^{e_i}$ over $\kappa(\mathfrak{m}_A)$ where we assume $\bar{g}_i(X)$ are distinct *separable* polynomials over $\kappa(\mathfrak{m}_A)$.

The Hilbert scheme $H_g = \mathrm{Spec}\, A[X_1,\ldots,X_m]_{\langle X_1 - b_1,\ldots,X_m - b_m, \mathfrak{m}_A\rangle}$ classifies the monic polynomials with reduction g mod $\mathfrak{m}_A$ in that there is a bijection:

$$H_g(\mathrm{Spec}\, A) \,\tilde{\to}\, \{\text{Monic polynomials } f(X) \text{ over } A \text{ with } f(X) \equiv g(X) \bmod \mathfrak{m}_A\}$$

given by

$$\{A[X_1,\ldots,X_m]_{\mathfrak{n}} \to A\colon X_i \mapsto c_i + b_i\} \to X^m + \sum_{i=0}^{m-1} (c_{i+1} + b_{i+1})X^i.$$

Let T be the tangent cone of Spec A, by definition $T = \mathrm{Proj}\, \mathrm{gr}_A(\mathfrak{m}_A)$ where $\mathrm{gr}_A(\mathfrak{m}_A) = \oplus_{i=0}^{\infty} \mathfrak{m}_A^i/\mathfrak{m}_A^{i+1}$; fix a basis, once and for all, $a_1,\ldots,a_n$ of $\mathfrak{m}_A$ so that $n = \mathrm{ed}(A)$. Let $T' = \mathbf{A}_\kappa^{nm} \times_\kappa T$ where $\kappa = \kappa(\mathfrak{m}_A) = A/\mathfrak{m}_A$ and $\mathbf{A}_\kappa^{nm} = \mathrm{Spec}\, \kappa[X_{ij}\colon 1 \leq i \leq m,\ 1 \leq j \leq n]$ is affine nm-space over κ. Regarding Spec κ as a T-scheme, via projection onto the 1st component $\mathrm{gr}_A(\mathfrak{m}_A) \to A/\mathfrak{m}_A = \kappa$, there is a bijection:

$$H_g(\mathrm{Spec}\, A/\mathfrak{m}_A^2) \,\tilde{\to}\, T\text{-sch}(\mathrm{Spec}\, \kappa, T')$$

given by

$$\{f\colon X_i \to c_i + b_i\} \to \left\{X_{ij} \to x_{ij} \in \kappa \text{ all } i,\, j \text{ where } c_i \equiv \sum_{j=1}^{n} a_j x_{ij} \bmod \mathfrak{m}_A^2\right\}.$$

Denote by * the composite of the maps:

$$H_g(\operatorname{Spec} A) \overset{\text{natural}}{\to} H_g(\operatorname{Spec} A/\mathfrak{m}_A^2) \tilde{\to} T\text{-sch}(\operatorname{Spec} \kappa, T').$$

PROPOSITION 1. (1) *The integer* $s = \Sigma_i(e_i - 1)[\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m}_A)]$ *is the same for all polynomials in* $H_g(\operatorname{Spec} A)$.

(2) *There are closed subschemes* $V \supset V'$ *of* T' *so that for any* $h \in H_g(\operatorname{Spec} A)$ *with associated polynomial* $f(X)$,

(a) $\nu_{\mathfrak{m}_A}(\operatorname{discr} f(X)) > s$ *if and only if* $h^* \in T$-sch-$(\operatorname{Spec} \kappa, V)$,

(b) $\operatorname{ed}(f(X)) > \operatorname{ed}(A)$ *if and only if* $h^* \in T$-sch$(\operatorname{Spec} \kappa, V')$.

(3) V *is a proper closed subscheme of* T *if and only if all polynomials in* $H_g(\operatorname{Spec} A)$ *are tamely ramified.*

Proof. (1) Clear.

(2) Recall $g(X)$ factorises as $\Pi_i\, \bar{g}_i^{e_i}$ in $\kappa[X]$ and choose representative monic polynomials $g_i(X) \in A[X]$ with residue $\bar{g}_i(X) \bmod \mathfrak{m}_A$ for all i.

