PACIFIC JOURNAL OF MATHEMATICS
Vol 107, No 1, 1983

TRANSFORMATIONS OF CERTAIN SEQUENCES OF RANDOM VARIABLES BY GENERALIZED HAUSDORFF MATRICES

DAVID BORWEIN AND AMNON JAKIMOVSKI

Sufficient conditions are established for a generalized Hausdorff matrix to transform certain sequences of random variables into almost surely convergent sequences.

1. Introduction. Suppose that $\{X_n\}(n = 0, 1, \ldots)$ is a sequence of random variables defined on a probability space $(\Omega, \mathfrak{F}, P)$, and that $A = \{a_{nk}\}(n, k = 0, 1, \ldots)$ is an infinite matrix. Let

$$T_n = \sum_{k=0}^{\infty} a_{nk} X_k.$$

The following theorem concerning the almost sure convergence to zero of the sequence $\{T_n\}$ is due to Borwein [**1**].

THEOREM A. *If* $1 < p \leq 2$, $0 < M < \infty$ *and*
(1) $|X_n| \leq M$ *a.s. for* $n = 0, 1, \ldots,$
(2) $\sum_{0 \leq i_1 < i_2 < \cdots < i_n} |E(X_{i_1} X_{i_2} \cdots X_{i_n})|^{p/(p-1)} \leq M^n$ *for* $n = 1, 2, \ldots,$
(3) $\sum_{k=0}^{\infty} |a_{nk}| < \infty$ *for* $n = 0, 1, \ldots,$ *and*

$$\lim_{n \to \infty} \log n \left(\sum_{k=0}^{\infty} |a_{nk}|^p \right)^{1/(p-n)} = 0,$$

then $T_n \to 0$ *a.s.*

The sequence $\{X_n\}$ is said to be multiplicative if the expectation $E(X_{i_1} X_{i_2} \cdots X_{i_n}) = 0$ whenever $0 \leq i_1 < i_2 < \cdots < i_n$; in particular, it is multiplicative if it is independent with $EX_n = 0$ for $n = 0, 1, \ldots$. Condition (2) is trivially satisfied when $\{X_n\}$ is multiplicative. The nature of Theorem A is clarified by comparison with Kolmogorov's classical strong law of large numbers which states that if $\{X_n\}$ is independent with $EX_n = 0$ for $n = 0, 1, \ldots,$ and if

$$\sum_{k=0}^{\infty} \frac{EX_k^2}{(k+1)^2} < \infty, \quad \text{then } \frac{1}{n+1} \sum_{k=0}^{n} X_k \to 0 \quad \text{a.s.}$$

We shall denote by Γ_p the set of matrices A such that $T_n \to 0$ a.s. whenever the sequence $\{X_n\}$ satisfies conditions (1) and (2). Our primary object in this paper is to establish conditions which are both sufficient and easy to verify for generalization Hausdorff matrices to be in Γ_p. Included in the class of generalized Hausdorff matrices are the matrices of such well-known methods of summability as the Cesàro, the Euler, and the weighted mean methods.

The matrix A is said to have the *Borel property* and we write $A \in (BP)$, if almost all sequences of zeros and ones are A-convergent to $1/2$. This amounts to (see **[5]**)

$$\frac{1}{2}\sum_{k=0}^{\infty} a_{nk}(1 - X_k) \to \frac{1}{2} \quad \text{a.s.}$$

when $\{X_n\}$ is the sequence of Rademacher functions on $\Omega = [0, 1]$ and P is Lebesgue measure. Since, in this case, $\{X_n\}$ satisfies conditions (1) and (2), it follows that

if $\sum_{k=0}^{\infty} a_{nk}$ *is convergent for* $n = 0, 1, \dots$ *and* $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1$, *and if* $A \in \Gamma_p$, *then* $A \in (BP)$.

Generalized Hausdorff matrices. Suppose in all that follows that $\lambda = \{\lambda_n\}$ is a sequence of real numbers satisfying

$$\lambda_0 \geq 0, \quad \lambda_n > 0 \quad \text{for } n = 1, 2, \dots, \lambda_n \to \infty, \quad \sum_{r=1}^{\infty} \frac{1}{\lambda_n} = \infty,$$

and that α is a function of bounded variation on $[0, 1]$.

