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TOPOLOGICAL PROPERTIES OF THE DUAL PAIR $\langle \mathring{\mathfrak{B}}(\Omega)', \mathring{\mathfrak{B}}(\Omega)'' \rangle$

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If $\Omega \neq \varnothing$ is an open subset of $\mathbf{R}^n$, various locally convex topologies have been proposed that make the bidual $\mathring{\mathfrak{B}}(\Omega)''$ a normal space of distributions with dual $\mathring{\mathfrak{B}}'$. It is shown that these topologies all coincide; in particular, the strict topology on $\mathring{\mathfrak{B}}(\Omega)''$ is a Mackey topology. Moreover, the dual $\mathring{\mathfrak{B}}(\Omega)'$ has the Schur property, and $\mathring{\mathfrak{B}}(\Omega)''$ is an Orlicz-Pettis space.

0. Introduction. The space $\mathring{\mathfrak{B}}(\mathbf{R}^n) := \{\varphi \in \mathfrak{E}(\mathbf{R}^n);\ \partial^\alpha \varphi \in C_0(\mathbf{R}^n)$ $(\alpha \in \mathbf{N}_0^n)\}$ provided with its natural Fréchet-space-topology $\mathfrak{T}_0$, its dual $\mathring{\mathfrak{B}}(\mathbf{R}^n)'$, and its bidual $\mathring{\mathfrak{B}}(\mathbf{R}^n)''$ are important for the theory of integrable distributions (Schwartz [**24**, exposé n° 21]) and for the definition of the convolution for distributions (Schwartz [**24**, exposé n° 22], cf. also Horváth [**19**] and Dierolf, Voigt [**11**]). Since $\mathfrak{D}(\mathbf{R}^n)$ is not dense in $(\mathring{\mathfrak{B}}(\mathbf{R}^n)'', \beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$, several locally convex topologies $\mathfrak{R}$ on $\mathring{\mathfrak{B}}(\mathbf{R}^n)''$ were suggested which make $(\mathring{\mathfrak{B}}(\mathbf{R}^n)'', \mathfrak{R})$ a normal space of distributions (in the sense of Horváth [**18**, p. 319, Def. 3]) and yield $(\mathring{\mathfrak{B}}(\mathbf{R}^n)'', \mathfrak{R})' = \mathring{\mathfrak{B}}(\mathbf{R}^n)'$:

$\mathfrak{T}_\sigma :=$ the finest locally convex topology which agrees on each $\beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$-bounded subset of $\mathring{\mathfrak{B}}''$ with $\beta(\mathfrak{E}, \mathfrak{E}')$ (Schwartz [**26**, p. 203]).

$\mathfrak{T}_{\beta c} :=$ the topology of uniform convergence on all $\beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}})$-compact subsets of $\mathring{\mathfrak{B}}'$ (Schwartz [**26**, p. 203], cf. also Schwartz [**25**, p. 100]).

$\tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') :=$ the Mackey-topology with respect to the dual pair $\langle \mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}' \rangle$.

Inspired by a similar problem in measure theory (Buck [**6**]) and by the paper of Kang, Richards [**20**], we also consider the *strict topology* $\mathfrak{S}$ on $\mathring{\mathfrak{B}}''$,

$\mathfrak{S} :=$ the locally convex topology on $\mathring{\mathfrak{B}}''$ which is generated by the semi-norms $q_{\alpha,f}$ $(\alpha \in \mathbf{N}_0^n, f \in C_0(\mathbf{R}^n))$,

$$q_{\alpha,f}(\psi) := \sup\{|f(x)\partial^\alpha \psi(x)|\ ;\ x \in \mathbf{R}^n\}.$$

In this article we treat the more general case of the Fréchet-space $\mathring{\mathfrak{B}}(\Omega) := \{\varphi \in \mathfrak{E}(\Omega);\ \partial^\alpha \varphi \in C_0(\Omega)\ (\alpha \in \mathbf{N}_0^n)\}$, where $\Omega \neq \varnothing$ is an arbitrary open subset of $\mathbf{R}^n$. Our main result is that on $\mathring{\mathfrak{B}}(\Omega)''$ all the above-mentioned topologies coincide. In particular, the strict topology on $\mathring{\mathfrak{B}}(\Omega)''$ is a Mackey-topology. In proving this result, we determine the

$\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$-compact subsets of $\mathring{\mathfrak{B}}(\Omega)'$, and we obtain that $(\mathring{\mathfrak{B}}(\Omega)', \beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}))$ has the Schur-property, i.e. the $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$-compact sets and the $\beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}})$-compact sets coincide. In particular, $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$ is sequentially complete. Moreover, the LB-space $(\mathring{\mathfrak{B}}(\Omega)', \beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}))$ is sequentially retractive: A $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$-convergent sequence is already norm-convergent in some Banach-space of the inductive sequence which generates $\beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}) = \beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$. As a consequence the LB-space $(\mathring{\mathfrak{B}}(\Omega)', \beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}))$ is strongly boundedly retractive (in the sense of Bierstedt, Meise [**3**, p. 100]), and thus the distinguished Fréchet-space $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ is even quasi-normable.

Finally we show that $(\mathring{\mathfrak{B}}(\Omega)'', \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$ is an Orlicz-Pettis-space (cf. [**10**]), i.e. $\tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ is the finest locally convex topology on $\mathring{\mathfrak{B}}(\Omega)''$ having the same subfamily-summable sequences as $\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$. Thus $(\mathring{\mathfrak{B}}(\Omega)'', \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$ — although it need not be quasi-barelled — is a suitable domain space for the measurable-graph theorem which was recently proved by Pfister [**23**].

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1. Notation, definitions, and basic facts. In this section we recall some results concerning $\mathring{\mathfrak{B}}(\Omega)'$ and $\mathring{\mathfrak{B}}(\Omega)''$ from [**12**] and prove some auxiliary facts which will be used later.

For spaces of functions and distributions, as well as for general locally convex spaces we use the standard notation of Horváth [**18**]. The 0-nbhd.-filter of a locally convex space $(E, \mathfrak{R})$ is denoted by $\mathfrak{U}_0(E, \mathfrak{R})$.

Ω always denotes an arbitrary non-empty open subset of $\mathbf{R}^n$. We define $r\colon \mathbf{R}^n \to [0, \infty)$ by

$$(1.1) \qquad r(x) := \begin{cases} \operatorname{dist}(x, \complement\Omega) & \text{if } \Omega \neq \mathbf{R}^n, \\ 1 & \text{if } \Omega = \mathbf{R}^n \end{cases}$$

and put

$$(1.2) \quad \rho(x) := \min\{r(x), 1\}, \quad \rho_k(x) := (\rho(x))^k \qquad (x \in \Omega, k \in \mathbf{Z}).$$

(1.3) LEMMA. *Given $\Omega = \mathring{\Omega} \subset \mathbf{R}^n$, $\Omega \neq \varnothing$, there exists a sequence $(\theta_k;$ $k \in \mathbf{N})$ in $\mathfrak{D}(\Omega)$ such that*

(a) $(\operatorname{supp}(\theta_k); k \in \mathbf{N})$ *is locally finite in* Ω.