As in the proof of Theorem 1, it is not difficult to see that $\nu_{\mathfrak{m}_A}(\operatorname{discr} f(X)) > s$ if and only if

$$\nu_{\mathfrak{m}_A}\left(R\left(f(X),\, \sum_i e_i g_i' g_i^{e_i-1} \prod_{j\neq i} g_j^{e_j}\right)\right) > s.$$

Now

$$R\left(f(X),\, \sum_i e_i g_i' g_i^{e_i-1} \prod_{j\neq i} g_j^{e_j}\right) = R\left(f,\, \prod_i g_i^{e_i-1}\right) R\left(f,\, \sum_i e_i g_i' \prod_{j\neq i} g_j\right);$$

as in the proof of Theorem 1, $\nu_{\mathfrak{m}_A}(R(f(X),\, \Sigma_i\, e_i g_i' \Pi_{j\neq i}\, g_j)) = 0$ if and only if p does not divide e_i for all i where p is the characteristic of $\kappa(\mathfrak{m}_A)$.

It follows from Theorem 1 that if $p \mid e_i$ for some e_i then $\nu_{\mathfrak{m}_A}(\operatorname{discr} f) > s$ for any f with reduction g so for this wildly ramified case $V = T'$ has the required properties. If now $p \nmid e_i$ for all i then $\nu_{\mathfrak{m}_A}(\operatorname{discr} f(X)) > s$ if and only if $\nu_{\mathfrak{m}_A}(R(f, g_j)) > \deg g_j(X)$ for some j. Putting

$$f(X) = X^m + \sum_{i=0}^{m-1} \left(b_i + \sum_{j=1}^{n} a_j X_{ij}\right) X^i,$$

with the notation as previously, $f(X)$ is the general polynomial of $H_g(\operatorname{Spec} A)$; $R(f(X), g_j(X))$ is a homogeneous polynomial in the X_{ij}'s of degree $\deg g_j(X)$. Moreover, the coefficient of each monomial in the X_{ij}'s is a monomial in the a_j's of degree $\deg g_j(X)$. Let $p_k(X_{ij}$: all $ij)$ be the polynomial $R(f(X), g_k(X))$-regarded as an element of $\mathrm{gr}(\mathfrak{m}_A)[X_{ij}]$ of degree $\deg g_k$ and put $p(X_{ij}) = \prod_k p_k(X_{ij}$: all $i, j)^{e_k}$. The ideal of $\mathrm{gr}(\mathfrak{m}_A)[X_{ij}$: all $i, j]$ generated by $p(X_{ij})$ clearly defines the closed subscheme V of T'.

(2b) With the notation above, let $f^*(X)$ be a polynomial from $H_g(\operatorname{Spec} A)$ then $f^*(X) = \prod_i g_i^{e_i} + \sum_{i=1}^n a_i p_i(X)$ for some $p_i(X) \in A[X]$. We assert $\mathrm{ed}(f^*(X)) = \mathrm{ed}(A)$ if and only if $\prod_i g_i^{e_i}, p_1(X), \ldots, p_n(X)$ have no common factor in the residue ring $\kappa(\mathfrak{m}_A)[X]$. For, without loss of generality A is complete as in the proof of Theorem 1, so $f^*(X) = \prod_i (g_i^{e_i} + \sum_{j=1}^n a_j p_{ij}(X))$ for some polynomials $p_{ij}(X) \in A[X]$ by Hensel's lemma. By the proof of Theorem 1 and Lemma 1, $\mathrm{ed}(f^*(X)) = \mathrm{ed}(A)$ if and only if $g_i(X), p_{i1}(X), \ldots, p_{in}(X)$ have no common factor in $\kappa(\mathfrak{m}_A)[X]$ for all i. The assertion easily follows on expanding the product for $f^*(X)$.