For $0 \leq k \leq n$, $0 < t \leq 1$, let

$$\text{(4)} \qquad \lambda_{nk}(t) = -\lambda_{k+1} \cdots \lambda_n \frac{1}{2\pi i} \int_C \frac{t^z\, dz}{(\lambda_k - z) \cdots (\lambda_n - z)};$$

$$\lambda_{nk}(0) = \lambda_{nk}(0+),$$

C being a positively sensed closed Jordan contour enclosing λ_k, $\lambda_{k+1}, \dots, \lambda_n$. We observe the convention that products such as $\lambda_{k+1} \cdots \lambda_n = 1$ when $k = n$. Let

$$\text{(5)} \quad \lambda_{nk} = \int_0^1 \lambda_{nk}(t)\, d\alpha(t) \quad \text{for } 0 \leq k \leq n; \quad \lambda_{nk} = 0 \quad \text{for } k > n,$$

and denote the triangular matrix $\{\lambda_{nk}\}$ by $H(\lambda, \alpha)$. This is called a *generalized Hausdorff matrix*.

Let

$$D_0 = (1 + \lambda_0)\, d_0 = 1,$$

$$D_n = \left(1 + \frac{1}{\lambda_1}\right) \cdots \left(1 + \frac{1}{\lambda_n}\right) = (1 + \lambda_n)\, d_n \quad \text{for } n \geq 1.$$

Then, for $n \geq 0$,

$$D_n = \lambda_{n+1} d_{n+1} = \frac{\lambda_0}{1+\lambda_0} + \sum_{k=0}^{n} d_k.$$

It is known (see [3]) that

$$0 \leq \lambda_{nj}(t) \leq \sum_{k=0}^{n} \lambda_{nk}(t) \leq 1 \quad \text{for } 0 \leq t \leq 1, 0 \leq j \leq n, \tag{6}$$

$$\int_0^1 \lambda_{nk}(t)\,dt = \frac{d_k}{D_n} \quad \text{for } 0 \leq k \leq n, \tag{7}$$

$$\sum_{k=0}^{n} |\lambda_{nk}| \leq \int_0^1 |d\alpha(t)|\,. \tag{8}$$

Let

$$\rho_{nk} = \sum_{j=k}^{n} \frac{1}{\lambda_j}, \qquad \sigma_{nk} = \left(\sum_{j=k}^{n} \frac{1}{\lambda_j^2} \right)^{1/2} \quad \text{for } 1 \leq k \leq n. \tag{9}$$

We shall prove the following theorems.

THEOREM 1. *Let M, m be positive constants. If $\alpha(0+) = \alpha(0)$ and $\alpha(1-) = \alpha(1)$, and if λ satisfies either*

$$M \log \lambda_k \geq \lambda_{k+1} - \lambda_k \geq m \quad \textit{for all sufficiently large } k \tag{10}$$

or

$$M \geq \lambda_{k+1} - \lambda_k > 0 \quad \textit{for all sufficiently large } k \textit{ and } \log n/\sqrt{\lambda_n} = o(1), \tag{11}$$

then $H(\lambda, \alpha) \in \Gamma_2$. If, in addition, $\alpha(1) - \alpha(0) = 1$, then $H(\lambda, \alpha) \in (BP)$.

THEOREM 2. *Let $\alpha(t) = \int_0^t \beta(u)\,du$ for $0 \leq t < 1$, and let $1 < p \leq 2$. If either*

$$\beta \in L^p[0,1] \quad \textit{and} \quad \max_{0 \leq k \leq n} d_k \cdot \frac{\log n}{D_n} = o(1), \tag{12}$$

or

$$\beta \in L^\infty[0,1] \quad \textit{and} \quad \log n \left(\sum_{k=0}^{n} \left(\frac{d_k}{D_n} \right)^p \right)^{1/(p-1)} = o(1), \tag{13}$$

then $H(\lambda, \alpha) \in \Gamma_p$. If, in addition, $\{\lambda_n\}$ is non-decreasing and $\alpha(1) = 1$, then $H(\lambda, \alpha) \in (BP)$.

