PACIFIC JOURNAL OF MATHEMATICS
Vol. 108, No. 1, 1983

A MINIMAL UPPER BOUND ON A SEQUENCE OF TURING DEGREES WHICH REPRESENTS THAT SEQUENCE

HAROLD T. HODES

Given a sequence of Turing degrees $\langle a_i \rangle_{i<\omega}$, $a_i < a_{i+1}$, is there a function of f such that (i) $\deg(f)$ is a minimal upper bound on $\langle a_i \rangle_{i<\omega}$, and (ii) $\{\deg((f)_n) \mid n < \omega\} = \{a_i \mid i < \omega\}$? In this note we show that the most natural minimal upper bound on $\langle a_i \rangle_{i<\omega}$ is of the form $\deg(f)$ for such an f.

Because there seem to be a cluster of interesting notions and question related to this problem, we start with some definitions. Fix a recursive pairing function $(x, y) \mapsto \langle x, y \rangle$; $(f)_x(y) = f(\langle x, y \rangle)$. Where I is a set of Turing degrees and $f \in {}^\omega\omega$, f represents (subrepresents) I iff $I = \{\deg((f)_n) \mid n < \omega\}$ ($I \subseteq \{\deg((f)_n) \mid n < \omega\}$). For $I' \subseteq I$, I' is cofinal in I iff for every $a \in I$ there is a $b \in I'$ with $a \leq b$. f weakly represents (weakly subrepresents) I iff f represents (subrepresents) some I' cofinal in I. A degree **a** represents (subrepresents, weakly represents, weakly subrepresents) I iff some $f \in$ **a** does so. I is an ideal iff I is non-empty closed downward and under join.

Terminology. A tree T is a total function from $2^{<\omega} = \mathrm{Str}$ into Str so that for any $\delta \in \mathrm{Str}$, $T(\delta\hat{\ }0)$ and $T(\delta\hat{\ }1)$ are incompatible extensions of $T(\delta)$. $\delta \in \mathrm{Str}(s)$ iff $\delta \in \mathrm{Str}$ and $\mathrm{dom}(\delta) = s$. A pre-tree of height s is a function T: $\mathrm{Str}(s) \to \mathrm{Str}$ where for all $\delta \in \mathrm{Str}(s-1)$, $T(\delta\hat{\ }\langle 0 \rangle)$ and $T(\delta\hat{\ }\langle 1 \rangle)$ are incompatible extensions of $T(\delta)$. For $\delta \in \mathrm{Str}$ and $A \in {}^\omega 2$, $\delta \subseteq A$ iff for all $i \in \mathrm{dom}(\delta)$, $\delta(i) = A(i)$. Where T is a tree, $B \in [T]$ iff for some $A \in {}^\omega 2$; for all n, $T(A \restriction n) \subset B$; (i.e. B is a path through T). Where T is a pre-tree of height s, $B \in [T]$ iff for some $\delta \in \mathrm{Str}$, $\mathrm{dom}(\delta) = s$ and $T(\delta) \subset B$.

Where T is a tree and $A \in {}^\omega 2$, let

$$\mathrm{Code}(T, A)(\delta) = T(\langle A(0), \delta(0), \ldots, \delta(n-1), A(n) \rangle),$$

where $n = \mathrm{dom}(\delta) - 1$. Notice: $\mathrm{Code}(T, A)(\langle\,\rangle) \supsetneqq T(\langle\,\rangle)$. Where T is a pre-tree of height $\leq 2n+1$ and $\tau \in \mathrm{Str}$, $\mathrm{dom}(\tau) \geq n$, $\mathrm{Code}(T, \tau)$ is defined similarly. For T a tree (pre-tree) and $B \in [T]$, let $\mathrm{Coded}(B, T)$ be the real $A \in {}^\omega 2$ (string τ) such that $A(e) = i$ ($\tau(e) = i$) iff for some δ, $T(\delta) \subseteq B$ and $\delta(2e) = i$. If T is a pre-tree of height $2n$ or $2n+1$, dom(Coded$(B, T)) = n$; so if T is a pre-tree, $B \in [T]$ and $\tau =$ Coded(B, T), Code(T, τ) is well defined.