For a general polynomial $f(X)$ in $H_g(\operatorname{Spec} A)$ put, as before, $f(X) = X^m + \sum_i (b_i + \sum_j a_j X_{ij}) X^i$. Let $f^*(X)$ denote the specialisation of $f(X)$ under $X_{ij} \to x_{ij} \in \kappa$, then $\mathrm{ed}(f^*(X)) = \mathrm{ed}(A)$ if and only if $\prod_i g_i^{e_i}$, $\sum_i x_{ij} X^i$, $j = 1, \ldots, n$, have no common factor in $\kappa(\mathfrak{m}_A)[X]$. Introducing arbitrary parameters $Z_1, \ldots, Z_n$, then $\mathrm{ed}(f^*(X)) > \mathrm{ed}(A)$ if and only if, by Lemma 1, $R(g(X), \sum_{j=1}^n Z_j \sum_i x_{ij} X^i)$ is the zero polynomial, regarded as a polynomial in $\kappa(\mathfrak{m}_A)[Z_1, \ldots, Z_n]$ by taking it mod $\mathfrak{m}_A^{\deg g + 1}$.

Thus $R(g(X), \sum_{j=1}^n Z_j \sum_{i=0}^{m_j} X_{ij} X^i)$ mod $\mathfrak{m}_A^{1+\deg g}$ is a homogeneous polynomial of degree $\deg g(X) = m$ in the Z_i, assuming it is non-zero.

Write $R(g(X), \sum_{ij} Z_j X_{ij} X^i) \equiv \sum Z_{\mathbf{i}} q_{\mathbf{i}}(X_{ij})$ mod $\mathfrak{m}_A^{1+\deg g}$ where $Z_{\mathbf{i}}$ runs over all monomials in Z_i of degree m and $q_{\mathbf{i}}(X_{ij}) \in \kappa(\mathfrak{m}_A)[X_{ij}$: $0 \le i \le m-1$, $1 \le j \le n]$ is an homogeneous polynomial of degree m. Thus $\mathrm{ed}(f^*(X)) > \mathrm{ed}(A)$ if and only if $q_{\mathbf{i}}(x_{ij}) = 0$ for all $\mathbf{i}$.

Let V' be the closed subscheme of T' defined by the ideal $< q_{\mathbf{i}}(X_{ij})$: all $\mathbf{i} >$, then clearly V' has the required properties. □

From the above proof we deduce:

COROLLARY 1. (1) *Either $V = T'$ or V is a union of hypersurfaces of T' of degree $t_k = [\kappa(\mathfrak{n}_k): \kappa(\mathfrak{m}_A)]$, with multiplicity e_k, for all k, and is defined by an homogeneous equation $\prod_k f_k(X_{ij}$: all $ij)^{e_k} = 0$ of degree $m = \deg g(x)$ with coefficients of X_{ij} in f_k homogeneous polynonials in $a_1, \ldots, a_n$ of degree t_k.*

(2) *V' is defined in T' by most $\binom{n-1}{n-1}$ equations of degree m in the variables X_{ij} and with coefficients in $\kappa(\mathfrak{m}_A)$.*

We relate the equations defining V, V' to those defining the tangent cone T in its embedding $T \to \mathbf{P}_k^n$ given by the very ample sheaf $O_T(1)$. Let $S^m(\mathfrak{m}_A/\mathfrak{m}_A^2)$ denote the mth symmetric power of $\mathfrak{m}_A/\mathfrak{m}_A^2$ and let K_m be the kernel of the natural map $S^m(\mathfrak{m}_A/\mathfrak{m}_A^2) \xrightarrow{k} \mathfrak{m}_A^m/\mathfrak{m}_A^{m+1}$; thus K_m is the set of "equations of degree m defining T".

Suppose g is tamely ramified and let $m = \deg g(x)$ and $s = \Sigma_i(e_i - 1)[\kappa(\mathfrak{n}_i): \kappa(\mathfrak{m}_A)]$, then there is a complex $C^{\cdot}$:

$$0 \to T - \mathrm{sch}(\kappa, V') \xrightarrow{i} T - \mathrm{sch}(\kappa, V) \xrightarrow{j} S^m(\mathfrak{m}_A/\mathfrak{m}_A^2) \xrightarrow{k} \mathfrak{m}_A^m/\mathfrak{m}_A^{m+1}$$

where i is the natural inclusion ('complex' meaning that composites of successive maps are zero: note that each component of the complex has a distinguished zero element).