(b) $\theta_k(x) \geq 0$ $(x \in \Omega, k \in \mathbf{N})$, $\Sigma_{k \in \mathbf{N}} \theta_k(x) = 1$ $(x \in \Omega)$.

(c) *For each* $\alpha \in \mathbf{N}_0^n$ *there exists* $a(\alpha) > 0$ *such that*

$$\sum_{k \in \mathbf{N}} |\partial^\alpha \theta_k(x)| \leq a(\alpha) \cdot \rho_{-|\alpha|}(x) \qquad (x \in \Omega).$$

It was shown in [**12**, Lemma (4.3)] that for $\Omega \neq \mathbf{R}^n$ such a sequence may be obtained as a slight modification of Whitney's partition of unity (cf. also Stein [**28**, p. 170]). For $\Omega = \mathbf{R}^n$ we proceed as follows: Let $\eta \in \mathfrak{D}(\mathbf{R}^n)$ satisfy $0 \leq \eta(x) \leq 1$ $(x \in \mathbf{R}^n)$, $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$, and define $\eta_k(x) := \eta(x/2^k)$ $(x \in \mathbf{R}^n,\ k \in \mathbf{N})$, $\theta_1(x) := \eta_1(x)$ $(x \in \mathbf{R}^n)$, $\theta_k(x) := \eta_k(x) - \eta_{k-1}(x)$ $(k \geq 2,\ x \in \mathbf{R}^n)$. Then the relations

$$\begin{aligned}
&\operatorname{supp}(\theta_k) \subset \{x \in \mathbf{R}^n;\ 2^{k-1} \leq |x| \leq 2^{k+1}\} \qquad (k \geq 2),\\
&|k - l| > 2 \Rightarrow \operatorname{supp}(\theta_k) \cap \operatorname{supp}(\theta_l) = \varnothing \qquad (k, l \in \mathbf{N}),\\
&|\partial^\alpha \eta_k(x)| \leq 2^{-k|\alpha|} \|\partial^\alpha \eta\|_\infty\\
&\qquad\qquad \leq 2^{-|\alpha|} \|\partial^\alpha \eta\|_\infty \qquad (x \in \mathbf{R}^n, \alpha \in \mathbf{N}_0^n, k \in \mathbf{N}),\\
&\qquad\qquad\qquad \sum_{l \leq k} \theta_l = \eta_k \qquad (k \in \mathbf{N})
\end{aligned}$$

show that this sequence $(\theta_k;\ k \in \mathbf{N})$ has all the properties stated in Lemma (1.3).

(1.4) For $m \in \mathbf{N}_0$ we provide the space

$$\mathfrak{B}^m(\Omega) := \{\psi \in \mathfrak{E}^m(\Omega);\ \partial^\alpha \psi \in L^\infty(\Omega)\ (|\alpha| \leq m)\}$$

with the topology $\mathfrak{T}^m$ which is generated by the norm

$$p_m(\psi) := \sup\{|\partial^\alpha \psi(x)|\ ;\ |\alpha| \leq m, x \in \Omega\},$$

and we define

$$\mathring{\mathfrak{B}}^m(\Omega) := \{\varphi \in \mathfrak{E}^m(\Omega);\ \partial^\alpha \varphi \in C_0(\Omega)\ (|\alpha| \leq m)\},$$
$$\mathfrak{T}_0^m := \mathfrak{T}^m \cap \mathring{\mathfrak{B}}^m(\Omega).$$

The space

$$\mathfrak{B}(\Omega) := \bigcap_{m \in \mathbf{N}_0} \mathfrak{B}^m(\Omega)$$

is provided with the Fréchet-space-topology $\mathfrak{T}$ which is generated by the norms p_m $(m \in \mathbf{N}_0)$. The closed subspace

$$\mathring{\mathfrak{B}}(\Omega) := \bigcap_{m \in \mathbf{N}_0} \mathring{\mathfrak{B}}^m(\Omega)$$

is given the relative topology $\mathfrak{T}_0 := \mathfrak{T} \cap \mathring{\mathfrak{B}}(\Omega)$.

(1.5) We use the partition of unity $(\theta_k;\ k \in \mathbf{N})$ from (1.3) to define $\eta_k := \sum_{j \leq k} \theta_j$ $(k \in \mathbf{N})$. We thus obtain a sequence $(\eta_k;\ k \in \mathbf{N})$ in $\mathfrak{D}(\Omega)$ which satisfies

(a) For all $K \Subset \Omega$ there exists $k(K) \in \mathbf{N}$ such that $\eta_k(x) = 1$ holds for all $x \in K$, $k \geq k(K)$.

(b) For all $\alpha \in \mathbf{N}_0^n$ the estimate $|\partial^\alpha \eta_k(x)| \leq a(\alpha) \cdot \rho_{-|\alpha|}(x)$ holds for all $x \in \Omega$, $k \in \mathbf{N}$.

(1.6) It was shown in **[12**, (4.10a)] that for all $m \in \mathbf{N}_0$, $\varphi \in \mathring{\mathfrak{B}}^m(\Omega)$ the sequence $(\eta_k \varphi;\ k \in \mathbf{N})$ converges to φ with respect to $\mathfrak{T}^m$. This shows that for all $m \in \mathbf{N}_0$ the space $\mathfrak{D}^m(\Omega)$ is dense in $(\mathring{\mathfrak{B}}^m(\Omega), \mathfrak{T}_0^m)$, and we obtain that $\mathfrak{D}(\Omega)$ is dense in $(\mathring{\mathfrak{B}}^m(\Omega), \mathfrak{T}_0^m)$ $(m \in \mathbf{N}_0)$ and in $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$. We thus may identify the dual $\mathring{\mathfrak{B}}(\Omega)'$ with a subspace of $\mathfrak{D}(\Omega)'$.

(1.7) We provide the space

$$\begin{aligned} \check{\mathfrak{B}}(\Omega) &:= \{\psi \in \mathfrak{E}(\Omega);\ \rho_{-m}\partial^\alpha\psi \in L^\infty(\Omega)\ (\alpha \in \mathbf{N}_0^n, m \in \mathbf{N}_0)\} \\ &\simeq \{\psi \in \mathfrak{B}(\mathbf{R}^n);\ \partial^\alpha\psi(x) = 0\ (\alpha \in \mathbf{N}_0^n, x \in \mathbf{R}^n \setminus \Omega)\} \end{aligned}$$

with the Fréchet-space-topology $\check{\mathfrak{T}}$ which is generated by the semi-norms $\psi \mapsto \|\rho_{-m}\partial^\alpha\psi\|_\infty$ $(\alpha \in \mathbf{N}_0^n,\ m \in \mathbf{N}_0)$. If $\Omega = \mathbf{R}^n$ we have $\check{\mathfrak{B}}(\mathbf{R}^n) = \mathfrak{B}(\mathbf{R}^n)$ and $\check{\mathfrak{T}} = \mathfrak{T}$.