We'll say that τ is on T iff $\tau \in$ Range(T). Let τ_0, τ_1 be an e-splitting of τ iff $\tau_0, \tau_1 \supseteq \tau$ and for some x and t, $\{e\}_t^{\tau_0}(x)$ and $\{e\}_t^{\tau_1}(x)$ are defined and different. By "the least e-splitting of τ", we mean that $\langle \tau_0, \tau_1, x, t\rangle$ is minimal. Where T is a tree, let e-Split$(T)(\langle\,\rangle) = T(\langle\,\rangle)$; if e-Split$(T)(\delta)$ is defined, e-Split$(T)(\delta\hat{}\langle 0\rangle)$, e-Split$(T)(\delta\hat{}\langle 1\rangle)$ is the least e-splitting of e-Split$(T)(\delta)$ on T, if such there be; otherwise they are undefined. Clearly e-Split(T) is partial-recursive in T.

Where T is a pre-tree, e-Split$_s(T)$ is defined like e-Split(T), except that (1) all searches for e-splittings on T are bounded by s; (2) e-Split$(T)(\delta)$ is defined iff for all τ with dom$(\tau) =$ dom(δ), e-Split$(T)(\tau)$ is defined. (2) insures that e-Split$_s(T)$ is a pre-tree. For T a tree or pre-tree, Full$(T, \delta)(\tau)$ $= T(\delta\hat{}\tau)$. (If $\delta \notin$ dom(T), Full$(T, \delta) = \varnothing$, which is still a pre-tree.)

THEOREM. *Suppose $I = \{\mathbf{a}_i \mid i < \omega\}$ is a sequence of Turing degrees, and for all i, $\mathbf{a}_i < \mathbf{a}_{i+1}$. Then some minimal upper-bound on I represents I.*

To prove this, we use the simplest construction of a minimal upper bound on I. Fix $\langle A_i\rangle_{i<\omega}$ so that for all i, $A_i \in \mathbf{a}_i$. Let $T_{-1} = \mathrm{Id}\restriction \mathrm{Str}$.

$$T_{2e} = \begin{cases} e\text{-Split}(T_{2e-1}) & \text{if } e\text{-Split}(T_{2e-1}) \text{ is total;} \\ \mathrm{Full}(T_{2e-1}, \tau_e) & \text{otherwise,} \end{cases}$$

where τ_e is the least τ such that $T_{2e-1}(\tau)$ is on e-Split$(T_{2e-1})(\tau)$ and has no e-splitting on T_{2e-1}.

$$T_{2e+1} = \mathrm{Code}(T_{2e}, A_e).$$

A tree T is uniformly recursively pointed iff for some e, $T = \{e\}^B$ for all $B \in [T]$. All T_e are uniformly recursively pointed, and so $T_{2e-1} \equiv_T T_{2e} \leq_T T_{2e+1} \leq_T A_e$. Let $\{B\} = \bigcap_{e<\omega}[T_e]$; where $\mathbf{b} = \deg(B)$, $\mathbf{b}$ is a minimal upper bound on I. We must show that B computes a g which represents I.

Let

$$f(e) = \begin{cases} 0 & \text{if } T_{2e} \text{ was defined by the first case;} \\ \tau_e + 1 & \text{otherwise.} \end{cases}$$

$$f^-(e) = 0 \quad \text{if } f(e) = 0; \qquad f^-(e) = 1 \quad \text{otherwise.}$$

We'll let $\delta \in$ Str represent the hypothesis that $\delta \subset f^-$. Assuming this hypothesis, for dom$(\delta) = n + 1$, B tries to recover $\langle T_e\rangle_{-1\leq e\leq 2n}$ and A_n.

If $\delta \subset f^-$, eventually B will have this right. If $\delta \not\subset f^-$, B will not be so fortunate. Where e is least so that $\delta(e) \neq f^-(e)$, e curses δ iff $f^-(e) = 1$ and $\delta(e) = 0$; e disrupts δ iff $f^-(e) = 0$ and $\delta(e) = 1$. If δ is cursed, by assuming δ B eventually finds himself waiting eternally for a splitting which never comes; if δ is disrupted, constant changes in B's guesses at a node beyond which there are no splits will prevent B's guesses from settling down.

At each stage s, on hypohtesis δ B constructs the sequence of pre-trees $T_e^{\delta,s}$, $-1 \leq e \leq 2n$, as follows: $T_{-1}^{\delta,s} = \mathrm{Id} \restriction \mathrm{Str}(s+1)$;

$$T_{2e}^{\delta,s} = \begin{cases} e\text{-Split}_s\left(T_{2e-1}^{\delta,s}\right) & \text{if } \delta(e) = 0, \\ \mathrm{Full}\left(T_{2e-1}^{\delta,s}, \tau_e^{\delta,s}\right) & \text{if } \delta(e) = 1, \end{cases}$$

where $\tau_e^{\delta,s}$ is the longest τ such that $e\text{-Split}_s(T_{2e-1}^{\delta,s})(\tau)$ is defined, $\subset B$, and has no e-splitting on $T_{2e-1}^{\delta,s}$ after s steps of searching. Let $F(e, \delta, s) = \mathrm{Coded}(B, T_{2e}^{\delta,s})$. $F(e, \delta, s)$ is B's stage s guess at $A_e \restriction k$, where $k = \mathrm{dom}(F(e, \delta, s))$, based on hypothesis δ.

$$T_{2e+1}^{\delta,s} = \mathrm{Code}\left(T_{2e}^{\delta,s}, F(e, \delta, s)\right).$$

By remarks after the definitions of Code and Coded, this is well-defined.