To define the complex it is only necessary to define j. Let $f(X_{ij}$: $0 \le i \le m-1,\ 1 \le j \le n) = 0$ be the equation defining V. Since $\mathrm{gr}_A(\mathfrak{m}_A)$ is without zero divisors, by Corollary 1(1) the coefficients of $f(X_{ij})$ are polynomials in the a_i's of degree m. The proof of Proposition 1(2b) actually constructs a polynomial $f^{\#}(X_{ij})$ in $S^{\cdot}(\mathfrak{m}_A/\mathfrak{m}_A^2)[X_{ij}$: $0 \le i \le m-1,\ 1 \le j \le n]$, $S^{\cdot}(\mathfrak{m}_A/\mathfrak{m}_A^2)$ denoting the symmetric algebra, whose image in $\mathrm{gr}(\mathfrak{m}_A)[X]$ is $f(X_{ij})$ under the canonical map. Denote by $a_1^{\#},\dots,a_n^{\#}$ the unique liftings of $a_1,\dots,a_n$ in $S^{\cdot}(\mathfrak{m}_A/\mathfrak{m}_A^2)$. Let $z \in T$-sch(κ, V) be given by $\{X_{ij} \to x_{ij} \in \kappa$ for all i, j with $f(x_{ij}) = 0\}$ and define $j(z) = f^{\#}(x_{ij}) \in S^m(\mathfrak{m}_A/\mathfrak{m}_A^2)$. Clearly $k \circ j = 0$ since $k \circ j(z) = k(f^{\#}(x_{ij})) = f(x_{ij}) = 0$. Note that the coefficients of $f^{\#}(X_{ij})$, regarded as a polynomial in $a_1^{\#},\dots,a_n^{\#} \in S^{\cdot}(\mathfrak{m}_A/\mathfrak{m}_A^2)$ with coefficients in $\kappa(\mathfrak{m}_A)[X_{ij}$: all $ij]$, are precisely the equations defining V'. Thus $j(z) = 0$ if and only if $z = i(y)$ for some $y \in T$-sch(κ, V') thus showing $j \circ i = 0$, and $C^{\cdot}$ is a complex. taking cohomology, we deduce $H^0(C^{\cdot}) = H^1(C^{\cdot}) = 0$.

From Proposition 1, $\{\nu(\mathrm{discr}\, f) = s$ if and only if $\mathrm{ed}(f) = \mathrm{ed}(A)$, for every $f(X)$ in $H_g(\mathrm{Spec}\, A)\}$ if and only if i is surjective, thus if and only if j is the zero map. We deduce:

Proposition 2. *$H^2(C^{\cdot}) \cong K_m$ if and only if $\{\nu(\mathrm{discr}\, f(x)) = s \Leftrightarrow \mathrm{ed}(f(x)) = \mathrm{ed}(A)$, for all $f(x)$ in $H_g(\mathrm{Spec}\, A)\}$.*

Corollary 2. *Suppose $f(X)$ has reduction $\Pi_i\, g_i(X)^{e_i}$ mod $\mathfrak{m}_A$ which has one of the following:*

(1) *f is totally ramified* i.e. $\deg g_i = 1$ *for all i,*

(2) *f is unramified* i.e. $e_i = 1$ *for all i,*

(3) $K_m = 0$ *where* $m = \deg f(x)$,

(4) $\deg f(X) \leq 3$,

then $\nu(\operatorname{discr} f(X)) = \Sigma_i(e_i - 1)[\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m}_A)]$ *if and only if f is tamely ramified and* $\operatorname{ed}(f(X)) = \operatorname{ed}(A)$.

Proof. In any case, if $f(X)$ is wildly ramified the result follows so we assume f is tamely ramified.

(1) Since f is totally ramified, the equation $p(X_{ij})$ defining V is, by Corollary 1, a product $\Pi_k\, p_k(X_{ij})$ of factors linear in the X_{ij}'s and a_i's. Let $z \in T\text{-sch}(\kappa, V)$ be given by $X_{ij} \to x_{ij} \in \kappa$ for all i, j, then $p(X_{ij}) = 0$ implies $p_k(x_{ij}) = 0$ for some k since $\operatorname{gr}_A(\mathfrak{m}_A)$ has no zero divisors. Thus $p_k(x_{ij}) = 0$ is a linear relation between the linearly independent a_i's so $j(z) = 0$. Since $H^1(C^{\cdot}) = 0$, $z = i(y)$ for some $y \in T - \operatorname{sch}(\kappa, V')$ proving the corollary in view of Proposition 1.