(1.8) LEMMA. (a) *For* $\alpha \in \mathbf{N}_0^n$, $m \in \mathbf{N}_0$ *there exists* $c(m, \alpha) > 0$ *such that*

$$|\partial^\alpha\psi(x)| \leq c(m,\alpha) \cdot \rho_m(x) \cdot \sup\{|\partial^\beta\psi(y)|\,;\ |\beta| \leq m + |\alpha|\,, y \in \Omega, \rho(y) \leq \rho(x)\}$$

holds for all $\psi \in \check{\mathfrak{B}}(\Omega)$, $x \in \Omega$.

(b) *For* $\psi \in \check{\mathfrak{B}}(\Omega)$, $\alpha \in \mathbf{N}_0^n$ *and* $m \in \mathbf{N}_0$ *we have*

$$\|\partial^\alpha\psi\|_\infty \leq \|\rho_{-m}\partial^\alpha\psi\|_\infty \leq c(m,\alpha) \cdot p_{m+|\alpha|}(\psi).$$

Thus the norms $(p_m;\ m \in \mathbf{N}_0)$ *generate the topology* $\check{\mathfrak{T}}$ *on* $\check{\mathfrak{B}}(\Omega)$, $\check{\mathfrak{T}} = \mathfrak{T} \cap \check{\mathfrak{B}}(\Omega)$.

The assertion (1.8a) follows from [**12**, Prop. (4.6a)], and (1.8b) is an immediate consequence of (1.8a).

(1.9) THEOREM. ([**12**, *Thm.* (2.6), *Thm.* (4.8) *and Cor.* (3.10)]).

(a) *The bitranspose of the canonical injection* $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0) \hookrightarrow (\mathfrak{E}(\Omega), \beta(\mathfrak{E}, \mathfrak{E}'))$ *induces a topological isomorphism from* $(\mathring{\mathfrak{B}}(\Omega)'', \beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$ *onto* $(\check{\mathring{\mathfrak{B}}}(\Omega), \check{\mathfrak{T}})$.

(b) *For* $\psi \in \mathring{\mathfrak{B}}(\Omega)''$ *and* $T \in \mathring{\mathfrak{B}}(\Omega)'$ *we have*

$$ {}_{\mathring{\mathfrak{B}}'}\langle T, \psi \rangle_{\mathring{\mathfrak{B}}''} = \lim_{k \to \infty} {}_{\mathring{\mathfrak{B}}'}\langle T, \eta_k \psi \rangle_{\mathring{\mathfrak{B}}}. $$

(c) $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ *is distinguished, and the LB-space-topology on* $\mathring{\mathfrak{B}}(\Omega)'$ *defined by* $\mathring{\mathfrak{B}}(\Omega)' = \bigcup_{m \in \mathbf{N}_0} (\mathring{\mathfrak{B}}^m(\Omega), p_m)'$ *coincides with* $\beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}})$.

(1.10) Let $(E, \mathfrak{R})$ be a Mackey-sequentially complete locally convex space (cf. [**8**]). Then $\beta(E, E')$ and $\sigma(E, E')$ have the same bounded sets, whence $\beta(E', E)$ and $\sigma(E', E)$ have the same bounded sets. If moreover $(E', \beta(E', E))$ is Mackey-sequentially complete, then $\beta(E', E'')$ and $\sigma(E', E'')$ have the same bounded sets which in turn implies that $\beta(E'', E')$ and $\sigma(E'', E')$ have the same bounded sets. Thus in particular, on $\mathring{\mathfrak{B}}(\Omega)'$ the topologies $\beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'') \supset \beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}) \supset \sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'') \supset \sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}})$ have the same bounded sets, and on $\mathring{\mathfrak{B}}(\Omega)''$ the topologies $\beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ and $\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ have the same bounded sets.

(1.11) Let $\langle E, F \rangle$ be a separating dual pair. By $\mathfrak{T}_{\tau c}$ and $\mathfrak{T}_{\beta c}$ we denote the topology (on E) of uniform convergence on all $\tau(F, E)$-compact subsets of F and the topology of uniform convergence on all $\beta(F, E)$-compact subsets of F, respectively. By $\mathfrak{T}_\sigma$ we denote the finest locally convex topology on E which satisfies $\mathfrak{T}_\sigma \cap B = \sigma(E, F) \cap B$ for all $\sigma(E, F)$-bounded subsets B of E. We obviously have $\sigma(E, F) \subset \mathfrak{T}_\sigma$ and $\sigma(E, F) \subset \mathfrak{T}_{\beta c} \subset \mathfrak{T}_{\tau c} \subset \beta(E, F)$.

(1.12) PROPOSITION. *Let the notation be as in* (1.11).

(a) $(F, \beta(F, E))$ *is complete if and only if* $\mathfrak{T}_\sigma \subset \tau(E, F)$.

(b) $\mathfrak{T}_\sigma \subset \tau(E, F)$ *implies* $\mathfrak{T}_\sigma \subset \mathfrak{T}_{\beta c}$.

(c) *If* $(F, \tau(F, E))$ *is barrelled, then* $\mathfrak{T}_{\tau c} \subset \mathfrak{T}_\sigma$.

Proof. We identify F with a subspace of E^*.

(a) A linear functional $f \in E^*$ belongs to $(E, \mathfrak{T}_\sigma)'$ if and only if $f | (B, \sigma(E, F) \cap B)$ is continuous for every $\sigma(E, F)$-bounded subset B of E (cf. Garling [**14**, p. 2]). Now Grothendieck's completeness theorem

(Horváth [**18**, p. 248, Thm. 1]) implies that $(F, \beta(F, E))$ is complete if and only if $(E, \mathfrak{T}_\sigma)' \subset F$, which in turn is equivalent to $\mathfrak{T}_\sigma \subset \tau(E, F)$.