It is known that $H(\lambda, \alpha) \in (BP)$ when α satisfies the conditions of Theorem 1 and $\lambda_n = n + c$, the case $c = 0$ of this result being due to Hill [6] and the case $c > 0$ to Liu and Rhoades [9]. On the other hand, Borwein and Cass [2] have shown that $H(\lambda, \alpha) \notin (BP)$ when $\alpha(t) = t$ and $\lambda_n = c\log(n+1)$, $0 < c < 1/\log 4$. Borwein and Cass [2] have also shown Theorem 2 to hold in the case $p = 2$, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$.

2. Preliminary results.

LEMMA 1. *If* $1 \le k \le n$, $0 < \lambda_k \le \lambda_{k+1} \le \cdots \le \lambda_n$ *and* $0 \le t \le 1$, *then*

$$\lambda_{nk}(t) \le \frac{\sqrt{2}}{\lambda_k \sigma_{nk}}.$$

Proof. Since $0 \le \lambda_{nk}(t) \le 1$, we may suppose that

$$\lambda_k^2 \sum_{j=k}^{n} \frac{1}{\lambda_j^2} > 2. \tag{14}$$

Jakimovski [6, Lemma 2.1] has shown that, for $u > 0$,

$$\lambda_{nk}(e^{-u}) = \frac{1}{2\pi\lambda_k} \int_{-\infty}^{\infty} \frac{e^{iuv}\, dv}{\prod_{j=k}^{n}(1 + iv/\lambda_j)},$$

from which it follows that

$$\lambda_{nk}(e^{-u}) \le \frac{1}{2\pi\lambda_k} \int_{-\infty}^{\infty} \frac{dv}{\prod_{j=k}^{n}\left(1 + v^2/\lambda_j^2\right)^{1/2}}.$$

Next, we have, by (14), that

$$\prod_{j=k}^{n}\left(1 + \frac{v^2}{\lambda_j^2}\right) \ge 1 + v^2 \sum_{j=k}^{n} \frac{1}{\lambda_j^2} + \frac{v^4}{2} \sum_{r=k}^{n} \frac{1}{\lambda_r^2}\left(\sum_{j=k}^{n} \frac{1}{\lambda_j^2} - \frac{1}{\lambda_r^2}\right)$$

$$\ge 1 + v^2 \sum_{j=k}^{n} \frac{1}{\lambda_j^2} + \frac{v^4}{4} \sum_{r=k}^{n} \frac{1}{\lambda_r^2} \sum_{j=k}^{n} \frac{1}{\lambda_j^2} = \left(1 + \frac{v^2}{2}\sigma_{nk}^2\right)^2.$$

Hence, for $u > 0$,

$$\lambda_{nk}(e^{-u}) \le \frac{1}{2\pi\lambda_k} \int_{-\infty}^{\infty} \frac{dv}{1 + v^2\sigma_{nk}^2/2} < \frac{\sqrt{2}}{\lambda_k \sigma_{nk}},$$

and this completes the proof of Lemma 1.

The case $s = 0$, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ of the following lemma is due to Hausdorff [4].

LEMMA 2. *Let* $\{\lambda_n\}$ *be non-decreasing, and let* s *be a non-negative integer. Then*

$$\lim_{n\to\infty}\sum_{k=s}^{n}\lambda_{nk}=\begin{cases}\alpha(1)-\alpha(0+) & \text{if } \lambda_s>0,\\ \alpha(1)-\alpha(0) & \text{if } \lambda_s=0;\end{cases} \tag{15}$$

and

$$\lim_{n\to\infty}\lambda_{ns}=\begin{cases}0 & \text{if } \lambda_s>0,\\ \alpha(0+)-\alpha(0) & \text{if } \lambda_s=0.\end{cases} \tag{16}$$

Proof. It is known [**3**, Theorem 1(iv) and Theorem 2] that (15) holds with $s=0$ when $\alpha(t)$ is non-decreasing, and the general case of (15) with $s=0$ follows by expressing $\alpha(t)$ as the difference of two non-decreasing functions.