(2) If f is unramified, then obviously $\nu_{\mathfrak{m}_A}(\operatorname{discr} f) = 0$ since $f(X) = 0$ has distinct roots mod m_A. Thus $\operatorname{ed}(f(X)) = \operatorname{ed}(A)$ by Theorem 1.

(3) The Corollary follows immediately from Proposition 2.

(4) If $\deg f(X) \leq 3$ then the only possibilities are that f is totally ramified or is unramified whence the result from (1) and (2). □

Since resultants are 'universally' defined it easily follows from the proof of Proposition 1 that the subschemes V, V' of T' have a 'universal' construction in that they are induced from $\mathbf{Z}$-schemes independent of T':

PROPOSITION 3. *Given non-negative integers* $n, f_1, \ldots, f_r, e_1, \ldots, e_r$ *there are affine* $\mathbf{Z}$*-schemes* Z, Z' *which are closed subschemes of* $\mathbf{A}^w_{\mathbf{Z}}$, *where* $w = n + \Sigma_{i=1}^r f_i(e_i i + 1)$, *with the following property. For any local ring A of embedding dimension n; any monic polynomial* $g(X) \in A[X]$ *with* $g(X) \equiv \Pi_{i=1}^r g_i(X)^{e_i} \bmod m_A$ *where* $g_i(X)$ *are distinct separable polynomials of degree* f_i, *there is a map* $T' \to \mathbf{A}^w_{\mathbf{Z}}$ *so that* $V = Z \times_{\mathbf{A}^w_{\mathbf{Z}}} T'$ *and* $V' = Z' \times_{\mathbf{A}^w_{\mathbf{Z}}} T'$.

EXAMPLE. We construct a quartic $f(X)$ over a 1-dimensional local ring A with f tamely ramified, $\operatorname{ed}(f) = \operatorname{ed}(A)$, $\nu_{\mathfrak{m}_A}(\operatorname{discr} f) = 3$ and $s = \Sigma_i(e_i - 1)[\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m}_A)] = 2$.

Let Q be the field of rational numbers and let a_1, a_2 be independent transcendentals over Q. Put $A = Q[a_1, a_2]_{\langle a_1, a_2\rangle}/\langle a_1^2 + a_2^2\rangle$. Let $f(X) = (X^2 + 1)^2 + a_2 X + a_1$ be a polynomial over A.

We claim A and $f(X)$ are our example.

For $f(X) \equiv (X^2+1)^2 \bmod \mathfrak{m}_A$ so $f(X)$ is tamely ramified with ramification index 2 and $s = 2$. The discriminant of $f(X)$ is

$$\left(256a_1^3 - 27a_2^4 + 288a_1a_2^2 + 256\left(a_1^2 + a_2^2\right)\right)/256 = \left(-32a_1^3 - 27a_1^4\right)/256$$

so that $\nu_{\mathfrak{m}_A}(\operatorname{discr} f) = 3$.

A is a 1-dimensional local ring of embedding dimension 2 and $\operatorname{gr}_A(\mathfrak{m}_A) \simeq Q[X_1, X_2]/\langle X_1^2 + X_2^2\rangle$ has no zero divisors.

The maximal ideal of $A[X]/(f(X))$ is $\langle a_1, a_2, X^2+1\rangle = \langle a_2, X^2+1\rangle$ since $a_1 = -(X^2+1)^2 - a_2X \in \langle a_2, X^2+1\rangle$ in $A[X]/(f(X))$. Thus $\operatorname{ed}(f(X)) = 2 = \operatorname{ed}(A)$.

4. Applications. For the translation of Theorem 1 to schemes, we have:

THEOREM 2. *Let Y be a regular scheme and $\mathbf{P}_Y^1$ the projective line bundle over Y*[4]. *Let X be a closed subscheme of $\mathbf{P}_Y^1$ so that for every irreducible component X_i of X, the induced $f_i\colon X_i \to Y$ is dominating and finite and all residue field extensions are separable. Then*:

(1) *$X \to Y$ is flat*;

(2) *X is regular and tamely ramified over Y if and only if $\nu_{\mathfrak{m}_y}(\mathfrak{d}_{X/Y,y}) = \sum_i (e_{x_i} - 1)[\kappa(\mathfrak{m}_{x_i})\colon \kappa(\mathfrak{m}_y)]$ for all points $x_1,\ldots,x_n$ in the fibre $f^{-1}(y)$ and for all points y of Y.*

By $\mathfrak{d}_{X/Y,y}$, we here mean the local discriminant of the finite free extension $O_{Y,y} \to \Gamma(X x_Y \operatorname{Spec} O_{Y,y}, O_{Xx_Y} \operatorname{Spec} O_{Y,y})$; e_{x_i} is defined similarly.