(b) Let $\mathfrak{T}_\sigma \subset \tau(E, F)$ be satisfied. Because of $\sigma(E, F) \subset \mathfrak{T}_\sigma$ the topology $\mathfrak{T}_\sigma$ is the topology of uniform convergence on all elements of $\mathfrak{N} := \{U^0;\ U \in \mathfrak{U}_0(E, \mathfrak{T}_\sigma)\}$. Moreover, each $\sigma(E, F)$-bounded subset B of E is precompact for $\sigma(E, F)$ and thus precompact for $\mathfrak{T}_\sigma$ (Grothendieck [**16**, Ch. II, n° 14, Lemme, p. 98]). According to Grothendieck [**16**, Ch. II, n° 18, Cor. 4, p. 123] each $\mathfrak{T}_\sigma$-equicontinuous subset of F is precompact with respect to $\beta(F, E)$. Since — by (a) from above — $\beta(F, E)$ is complete, each $\mathfrak{T}_\sigma$-equicontinuous subset of F is relatively compact with respect to $\beta(F, E)$. This implies $\mathfrak{T}_\sigma \subset \mathfrak{T}_{\beta c}$.

(c) Let $\tau(F, E)$ be barrelled. We have to show that $\mathfrak{T}_{\tau c} \cap B = \sigma(E, F) \cap B$ holds for each $\sigma(E, F)$-bounded subset B of E. Each $\mathfrak{T}_{\tau c}$-equicontinuous subset of F is $\tau(F, E)$-precompact. Therefore each $\tau(F, E)$-equicontinuous subset of E is $\mathfrak{T}_{\tau c}$-precompact (Grothendieck, loc. cit.). This implies that each absolutely convex $\sigma(E, F)$-compact subset B of E is $\mathfrak{T}_{\tau c}$-precompact and thus $\mathfrak{T}_{\tau c}$-compact, since $\mathfrak{T}_{\tau c}$ has a 0-nbhd.-base consisting of $\sigma(E, F)$-closed sets (Köthe [**21**, §18.4 (4), p. 210]). We now use the barelledness of $\tau(F, E)$ to conclude that for each bounded subset B of E the set $K := \overline{\Gamma(B)}$ is $\mathfrak{T}_{\tau c}$-compact which implies $\mathfrak{T}_{\tau c} \cap K = \sigma(E, F) \cap K$. □

(1.13) REMARK. Let $(H, \mathfrak{R})$ be a distinguished Fréchet-space. We then may apply (1.12) to the dual pair $\langle H'', H' \rangle$. Since $(H', \tau(H', H'')) = (H', \beta(H', H''))$ is barrelled and complete, we obtain $\sigma(H'', H') \subset \mathfrak{T}_{\tau c} = \mathfrak{T}_{\beta c} = \mathfrak{T}_\sigma \subset \tau(H'', H')$.

(1.14) PROPOSITION. *On $\mathring{\mathfrak{B}}(\Omega)''$ the following topologies coincide*:

$\mathfrak{T}_\sigma =$ *the finest locally convex topology which satisfies* $\mathfrak{T}_\sigma \cap B = \sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \cap B = \sigma(\mathfrak{E}, \mathfrak{E}') \cap B = \beta(\mathfrak{E}, \mathfrak{E}') \cap B$ *for each* $\beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$-*bounded subset* B *of* $\mathring{\mathfrak{B}}(\Omega)''$.

$\mathfrak{T}_{\beta c} =$ *the topology of uniform convergence on all* $\beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}})$-*compact subsets of* $\mathring{\mathfrak{B}}(\Omega)'$.

Moreover, we have $\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \subset \mathfrak{T}_{\beta c} = \mathfrak{T}_\sigma \subset \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$.

Proof. Let $B \subset \mathring{\mathfrak{B}}(\Omega)''$ be bounded with respect to $\beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$. Since $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ is distinguished, there exists a bounded subset D of $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ such that $B \subset \overline{D}^{\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')} =: M$. Since $\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \cap M$ is compact, the inclusion $\mathfrak{E}(\Omega)' \subset \mathring{\mathfrak{B}}(\Omega)'$ implies $\sigma(\mathfrak{E}, \mathfrak{E}') \cap M = \sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \cap M$. Since $(\mathfrak{E}(\Omega), \beta(\mathfrak{E}, \mathfrak{E}'))$ is a Montel-space and M is also $\beta(\mathfrak{E}, \mathfrak{E}')$-bounded, we obtain $\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \cap B = \sigma(\mathfrak{E}, \mathfrak{E}') \cap B = \beta(\mathfrak{E}, \mathfrak{E}') \cap B$. Now the result follows from (1.13). □

(1.15) REMARKS. (a) For $\Omega = \mathbf{R}^n$ the assertions of Proposition (1.14) are stated by Schwartz [**26**, p. 203].

(b) As $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ is distinguished, the closed bounded subsets of $(\mathring{\mathfrak{B}}(\Omega)'', \sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$ are compact. Thus we may replace the topology $\beta(\mathfrak{E}, \mathfrak{E}')$ in the definition of $\mathfrak{T}_\sigma$ by any Hausdorff-topology coarser than $\beta(\mathfrak{E}, \mathfrak{E}')$. In particular, we could use the topology of pointwise convergence on Ω (= relative topology from $\mathbf{K}^\Omega$) or the topology of convergence in measure (cf. Bourbaki [**5**, Ch. IV, §5, n° 11, p. 194]).

2. The strict topology $\mathfrak{S}$ on $\mathring{\mathfrak{B}}(\Omega)''$. From now on we identify the spaces $(\mathring{\mathfrak{B}}(\Omega), \check{\mathfrak{T}})$ and $(\mathring{\mathfrak{B}}(\Omega)'', \beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$ according to Theorem (1.9).

(2.1) For $\alpha \in \mathbf{N}_0^n$, $m \in \mathbf{N}_0$, and $f \in C_0(\Omega)$ we define the semi-norm $q_{\alpha,m,f}: \mathring{\mathfrak{B}}(\Omega)'' \to \mathbf{R}$ by

$$q_{\alpha,m,f}(\psi) := \sup\{|f(x)\cdot \rho_{-m}(x)\cdot \partial^\alpha \psi(x)|\,;\, x \in \Omega\}$$

and put $q_{m,f}(\psi) := \max\{q_{\alpha,m,f}(\psi);\ |\alpha| \leq m\}$. The locally convex topology $\mathfrak{S}$ on $\mathring{\mathfrak{B}}(\Omega)''$ generated by the *directed* system of semi-norms $\{q_{m,f};\ m \in \mathbf{N}_0, f \in C_0(\Omega)\}$ will be called the *strict topology* on $\mathring{\mathfrak{B}}(\Omega)''$. The name "strict topology" is justified by the analogy to the construction of Buck [**6**].

(2.2) REMARKS. (a) An obvious modification of the proof of Lemma 4 of Buck [**6**, p. 97] yields:

For $\psi \in \mathfrak{E}(\Omega)$ the following are equivalent:
(a.1) $\rho_{-m} \cdot \partial^\alpha \psi \in L^\infty(\Omega)$ for all $\alpha \in \mathbf{N}_0^n$, $m \in \mathbf{N}_0$.
(a.2) $f \cdot \rho_{-m} \cdot \partial^\alpha \psi \in L^\infty(\Omega)$ for all $\alpha \in \mathbf{N}_0^n$, $m \in \mathbf{N}_0$, $f \in C_0(\Omega)$.