Next, suppose $s\geq 1$ and let $\tilde{\lambda}_k=\lambda_{k+s}$ for $k=0,1,\cdots$. Then, for $s\leq k\leq n$,

$$\lambda_{nk}=\tilde{\lambda}_{n-s,k-s},$$

$\tilde{\lambda}_{nk}$ being defined by (4) and (5) with $\{\lambda_k\}$ replaced by $\{\tilde{\lambda}_k\}$. Hence, as $n\to\infty$,

$$\sum_{k=s}^{n}\lambda_{nk}=\sum_{k=0}^{n-s}\tilde{\lambda}_{n-s,k}\to\alpha(1)-\alpha(0+)$$

by (15) with $s=0$, since $\tilde{\lambda}_0=\lambda_s>0$. This establishes (15) with $s\geq 0$.

To complete the proof of Lemma 2 we can deduce (16) from (15) by observing that, for $n>s\geq 0$,

$$\lambda_{ns}=\sum_{k=s}^{n}\lambda_{nk}-\sum_{k=s+1}^{n}\lambda_{nk}.$$

LEMMA 3. *Let* $0\leq\lambda_0<\lambda_1<\lambda_2<\cdots$, $0<\delta<1/2$, *and let* s *be a positive integer. Then there is an integer* N *and a positive constant* M *such that, for* $n\geq N$,

$$\sum_{k=s}^{n}\left(\int_{\delta}^{1-\delta}\lambda_{nk}(t)\,|\,d\alpha(t)\,|\right)^2\leq M\max(M_1(n,s),M_2(n,s))$$

where

$$M_1(n,s)=\max_{s\leq k\leq n}\frac{e^{-\lambda_s\rho_{nk}}}{\lambda_k} \tag{17}$$

and

$$M_2(n,s) = \max_{\substack{s\le k\le n\\ \delta/2\le e^{-\rho_{nk}}\le 1-\delta/2}} \frac{1}{\lambda_k\sigma_{nk}}. \tag{18}$$

Proof. *Case* 1. Suppose that $\lambda_0 = 0$, $s = 1$. Let

$$\omega_{nk} = \left(\left(1-\frac{\lambda_1}{\lambda_{k+1}}\right)\cdots\left(1-\frac{\lambda_1}{\lambda_n}\right)\right)^{1/\lambda_1} \quad \text{for } 0\le k<n,\ \omega_{nn}=1.$$

Then, in view of (6), we have

$$\sum_{k=1}^{n}\left(\int_\delta^{1-\delta}\lambda_{nk}(t)\,|\,d\alpha(t)\,|\right)^2 \le \int_\delta^{1-\delta}|\,d\alpha(t)\,|\cdot\max_{1\le k\le n}\int_\delta^{1-\delta}\lambda_{nk}(t)\,|\,d\alpha(t)\,|$$
$$\le V_\delta\max(I_1, I_2)$$

where $V_\delta = \int_\delta^{1-\delta}|\,d\alpha(t)\,|$,

$$I_1 = \max_{\substack{1\le k\le n\\ |\omega_{nk}-1/2|\ge 1/2-3\delta/4}} \int_\delta^{1-\delta}\lambda_{nk}(t)\,|\,d\alpha(t)\,|,$$

and

$$I_2 = \max_{\substack{1\le k\le n\\ |\omega_{nk}-1/2|< 1/2-3\delta/4}} \int_\delta^{1-\delta}\lambda_{nk}(t)\,|\,d\alpha(t)\,|.$$

To deal with I_1, let $f(t)$ be a twice continuously differentiable function on [0, 1] satisfying $0\le f(t)\le 1$, $f(t)=1$ for $|\,t-\frac{1}{2}\,|\ge\frac{1}{2}-\frac{3\delta}{4}$, $f(t) = 0$ for $\delta\le t\le 1-\delta$, and let

$$B_n(f,t) = \sum_{k=0}^{n}\lambda_{nk}(t)f(\omega_{nk}).$$

Then, by a result proved by Leviatan [**8**, Theorem 7],

$$I_1 \le V_\delta \max_{\delta\le t\le 1-\delta}|\,B_n(f,t)-f(t)\,|\le V_\delta K M_1(n,1)$$

where K is a constant.