Let $\mathrm{dom}(\delta) = n + 1$. If $T_{2n}^{\delta,s} \neq \varnothing$, for all e with $-1 \leq e \leq 2n$, $T_e^{\delta,s} \neq \varnothing$; let $f^{\delta,s}\colon n + 1 \to \omega$ be given by:

$$f^{\delta,s}(e) = \begin{cases} 0 & \text{if } \delta(e) = 0 \\ \tau_e^{\delta,s} + 1 & \text{if } \delta(e) = 1. \end{cases}$$

$f^{\delta,s}$ is B's guess at $f \restriction n + 1$ at stage s, assuming δ. If $T_{2n}^{\delta,s} = \varnothing$, at stage s B hasn't enough information to make a guess. If $\delta \not\subseteq \delta'$, $T_e^{\delta,s} = T_e^{\delta',s}$ for $e \leq 2n$, and $f^{\delta,s} = f^{\delta',s} \restriction n + 1$.

We now consider the possible behavior of $f^{\delta,s}$ as s increases.

(1) If $\delta \subset f^-$ there is an s such that for all $t \geq s$, $f^{\delta,t}$ is defined, $f^{\delta,t} = f^{\delta,s} = f \restriction n + 1$, $T_e^{\delta,t} = T_e \restriction \mathrm{Str}(l_e^t)$ for $-1 \leq e \leq 2n$, where l_e^t is nondecreasing in t and approaches ω for $t \geq s$; furthermore for $t \geq s$, $F(n, \delta, t) \subset A_n$, and so $\bigcup_{t \geq s} F(n, \delta, t) = A_n$. All this follows by induction on n.

(2) If δ is cursed, there is an s such that either (a) for all $t \geq s$, $f^{\delta,t}$ is defined and $f^{\delta,t} = f^{\delta,s}$, or (b) for all $t \geq s$, $f^{\delta,t}$ is undefined. Furthermore, in case (a), for all $t \geq s$, $F(n, \delta, t) = F(n, \delta, s)$. To see this, suppose e curses δ; by (1) there is a stage s_0 by which $f^{\delta \restriction e,t}$ is defined and equal to

$f \upharpoonright e$ for all $t \geq s_0$; furthermore $T_{2e-1}^{\delta,t} = T_{2e-1} \upharpoonright \mathrm{Str}(l_{2e-1}^t)$. Fix the least level l such that for some δ with $\mathrm{dom}(\delta) = l$, e-Split$(T_{2e-1})(\delta)$ is undefined. In building $T_{2e}^{\delta,t}$, B gets stuck at level l; so eventually B is waiting for e-splittings on $T_{2e-1}^{\delta,t}$ of a string with no such e-splittings. So for some $s_1 \geq s_0$, for all $t \geq s_1$, $T_{2e}^{\delta,t} = T_{2e}^{\delta,s_1}$. Clearly for $-1 \leq j < j' \leq 2n$, $\mathrm{Range}(T_j^{\delta,t}) \subseteq \mathrm{Range}(T_{j'}^{\delta,t})$. So by induction we find s so that for all $j \leq 2n$ and $t \geq s$, $T_j^{\delta,t} = T_j^{\delta,s}$. If $T_j^{\delta,s} = \varnothing$, for $t \geq s$, $f^{\delta,t}$ is undefined. Otherwise $f^{\delta,t}(e) = 0$.

(3) If δ is disrupted and $f^{\delta,s}$ is defined, for some $t > s$ either $f^{\delta,t}$ is undefined or $f^{\delta,t} \neq f^{\delta,s}$. To see this, suppose e disrupts δ and select s_0 as above. Once $t \geq s_0$, $\tau_e^{\delta,t}$ goes to ω with t, since e-splittings for e-Split$_t(T_{2e-1}^{\delta,t})(\tau_e^{\delta,t}) = e$-Split$(T_{2e-1})(\tau_e^{\delta,t})$ eventually turn up on T_{2e-1}, and thus on $T_{2e-1}^{\delta,t'}$ for sufficiently large $t' \geq t$; when this happens, $\tau_e^{\delta,t'} \supsetneq \tau_e^{\delta,t}$. Fixing s, for sufficiently large $t \geq s$, if $f^{\delta,t}$ is defined, $f^{\delta,t}(e) > f^{\delta,s}(e)$.