(b) In view of the definition of the strict topology of Buck [**6**, p. 97] it seems more natural to define the strict topology $\mathfrak{S}$ on $\mathring{\mathfrak{B}}(\Omega)''$ by the semi-norms

$$\bar{q}_{\alpha,f}(\psi) := \sup\{|f(x)\partial^\alpha \psi(x)|\,;\, x \in \Omega\} \quad (\alpha \in \mathbf{N}_0^n, f \in C_0(\Omega)).$$

This is in fact possible: *The systems of semi-norms* $\{q_{m,f};\ m \in \mathbf{N}_0,\ f \in C_0(\Omega)\}$ *and* $\{\bar{q}_{\alpha,f};\ \alpha \in \mathbf{N}_0^n, f \in C_0(\Omega)\}$ *are equivalent on* $\mathring{\mathfrak{B}}(\Omega)''$. (We note that the latter semi-norms — in contrast to the former — are defined on all of $\mathfrak{B}(\Omega)$.) Since the construction used in the proof of this equivalence might be of independent interest, we give the proof in an appendix at the end of the paper.

Our estimates, however, will be based on the semi-norms defined in (2.1), since — by (1.3) and (1.8) — the weight-functions ρ_{-m} $(m \in \mathbf{N}_0)$ occur quite naturally.

(2.3) PROPOSITION. (a) *For* $m \in \mathbf{N}_0$ *there exists* $d(m) > 0$ *such that the estimates*

$$p_m\Big(\sum_{k\in L} t_k\theta_k\psi\Big) \leq d(m)\cdot \|t\|_\infty \cdot p_m(\psi),$$

$$q_{m,f}\Big(\sum_{k\in L} t_k\theta_k\psi\Big) \leq d(m)\cdot \|t\|_\infty \cdot q_{2m,f}(\psi)$$

hold for all $t = (t_k;\ k \in \mathbf{N}) \in l^\infty$, $\psi \in \mathring{\mathfrak{B}}(\Omega)''$, $L \subset \mathbf{N}$, *and all* $f \in C_0(\Omega)$. *In particular, the estimates*

$$p_m(\eta_k\psi) \leq d(m)\cdot p_m(\psi) \quad \textit{and} \quad q_{m,f}(\eta_k\psi) \leq d(m)\cdot q_{2m,f}(\psi)$$

hold for all $\psi \in \mathring{\mathfrak{B}}(\Omega)''$, $k \in \mathbf{N}$, $m \in \mathbf{N}_0$, *and all* $f \in C_0(\Omega)$.

(b) *For every* $\psi \in \mathring{\mathfrak{B}}(\Omega)''$ *the net* $(\sum_{k\in J}\theta_k\psi;\ J \subset \mathbf{N},\ J$ *finite*) $\mathfrak{S}$-*converges to* ψ. *In particular, the sequence* $(\eta_k\psi;\ k \in \mathbf{N})$ $\mathfrak{S}$-*converges to* ψ *for every* $\psi \in \mathring{\mathfrak{B}}(\Omega)''$.

(c) $\mathfrak{S} \subset \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$.

Proof. (a) This is a straightforward estimate which only uses (1.3c) and (1.8b).

(b) Let $m \in \mathbf{N}_0$, $f \in C_0(\Omega)$, and $\varepsilon > 0$ be given. Using the constants $c(m,\alpha)$ $(|\alpha| \leq m)$ from (1.8a) and $d(2m)$ from above we define for $\psi \in \mathring{\mathfrak{B}}(\Omega)''$

$$b(m,\psi) := 1 + (1 + d(2m))\cdot p_{2m}(\psi)\cdot \max\{c(m,\alpha);\ |\alpha| \leq m\}.$$

Then $K := \{x \in \Omega;\ |f(x)| \geq \varepsilon\cdot b(m,\psi)^{-1}\}$ is a compact subset of Ω. Since $(\operatorname{supp}(\theta_k);\ k \in \mathbf{N})$ is locally finite in Ω there exists $J_0 \subset \mathbf{N}$, J_0 finite, such that $\operatorname{supp}(\theta_k) \cap K = \varnothing$ holds for all $k \notin J_0$. For every finite subset $J \supset J_0$ of $\mathbf{N}$ we have $\sum_{k\in J}\theta_k(x) = 1$ for all x in a neighbourhood of K.

Thus we obtain the following estimate

$$\begin{aligned} q_{m,f}\Big(\psi - \sum_{k\in J} \theta_k\psi\Big) &= \sup\left\{\left|f(x)\cdot \rho_{-m}(x)\partial^\alpha\Big(\psi - \sum_{k\in J}\theta_k\psi\Big)(x)\right|;\, x\in\Omega\setminus K, |\alpha|\le m\right\} \\ &\le \varepsilon\cdot b(m,\psi)^{-1} \\ &\quad\cdot\sup\left\{\rho_{-m}(x)\left|\partial^\alpha\Big(\psi-\sum_{k\in J}\theta_k\psi\Big)(x)\right|;\, x\in\Omega, |\alpha|\le m\right\} \\ &\le \varepsilon\cdot b(m,\psi)^{-1}\cdot\max\left\{c(m,\alpha)p_{m+|\alpha|}\Big(\psi-\sum_{k\in J}\theta_k\psi\Big);\, |\alpha|\le m\right\} \\ &\le \varepsilon\cdot b(m,\psi)^{-1}\cdot p_{2m}\Big(\psi-\sum_{k\in J}\theta_k\psi\Big)\cdot\max\{c(m,\alpha); |\alpha|\le m\}\le\varepsilon \end{aligned}$$

for all $J\subset\mathbf{N}$, J finite, $J\supset J_0$.