To deal with I_2 we note that, by Lemma 1,

$$I_2 \le \max_{\substack{1\le k\le n\\ |\omega_{nk}-1/2|< 1/2-3\delta/4}} \frac{V_\delta\sqrt{2}}{\lambda_k\sigma_{nk}} = \frac{V_\delta\sqrt{2}}{\lambda_{k(n)}\sigma_{n,k(n)}}$$

where $k(n)$ is an integer satisfying $1\le k(n)\le n$, $3\delta/4<\omega_{n,k(n)}<1-3\delta/4$. Since $\sum_{j=1}^{\infty}1/\lambda_j=\infty$, it follows that, for every fixed integer j, $\lim_{n\to\infty}\omega_{nj}=0$ and hence that $\lim_{n\to\infty}k(n)=\infty$. Further, since

$$\log(1-x) = x + O(x^2) \quad \text{for } |\,x\,|\le 1/2,$$

we have that, for $k = k(n)$,

$$\omega_{nk} \sim \omega_{n,k-1} = e^{-\rho_{nk} + O(\sigma_{nk}^2)}$$
$$= e^{-\rho_{nk} + O(\rho_{nk}/\lambda_k)} = e^{-\rho_{nk}(1+o(1))}.$$

Hence, for n sufficiently large,

$$\delta/2 < e^{-\rho_{n,k(n)}} < 1 - \delta/2,$$

and thus

$$I_2 \le V_\delta \sqrt{2}\, M_2(n,1).$$

This completes the proof of Case 1.

Case 2. Suppose that $\lambda_0 \ge 0$, $s \ge 1$. Let

$$\tilde{\lambda}_0 = 0, \qquad \tilde{\lambda}_k = \lambda_{k+s-1} \quad \text{for } k = 1,2,\ldots,$$

and define $\tilde{\lambda}_{n,k}(t)$, $\tilde{M}_1(n,s)$, $\tilde{M}_2(n,s)$ by means of (4), (9), (17) and (18) with $\{\lambda_k\}$ replaced by $\{\tilde{\lambda}_k\}$. Then, for $n \ge k \ge s$, $0 \le t \le 1$, we have

$$\tilde{\lambda}_{n-s+1,k-s+1}(t) = \lambda_{nk}(t),$$

and hence, by Case 1,

$$\sum_{k=s}^{n} \left(\int_\delta^{1-\delta} \lambda_{nk}(t) \mid d\alpha(t) \mid \right)^2 = \sum_{r=1}^{n-s+1} \left(\int_\delta^{1-\delta} \tilde{\lambda}_{n-s+1,r}(t) \mid d\alpha(t) \mid \right)^2$$
$$\le M \max\left(\tilde{M}_1(n-s+1,1), \tilde{M}_2(n-s+1,1) \right)$$
$$= M \max\left(M_1(n,s), M_2(n,s) \right).$$

This completes the proof of Lemma 3.

LEMMA 4. *Let* $0 \le \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, $0 < \delta < 1/2$, $s \ge 2$, $\lambda_s > M + 1$, *and let* λ *satisfy either* (10) *or* (11) *with the same* M *for* $k \ge s-1$. *Then*

$$\lim_{n\to\infty} \log n \sum_{k=s}^{n} \left(\int_\delta^{1-\delta} \lambda_{nk}(t) \mid d\alpha(t) \mid \right)^2 = 0.$$

Proof. Case 1. Suppose that λ satisfies (10) for $k \ge s-1$, and that $n \ge k \ge s$. Then $\lambda_n \ge \lambda_s + m(n-s)$, and

$$M\rho_{nk} \ge \sum_{j=k}^{n} \frac{\lambda_{j+1} - \lambda_j}{\lambda_j \log \lambda_j} \ge \sum_{j=k}^{n} \int_{\lambda_j}^{\lambda_{j+1}} \frac{dx}{x \log x} = \log \frac{\log \lambda_{n+1}}{\log \lambda_k}. \tag{19}$$