For the proof of the theorem, note that the question is local on Y so we may assume Y is affine. Moreover, we may replace Y by $\operatorname{Spec} O_{Y,y}$, by flat base change, and prove the theorem when $Y = \operatorname{Spec} A$ with A a regular local ring. In this case, $\mathbf{P}_A^1 = \operatorname{Proj} A[X_0, X_1]$ has every finite subset of points contained in an open affine subscheme isomorphic to $\operatorname{Spec} A[X]$; since condition (2) of the theorem is applied to such finite sets, we may assume X is a closed subscheme of some $\operatorname{Spec} A[X]$. Let $X = \operatorname{Spec} A[X]/I$.

Let $I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cdots \cap \mathfrak{q}_n$ be the primary decomposition of I in $A[X]$ and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, $i = 1,\ldots,n$, be the prime ideals associated to $\mathfrak{q}_i$. Then $\operatorname{Spec} A[X]/_{\mathfrak{p}_i} \to \operatorname{Spec} A$ is dominating and finite for each i. Therefore

$\mathfrak{p}_i \cap A = \{0\}$ so height $\mathfrak{p}_i = 1$ for each i. It easily follows that $\mathfrak{p}_i$ is a principal ideal, generated by $p_i(X)$, say where $p_i(X)$ is a non-constant polynomial in $A[X]$.

Since $\mathfrak{p}_i \cap A = \{0\}$ for all i, we have $\mathfrak{q}_i = \langle p_i(X)^{n_i} \rangle$ for some integers n_i. Therefore $\langle \prod_i p_i(X)^{n_i} \rangle \subseteq I = \bigcap_{i=1}^{n} \mathfrak{q}_i$. In the fibre $A[X] \otimes_A \mathrm{fract}(A)$, I and $\langle \prod_i p_i(X)^{n_i} \rangle$ coincide since $A[X] \otimes_A \mathrm{fract}(A)$ is a principal ideal domain. It follows that for every $q(X) \in I$ there are $a, b \in A$ with $aq(X) = bp(X)$ where $p(X) = \prod_i p_i(X)^{n_i}$.

But $A[X]/_{\langle p_i(X) \rangle}$ is a finite A-module and, since A is normal, Kronecker's Theorem [**1**] shows that $p_i(X)$ has invertible leading coefficient for all i; thus we may suppose $p(X)$, $p_1(X), \ldots, p_n(X)$ are monic polynomials. Consequently, if $aq(X) = bp(X)$ then $q(X) \in \langle p(X) \rangle$ thus $I = \langle p(X) \rangle$ and so $X = \mathrm{Spec}\, A[X]/\langle p(X) \rangle$ where $p(X)$ is a monic polynomial; consequently $X \to Y$ is flat. The second part of the theorem now follows from Theorem 1. $\square$

COROLLARY. *With $f\colon X \to Y$ as in Theorem* 2. *Suppose that* $\mathrm{Reg}(Y)$ *is open (resp. contains a non-empty open set). Then the set of points $\{x \in X \mid X$ is regular and tamely ramified over Y at every point of the fibre $f^{-1}f(x)\}$ is open (resp. contains a non-empty open set).*

Proof. By replacing Y by a regular open subscheme we may assume Y is regular. Now, $\nu_{m_y}(\mathfrak{d}_{X/Y,y})$, $\sum_i [\kappa(\mathfrak{m}_{x_i})\colon \kappa(\mathfrak{m}_y)]$ are upper semi-continuous on Y and $\sum_i e_i[\kappa(\mathfrak{m}_{x_i})\colon \kappa(\mathfrak{m}_y)]$ is locally constant; the corollary now follows. $\square$

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