(c) It suffices to show that each $T\in(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})'$ is $\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$-continuous. Let $T\in(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})'$. From (1.8b) we obtain the estimate $q_{\alpha,m,f}(\psi)\le\|f\|_\infty\cdot c(m,\alpha)p_{m+|\alpha|}(\psi)$ ($\alpha\in\mathbf{N}_0^n$, $m\in\mathbf{N}_0$, $f\in C_0(\Omega)$, $\psi\in\mathring{\mathfrak{B}}(\Omega)''$), which shows that $\mathfrak{T}_0=\beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')\cap\mathring{\mathfrak{B}}(\Omega)\supset\mathfrak{S}\cap\mathring{\mathfrak{B}}(\Omega)$. Thus the restriction $T|\mathring{\mathfrak{B}}(\Omega)$ belongs to $\mathring{\mathfrak{B}}(\Omega)'=(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)'$. Let R denote the unique $\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$-continuous extension of $T|\mathring{\mathfrak{B}}(\Omega)$ to all of $\mathring{\mathfrak{B}}(\Omega)''$. Using Theorem (1.9b) and (a) from above we obtain for every $\psi\in\mathring{\mathfrak{B}}(\Omega)''$:

$$\langle R,\psi\rangle=\lim_{k\to\infty}\langle R,\eta_k\psi\rangle=\lim_{k\to\infty}\langle T,\eta_k\psi\rangle=\langle T,\psi\rangle.$$

Thus $R=T$ is $\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$-continuous. □

(2.4) According to (2.3c) the dual $(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})'$ can be identified with a subspace of $\mathring{\mathfrak{B}}(\Omega)'$. Using (1.9b) and (2.3a) it is easily seen that for $H\subset\mathring{\mathfrak{B}}(\Omega)'$ and $m\in\mathbf{N}_0$ the following statements are equivalent:

(a) $\forall\,\varepsilon>0\ \ \exists\,K\Subset\Omega\ \ \forall\,\varphi\in\mathfrak{D}(\Omega)$, $\operatorname{supp}(\varphi)\cap K=\varnothing\ \ \forall\,T\in H$: $|\langle T,\varphi\rangle|\le\varepsilon\cdot p_m(\varphi)$.

(b) $\forall\,\varepsilon>0\,\exists\,K\Subset\Omega\ \ \forall\,\psi\in\mathring{\mathfrak{B}}(\Omega)''$, $\operatorname{supp}(\psi)\cap K=\varnothing\ \ \forall\,T\in H$: $|\langle T,\psi\rangle|\le\varepsilon\cdot p_m(\psi)$.

Moreover, from the *proof* of (a) $\Rightarrow$ (b) in Lemma (2.3) of **[12]** we obtain:

(c) Let $T\in\mathring{\mathfrak{B}}(\Omega)'$ satisfy $|\langle T,\varphi\rangle|\le c\cdot p_m(\varphi)$ ($\varphi\in\mathfrak{D}(\Omega)$) for some $m\in\mathbf{N}_0$ and some $c>0$. Then $H:=\{T\}$ also satisfies (a) from above with the same m.

Next we characterize the $\mathfrak{S}$-equicontinuous subsets of $\mathring{\mathfrak{B}}(\Omega)'$.

(2.5) THEOREM. *For a subset H of $\mathring{\mathfrak{B}}(\Omega)'$ the following statements are equivalent*:

(a) *H is $\mathfrak{S}$-equicontinuous.*

(b) *H is $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}})$-bounded and $\exists\, m \in \mathbf{N}_0\ \forall\, \varepsilon > 0\ \exists\, K \Subset \Omega\ \forall\, \psi \in \mathring{\mathfrak{B}}(\Omega)''$, $\operatorname{supp}(\psi) \cap K = \varnothing\ \forall\, T \in H$: $|\langle T, \psi\rangle| \leq \varepsilon \cdot p_m(\psi)$.*

(c) *Same statement as in* (b) *with $\mathring{\mathfrak{B}}(\Omega)''$ replaced by $\mathfrak{D}(\Omega)$.*

Proof. (a) $\Rightarrow$ (b): Because of (2.3c) the set H is $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$-bounded and thus $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}})$-bounded.

Since $\{q_{m,f};\ m \in \mathbf{N}_0, f \in C_0(\Omega)\}$ is directed, there exists $m \in \mathbf{N}_0$ and $f \in C_0(\Omega)$ such that $|\langle T, \psi\rangle| \leq q_{m,f}(\psi)$ holds for all $\psi \in \mathring{\mathfrak{B}}(\Omega)''$, $T \in H$. Let $\varepsilon > 0$ be given and put $c(m) := \max\{c(m, \alpha);\ |\alpha| \leq m\}$. Then $K := \{x \in \Omega;\ |f(x)| \geq \varepsilon \cdot c(m)^{-1}\}$ is a compact subset of Ω and for every $\psi \in \mathring{\mathfrak{B}}(\Omega)''$ satisfying $K \cap \operatorname{supp}(\psi) = \varnothing$ we obtain the estimate

$$\begin{aligned} |\langle T, \psi\rangle| &\leq \sup\{|f(x)\rho_{-m}(x)\partial^\alpha\psi(x)|;\ x \in \Omega\backslash K, |\alpha| \leq m\} \\ &\leq \varepsilon \cdot c(m)^{-1} \cdot \max\{\|\rho_{-m}\partial^\alpha\psi\|_\infty;\ |\alpha| \leq m\} \\ &\leq \varepsilon \cdot c(m)^{-1} \cdot p_{2m}(\psi) \max\{c(m, \alpha);\ |\alpha| \leq m\} = \varepsilon \cdot p_{2m}(\psi) \end{aligned}$$

for all $T \in H$.

(b) $\Rightarrow$ (a): We first construct inductively a sequence $(K(k);\ k \in \mathbf{N})$ of compact subsets of Ω and a map l: $\mathbf{N} \to \mathbf{N}$ such that for all $k \in \mathbf{N}$ the following conditions are satisfied:

(α) $K(j) \subset K(k)$ and $l(j) < l(k)$ for $j < k$.

(β) $|\langle T, \psi\rangle| \leq 4^{-k} p_m(\psi)$ for all $T \in H$ and all $\psi \in \mathring{\mathfrak{B}}(\Omega)''$ satisfying $K(k) \cap \operatorname{supp}(\psi) = \varnothing$.

(γ) There exists an open neighbourhood $U(k)$ of $K(k)$ in Ω such that $\eta_j(x) = 1$ holds for all $x \in U(k), j \geq l(k)$.

(δ) $\operatorname{supp}(\eta_{l(j)}) \subset K(k)$ $(j < k)$.