Hence

$$\frac{e^{-\lambda_s \rho_{nk}}}{\lambda_k} \le \frac{1}{\lambda_k}\left(\frac{\log \lambda_k}{\log \lambda_{n+1}}\right)^{\lambda_s/M},$$

and so

$$M_1(n, s) = O\left((\log \lambda_{n+1})^{-\lambda_s/M}\right) = o\left(\frac{1}{\log n}\right). \tag{20}$$

Suppose now that

$$\frac{\delta}{2} \le e^{-\rho_{nk}} \le 1 - \frac{\delta}{2}. \tag{21}$$

Then

$$m \log \frac{2}{2-\delta} \le m\rho_{nk} \le \sum_{j=k}^{n} \frac{\lambda_j - \lambda_{j-1}}{\lambda_j} \le \sum_{j=k}^{n} \int_{\lambda_{j-1}}^{\lambda_j} \frac{dx}{x} = \log \frac{\lambda_n}{\lambda_{k-1}}$$

so that $\lambda_{k-1} \le (1 - \delta/2)^m \lambda_n$ and hence, by (10), we have that

$$\lambda_k \le \lambda_{k-1} + M \log \lambda_k \le \left(1 - \frac{\delta}{2}\right)^m \lambda_n + M \log \lambda_n. \tag{22}$$

Further, by (19) and (21),

$$M \log \frac{2}{\delta} \ge \log \frac{\log \lambda_{n+1}}{\log \lambda_k},$$

and so

$$\lambda_k \ge \lambda_{n+1}^{\varepsilon} \tag{23}$$

where $\varepsilon = (\delta/2)^M$.

Next, let $f(x) = 1/x \log x$ so that

$$f'(x) = \frac{1}{x^2 \log x}\left(1 + \frac{1}{\log x}\right) \le \frac{c}{x^2 \log x}$$

for $x \ge \lambda_s$ where $c = 1 + 1/\log \lambda_s > 0$. Hence, by (10), (22) and (23),

$$\begin{aligned} cM(\lambda_k \sigma_{nk})^2 &\ge c\lambda_k^2 \sum_{j=k}^{n} \frac{\lambda_{j+1} - \lambda_j}{\lambda_j^2 \log \lambda_j} \\ &\ge \lambda_k^2 \sum_{j=k}^{n} \int_{\lambda_j}^{\lambda_{j+1}} \frac{c\,dx}{x^2 \log x} \ge \lambda_k^2 \int_{\lambda_k}^{\lambda_{n+1}} f'(x)\,dx \\ &= \frac{\lambda_k}{\log \lambda_k}\left(1 - \frac{\lambda_k \log \lambda_k}{\lambda_{n+1} \log \lambda_{n+1}}\right) \\ &\ge \frac{\lambda_n^{\varepsilon}}{\log \lambda_n}\left(1 - (1 - \delta/2)^m - \frac{M \log \lambda_n}{\lambda_n}\right). \end{aligned}$$

Consequently

$$(24)\qquad M_2(n,s) = O\left(\lambda_n^{-\varepsilon/2}\log^{1/2}\lambda_n\right) = O\left(\lambda_n^{-\varepsilon/4}\right) = O\left(n^{-\varepsilon/4}\right)$$
$$= o\left(\frac{1}{\log n}\right).$$

The desired conclusion in Case 1 now follows from (20) and (24), by Lemma 3.

Case 2. Suppose that λ satisfies (11) for $k \geq s-1$ and that $n \geq k \geq s$. Then

$$(25)\qquad M\rho_{nk} \geq \sum_{j=k}^{n} \frac{\lambda_{j+1}-\lambda_j}{\lambda_j} \geq \sum_{j=k}^{n} \int_{\lambda_j}^{\lambda_{j+1}} \frac{dx}{x} = \log\frac{\lambda_{n+1}}{\lambda_k}.$$