We now view $h \in \omega^{<\omega}$ as a guess at $f \upharpoonright \mathrm{dom}(h)$. Let $h^-(e) = 0$ if $h(e) = 0$, $h^-(e) = 1$ otherwise. An h-block is a maximal interval $[s_0, s_1] = \{t \mid s_0 \leq t \leq s_1\}$ or $[s_0, \infty] = \{t \mid s_0 \leq t\}$ such that for all s in that interval, $h = f^{h^-,s}$. For any h there are finitely many h-blocks. If $h^- \subset f^-$, this follows from (1); if h^- is cursed, this follows from (2). Note that if $h^- \subset f^-$ or if h^- is cursed and (2a) is true, the final h-block is of the form $[s, \infty]$. If h^- is disrupted by e, this follows from (3) and the previous observation that for sufficiently large t, $\tau_e^{h^-,t}$ increases non-decreasingly with t. If s and t belong to one h-block and $s \leq t$, $F(e, h^-, s) \subset F(e, h^-, t)$ for $-1 \leq e < \mathrm{dom}(h)$. For the moment, assume that $\mathbf{a}_0 = \mathbf{0}$. For $h \in \omega^{<\omega}$, $k \in \omega$ and $\mathrm{dom}(h) = n + 1$, let

$$(g)_{\langle h,k\rangle}(s) = \begin{cases} F(n, h^-, s) + 1 & \text{if } s \text{ belongs to the } k\text{th } h\text{-block;} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $g \leq_T B$. If $h \not\subset f$, or if the kth h-block is not of the form $[s, \infty]$, $(g)_{\langle h,k\rangle}$ differs only finitely from $\lambda s.0$. If $h \subset f$ and the kth h-block is of the form $[s, \infty]$, since $A_n = \bigcup_{t \geq s} F(n, h^-, t)$, $A_n \leq_T (g)_{\langle h,k\rangle}$. Furthermore, $\lambda s.F(n, h^-, s) \leq_T A_0 \oplus \cdots \oplus A_n \leq_T A_n$; thus $(g)_{\langle h,k\rangle} \leq_T A_n$. So either $\deg((g)_{\langle h,k\rangle}) = \mathbf{a}_n$ or $= \mathbf{0} = \mathbf{a}_0$. Thus g represents I.

Now suppose $\mathbf{a}_0 \neq \mathbf{0}$. Select $D \in \mathbf{a}_0$. Suppose we revised our definition of $(g)_{\langle h,k\rangle}(s)$ by requiring in the "otherwise" case that $(g)_{\langle h,k\rangle}(s) = D(s)$. If $h^- \subset f^-$ and the kth block is of the form $[s_0, \infty]$, we still have $\deg((g)_{\langle h,k\rangle}) = \mathbf{a}_n$; if otherwise and if h^- is not cursed, $\deg((g)_{\langle h,k\rangle}) = \mathbf{a}_0$. But if h^- is cursed and the kth block is of the form $[s_0, \infty]$,

$\deg((g)_{\langle h,k\rangle}) = \mathbf{0}$. To remedy this, we slightly hair-up the definition of $(g)_{\langle h,k\rangle}$:

$$(g)_{\langle h,k\rangle}(2s) = \begin{cases} F(h, h^{-}, s) + 1 & \text{if } s \text{ belongs to the } k\text{th } h\text{-block.} \\ D(s) & \text{otherwise} \end{cases}$$

$$(g)_{\langle h,k\rangle}(2s+1) = D(s).$$

g is now as desired.

COROLLARY. *If I is a countable ideal, some minimal upper bound on I weakly represent I.*

Proof. There is an $I' \subseteq I$ cofinal in I and linearly ordered; apply *Theorem* 1 to I' and notice that a minimal upper bound on I' is also one for I.

Questions. Does every ideal have a representing minimal upper bound?

Does a sequence $\langle a_i \rangle_{i<\omega}$ as above have a minimal upper bound which does not represent it?

The author thanks Richard Shore for fruitful discussions on matters related to the subject of this paper.

Received November 3, 1981 and in revised form July 23, 1982.

DEPARMENT OF PHILOSOPHY
CORNELL UNIVERSITY
ITHACA, NY 14853