Let $k \in \mathbf{N}$. According to (δ) we have $\operatorname{supp}(\eta_{l(k+1)} - \eta_{l(k)}) \subset K(k+2)$ whereas (γ) implies $\eta_{l(k+1)}(x) - \eta_{l(k)}(x) = 0$ for all $x \in U(k)$. Thus we have

$$\operatorname{supp}(\eta_{l(k+1)} - \eta_{l(k)}) \subset K(k+2)\backslash U(k) \subset K(k+2)\backslash K(k).$$

For $k \in \mathbf{N}_0$ we choose a continuous function f_k: $\Omega \to [0, 1]$ with compact support in Ω such that $f_0(x) = 1$ $(x \in K(2))$ and $f_k(x) = 1$ $(x \in K(k+2)\backslash K(k))$ holds for all $k \in \mathbf{N}$. Then $f := \sum_{k \in \mathbf{N}_0} 2^{-k} f_k$ belongs to

$C_0(\Omega)$ and we obtain the estimates $f(x) \geq 1$ ($x \in K(2)$), $f(x) \geq 2^{-k}$ ($x \in K(k+2) \setminus K(k)$) for all $k \in \mathbf{N}$. Since $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ is a Fréchet-space, the $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}})$-bounded set H is $\mathfrak{T}_0$-equicontinuous. Thus there exists $j \in \mathbf{N}_0$ and $c > 0$ such that $|\langle T, \varphi \rangle| \leq c \cdot p_j(\varphi)$ ($\varphi \in \mathring{\mathfrak{B}}(\Omega)$, $T \in H$) holds. Let $T \in H$ and $\psi \in \mathring{\mathfrak{B}}(\Omega)''$. By Theorem (1.9b) and (β) from above we obtain:

$$\begin{aligned}
|\langle T, \psi \rangle| &= \lim_{i \to \infty} |\langle T, \eta_{l(i)} \psi \rangle| \\
&= \lim_{i \to \infty} \left| \left\langle T, \sum_{k=1}^{i-1} (\eta_{l(k+1)} - \eta_{l(k)}) \psi + \eta_{l(1)} \psi \right\rangle \right| \\
&\leq \sum_{k=1}^{\infty} |\langle T, (\eta_{l(k+1)} - \eta_{l(k)}) \psi \rangle| + |\langle T, \eta_{l(1)} \psi \rangle| \\
&\leq \sum_{k=1}^{\infty} 4^{-k} p_m\big((\eta_{l(k+1)} - \eta_{l(k)}) \psi\big) + c \cdot p_j\big(\eta_{l(1)} \psi\big).
\end{aligned}$$

For $k \in \mathbf{N}$ we have the estimate

$$\begin{aligned}
&p_m\big((\eta_{l(k+1)} - \eta_{l(k)}) \psi\big) \\
&\quad = \sup\big\{ |\partial^\alpha\big((\eta_{l(k+1)} - \eta_{l(k)}) \psi\big)(x)| ;\ x \in K(k+2) \setminus K(k), |\alpha| \leq m \big\} \\
&\quad \leq \sup\big\{ 2^k f(x) |\partial^\alpha\big((\eta_{l(k+1)} - \eta_{l(k)}) \psi\big)(x)| ;\ x \in \Omega, |\alpha| \leq m \big\} \\
&\quad \leq 2^k \cdot \max\big\{ q_{\alpha,0,f}\big((\eta_{l(k+1)} - \eta_{l(k)}) \psi\big);\ |\alpha| \leq m \big\} \\
&\quad \leq 2^k \cdot 2 \cdot d(m) \cdot q_{2m,f}(\psi) \quad \text{by (2.3a)},
\end{aligned}$$

$$\begin{aligned}
p_j\big(\eta_{l(1)} \psi\big) &= \sup\big\{ |\partial^\alpha\big(\eta_{l(1)} \psi\big)(x)| ;\ x \in K(2), |\alpha| \leq j \big\} \\
&\leq \sup\big\{ f(x) \cdot |\partial^\alpha\big(\eta_{l(1)} \psi\big)(x)| ;\ x \in \Omega, |\alpha| \leq j \big\} \\
&\leq \max\big\{ q_{\alpha,0,f}\big(\eta_{l(1)} \psi\big);\ |\alpha| \leq j \big\} \leq d(j) \cdot q_{2j,f}(\psi).
\end{aligned}$$

Taking these estimates together we obtain

$$\begin{aligned}
|\langle T, \psi \rangle| &\leq \sum_{k=1}^{\infty} 2^{-k} \cdot 2 \cdot d(m) \cdot q_{2m,f}(\psi) + c \cdot d(j) \cdot q_{2j,f}(\psi) \\
&\leq q_{2(m+j),f}(\psi) \cdot (2 \cdot d(m) + c \cdot d(j)) \qquad \big(T \in H, \psi \in \mathring{\mathfrak{B}}(\Omega)''\big).
\end{aligned}$$

This shows that H is $\mathfrak{S}$-equicontinuous.

The equivalence of (b) and (c) is an immediate consequence of (2.4). □

(2.6) COROLLARY. (a) $\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \subset \mathfrak{S}$.

(b) *On* $\mathring{\mathfrak{B}}(\Omega)''$ *the topologies*

$$\beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \supset \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \supset \mathfrak{S} \supset \sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$$

have the same bounded sets.

Proof. (a) According to [**12**, Lemma (2.3)] every $T \in \mathring{\mathfrak{B}}(\Omega)'$ (considered as a continuous linear form on $(\mathring{\mathfrak{B}}(\Omega)'', \sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')))$ satisfies (2.4a) and thus (2.4b) for some $m \in \mathbf{N}_0$. Therefore Theorem (2.5) shows that T is $\mathfrak{S}$-continuous.

(b) The inclusions follow from (2.3c) and (a) from above. It was shown in (1.10) that $\beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ and $\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ have the same bounded sets. □

(2.7) REMARK. Condition (2.5b) is somewhat similar to the Prokhorov-condition (cf. N. Bourbaki: Intégration, Ch. IX, p. 63) which characterizes the $\sigma(\mathfrak{M}^b(X), \mathcal{C}^b(X))$-relatively compact sets in the space of bounded measures on a completely regular space X (cf. also Conway [**7**, p. 476, Thm. 2.2]).

(2.8) THEOREM. (a) $\mathfrak{S} \supset \beta(\mathfrak{E}, \mathfrak{E}') \cap \mathring{\mathfrak{B}}(\Omega)''$.

(b) $\mathfrak{S} \cap B = \beta(\mathfrak{E}, \mathfrak{E}') \cap B$ *holds for every bounded subset* B *of* $(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})$, *hence* $\mathfrak{S} \subset \mathfrak{T}_\sigma$.

(c) $(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})$ *is a semi-Montel-space.*

(d) *The following conditions are equivalent*:

(d.1) $\beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ *and* $\mathfrak{S}$ *have the same convergent sequences.*

(d.2) Ω *is quasi-bounded*, i.e. $\rho \in C_0(\Omega)$.

(d.3) $\beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') = \mathfrak{S}$.

(d.4) $\mathfrak{S}$ *is quasi-barrelled* (*or metrizable*).

Proof. (a) Let $\alpha \in \mathbf{N}_0^n$ and a compact subset K of Ω be given. Then there exists $f \in C_0(\Omega)$ satisfying $|f(x)| \geq 1$ for all $x \in K$. Thus we obtain

$$\sup\{|\partial^\alpha \psi(x)|\,;\, x \in K\} \leq \sup\{|f(x)\partial^\alpha \psi(x)|\,;\, x \in \Omega\} \leq q_{\alpha,0,f}(\psi)$$

for all $\psi \in \mathring{\mathfrak{B}}(\Omega)''$.