Hence, since $\lambda_s > M+1$,

$$\frac{e^{-\lambda_s\rho_{nk}}}{\lambda_k} \leq \frac{1}{\lambda_k}\left(\frac{\lambda_k}{\lambda_{n+1}}\right)^{\lambda_s/M} \leq \frac{1}{\lambda_n}$$

and so

$$(26)\qquad M_1(n,s) \leq \frac{1}{\lambda_n} = o\left(\frac{1}{\log n}\right).$$

Suppose now that (21) holds. Then, by (25),

$$\lambda_k \geq \lambda_{n+1}(\delta/2)^M,$$

and hence

$$\lambda_k\sigma_{nk} \geq \lambda_k\left(\frac{\rho_{nk}}{\lambda_n}\right)^{1/2} \geq \lambda_k\left(\frac{1}{\lambda_n}\log\frac{2}{2-\delta}\right)^{1/2}$$
$$\geq \left(\frac{\delta}{2}\right)^M\left(\log\frac{2}{2-\delta}\right)^{1/2}\lambda_n^{1/2}.$$

Consequently

$$(27)\qquad M_2(n,s) = O\left(\lambda_n^{-1/2}\right) = o\left(\frac{1}{\log n}\right).$$

The desired conclusion now follows from (26) and (27), by Lemma 3, and this completes the proof of Lemma 4.

3. Proof of Theorem 1. Suppose that $n \geq k \geq s$ and that $r = 3, 4, \ldots$. Let

$$\lambda_{nk}^r = \int_{1/r}^{1-1/r} \lambda_{nk}(t)\,d\alpha(t).$$

Let $\{X_n\}$ be a sequence of random variables satisfying (1) and (2) with $p = 2$, and let

$$T_n = \sum_{k=s}^{n} \lambda_{nk} X_k, \qquad T_n^r = \sum_{k=s}^{n} \lambda_{nk}^r X_k.$$

By Lemma 4, we have, subject to either (10) or (11), that

$$\log n \sum_{k=s}^{n} (\lambda_{nk}^r)^2 \to 0 \quad \text{as } n \to \infty.$$

Hence, by Theorem A,

$$T_n^r \to 0 \text{ a.s.} \quad \text{as } n \to \infty.$$

Let Ω_r be the subset of Ω on which $T_n^r \to 0$ and $|X_r| \le M$, and let $\Omega_0 = \bigcap_{r=3}^{\infty} \Omega_r$. Then

$$\begin{aligned} T_n - T_n^r &= \sum_{k=s}^{n} X_k \left\{ \int_0^1 \lambda_{nk}(t)\, d\alpha(t) - \int_{1/r}^{1-1/r} \lambda_{nk}(t)\, d\alpha(t) \right\} \\ &= \sum_{k=s}^{n} X_k \left(\int_0^{1/r} + \int_{1-1/r}^{1} \right) \lambda_{nk}(t)\, d\alpha(t), \end{aligned}$$

and hence, in view of (6), on Ω_0

$$|T_n - T_n^r| \le M \left(\int_0^{1/r} + \int_{1-1/r}^{1} \right) |d\alpha(t)| \to 0 \quad \text{as } r \to \infty,$$

since $\alpha(0+) = \alpha(0)$ and $\alpha(1-) = \alpha(1)$. Thus

$$\lim_{r \to \infty} T_n^r = T_n \quad \text{on } \Omega_0 \text{ uniformly in } n \text{ for } n \ge s.$$

On the other hand

$$\lim_{n \to \infty} T_n^r = 0 \quad \text{on } \Omega_0 \text{ for } r \ge 3.$$

It follows that

$$\lim_{n \to \infty} T_n = \lim_{n \to \infty} \lim_{r \to \infty} T_n^r = \lim_{r \to \infty} \lim_{n \to \infty} T_n^r = 0 \quad \text{on } \Omega_0.$$

i.e., $T_n \to 0$ a.s.

Since $\alpha(0) = \alpha(0+)$ we have, by Lemma 2, that $\lim_{n\to\infty} \lambda_{nk} = 0$ for $k \ge 0$. Consequently

$$\sum_{k=0}^{n} \lambda_{nk} X_k \to 0 \quad \text{a.s.}$$

and so $H(\lambda, \alpha) \in \Gamma_2$.

Finally, the additional condition $\alpha(1) - \alpha(0) = 1$ ensures, by Lemma 2, that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \lambda_{nk} = 1,$$

and hence that $H(\lambda, \alpha) \in (BP)$.

4. **Proof of Theorem 2.** Let $0 \le k \le n$. By (5), we have that

$$\lambda_{nk} = \int_0^1 \lambda_{nk}(t)\beta(t)\,dt.$$

First, suppose that (12) holds. Then, by Hölder's inequality and (7),

$$|\lambda_{nk}|^p \le \left(\int_0^1 \lambda_{nk}(t)\,|\beta(t)|^p\,dt\right)\left(\int_0^1 \lambda_{nk}(t)\,dt\right)^{p-1}$$
$$= \left(\frac{d_k}{D_n}\right)^{p-1} \int_0^1 \lambda_{nk}(t)\,|\beta(t)|^p\,dt.$$

Hence, by (6) and (12),

$$\left(\sum_{k=0}^{n} |\lambda_{nk}|^p\right)^{1/(p-1)} \le \frac{1}{D_n}\left(\int_0^1 |\beta(t)|^p\,dt \sum_{k=0}^{n} d_k^{p-1}\lambda_{nk}(t)\right)^{1/(p-1)}$$
$$\le \max_{0\le k\le n} d_k \cdot \frac{\|\beta\|_p^{p/(p-1)}}{D_n} = o\left(\frac{1}{\log n}\right).$$

It follows, by Theorem A, that $H(\lambda, \alpha) \in \Gamma_p$.

Next, suppose that (13) holds. Then, by (7),

$$|\lambda_{nk}| \le \|\beta\|_\infty \int_0^1 \lambda_{nk}(t)\,dt = \|\beta\|_\infty \frac{d_k}{D_n},$$

and hence

$$\left(\sum_{k=0}^{n} |\lambda_{nk}|^p\right)^{1/(p-1)} \le \|\beta\|_\infty^{p/(p-1)}\left(\sum_{k=0}^{n}\left(\frac{d_k}{D_n}\right)^p\right)^{1/(p-1)} = o\left(\frac{1}{\log n}\right).$$

Thus, by Theorem A, we have that $H(\lambda, \alpha) \in \Gamma_p$.

In view of Lemma 2, the additional conditions $\{\lambda_n\}$ monotonic and $\alpha(1) = 1$, ensure that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \lambda_{nk} = 1,$$

and hence that $H(\lambda, \alpha) \in (BP)$.

This completes the proof of Theorem 2.

References

1. D. Borwein, *Matrix transformations of weakly multiplicative sequences of random variables*, J. London Math. Soc., (2) **23** (1981), 363–371.
2. D. Borwein and F. P. A. Cass, *Weighted means, generalised Hausdorff matrices and the Borel property*, Acta Sci. Math., **37** (1981), 29–34.
3. D. Borwein and A. Jakimovski, *Generalization of the Hausdorff moment problem*, Canad. J. Math., **33** (1981), 946–960.
4. F. Hausdorff, *Summationsmethoden und Momentfolgen II*, Math. Z., **9** (1921), 280–299.
5. J. D. Hill, *The Borel property of summability methods*, Pacific J. Math., **1** (1951), 399–409.
6. _____, *Remarks on the Borel property*, Pacific J. Math., **4** (1954), 227–242.
7. A. Jakimovski, *Generalized Bernstein polynomials for discontinuous and convex functions*, J. D'Analyse Mathématique, **23** (1970), 171–183.
8. D. Leviatan, *On the remainder in the approximation of functions by Bernstein-type operators*, J. Approximation Theory, **2** (1969), 400–409.
9. M. Liu and B. E. Rhoades, *Some properties of generalised Hausdorff matrices*, Houston J. Math., **2** (1976), 239–256.

Received September 8, 1981 and in revised form March 11, 1982. Supported in part by the Natural Sciences and Engineering Research Council of Canada, Grant A-2983.

THE UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO N6A 5B7
AND
TEL-AVIV UNIVERSITY
RAMAT-AVIV, TEL-AVIV, ISRAEL