(b) Let B be a bounded absolutely convex subset of $(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})$. In view of (a) and of the metrizability of $\beta(\mathfrak{E}, \mathfrak{E}')$ it suffices to show that each sequence $(\psi_k;\ k \in \mathbf{N})$ in B which converges to 0 with respect to $\beta(\mathfrak{E}, \mathfrak{E}')$ also converges to 0 with respect to $\mathfrak{S}$ (cf. Grothendieck [**16**, Ch. II, n° 14, Lemme, p. 98]). Let $m \in \mathbf{N}_0$, $f \in C_0(\Omega)\backslash\{0\}$, and $\varepsilon > 0$ be given. According to (2.6b) we have

$$M := \sup\{c(m, \alpha)\cdot p_{2m}(\psi);\ |\alpha| \leq m, \psi \in B\} < \infty.$$

$K := \{x \in \Omega;\ |f(x)| \geq \varepsilon\cdot(M+1)^{-1}\}$ is a compact subset of Ω, and $N := \sup\{\rho_{-m}(x);\ x \in K\}$ is finite. Since $(\psi_k;\ k \in \mathbf{N})$ converges to 0 with respect to $\beta(\mathfrak{E}, \mathfrak{E}')$ there exists $k_0 \in \mathbf{N}$ such that

$$\sup\{|\partial^\alpha \psi_k(x)|\ ;\ x \in K, |\alpha| \leq m\} \leq \varepsilon\cdot(\|f\|_\infty \cdot N)^{-1}$$

holds for all $k \geq k_0$. Now we estimate

$$|f(x)|\ \cdot \rho_{-m}(x)\,|\partial^\alpha \psi_k(x)| \leq \varepsilon\cdot(M+1)^{-1}\cdot c(m, \alpha)\cdot p_{m+|\alpha|}(\psi_k) \leq \varepsilon$$

for all $x \in \Omega\backslash K$, $|\alpha| \leq m$, $k \in \mathbf{N}$.

$$|f(x)|\ \cdot \rho_{-m}(x)\cdot|\partial^\alpha \psi_k(x)| \leq \|f\|_\infty \cdot N \cdot \varepsilon\cdot(\|f\|_\infty \cdot N)^{-1} = \varepsilon$$

for all $x \in K$, $|\alpha| \leq m$, $k \geq k_0$. This implies $q_{m,f}(\psi_k) \leq \varepsilon$ for all $k \geq k_0$.

(c) Let B be an absolutely convex closed and bounded subset of $(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})$. Because of $\tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \supset \mathfrak{S} \supset \sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ the set B is also bounded and closed with respect to $\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$. Since $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ is distinguished (cf. Thm. (1.9c)), $(B, \sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \cap B)$ is compact. Now it follows from Proposition (1.14) and (b) from above that $\mathfrak{S} \cap B = \beta(\mathfrak{E}, \mathfrak{E}') \cap B = \sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \cap B$ is compact.

(d.1) $\Rightarrow$ (d.2): Let B be a bounded closed subset of $(\mathring{\mathfrak{B}}(\Omega)'', \beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$. Then (b), (c), and (d.1) imply that $(B, \beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \cap B)$ is compact. As $(\mathring{\mathfrak{B}}(\Omega)'', \beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$ is barrelled, we obtain that $(\mathring{\mathfrak{B}}(\Omega)'', \beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$ is a Montel-space, whence the closed subspace $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ is a Montel-space too. Now it follows from [**12**, Thm. (4.11)] that Ω is quasi-bounded, i.e. $\rho \in C_0(\Omega)$.

(d.2) $\Rightarrow$ (d.3): Since $\rho \in C_0(\Omega)$ we have the estimate

$$\begin{aligned} p_m(\psi) &= \sup\{|\partial^\alpha \psi(x)|\ ;\ x \in \Omega, |\alpha| \leq m\} \\ &= \sup\{\rho(x)\cdot\rho_{-1}(x)\,|\partial^\alpha \psi(x)|\ ;\ x \in \Omega, |\alpha| \leq m\} \\ &\leq \max\{q_{\alpha,1,\rho}(\psi);\ |\alpha| \leq m\} \qquad \left(m \in \mathbf{N}_0, \psi \in \mathring{\mathfrak{B}}(\Omega)''\right) \end{aligned}$$

which shows $\beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') = \mathfrak{S}$.

(d.3) $\Rightarrow$ (d.4) is evident.

(d.4) $\Rightarrow$ (d.1): If $\mathfrak{S}$ is quasi-barrelled (or metrizable) we obtain $\mathfrak{S} = \beta^*(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') = \beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ from (2.6b) (cf. Köthe [**21**, §27, n° 1, p. 367 ff]). In particular, $\mathfrak{S}$ and $\beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ have the same convergent sequences. □

(2.9) Until now we have proved the following relations between the various topologies on $\mathring{\mathfrak{B}}(\Omega)''$:

$$\beta(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \supset \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \supset \mathfrak{T}_{\beta c} = \mathfrak{T}_{\sigma} \supset \mathfrak{S} \supset \sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}').$$

3. A characterization of the $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$-compact sets.

(3.1) According to (1.6) the space $\mathfrak{D}(\Omega)$ is dense in $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$. It follows from the *proof* of Lemma 1 in Horváth [**18**, p. 368] that $(\mathfrak{D}(\Omega), \beta(\mathfrak{D}, \mathfrak{D}'))$ is separable. Thus there exists a metrizable locally convex topology $\mathfrak{R}$ on $\mathring{\mathfrak{B}}(\Omega)'$ which is coarser than $\sigma(\mathfrak{D}', \mathfrak{D}) \cap \mathring{\mathfrak{B}}(\Omega)'$. Therefore $(\mathring{\mathfrak{B}}(\Omega)', \sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}''))$ is an *angelic* space (cf. Floret [**13**, p. 39 (2)]). In particular, the notions "compact", "sequentially compact", and "countably compact" coincide for subsets of $(\mathring{\mathfrak{B}}(\Omega)', \sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}''))$ (Floret [**13**, p. 31, Thm.]).

(3.2) THEOREM. *Let $H \subset \mathring{\mathfrak{B}}(\Omega)'$ be $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$-compact. Then H is $\mathfrak{S}$-equicontinuous. More precisely, if H satisfies $|\langle T, \psi\rangle| \leq c \cdot p_m(\psi)$ ($\psi \in \mathring{\mathfrak{B}}(\Omega)''$, $T \in H$) for some $m \in \mathbf{N}_0$ and some $c > 0$, then H also satisfies*

$$(*) \quad \begin{cases} \forall\, \varepsilon > 0\ \exists\, K \Subset \Omega\ \forall\, \psi \in \mathring{\mathfrak{B}}(\Omega)'', K \cap \operatorname{supp}(\psi) = \varnothing\ \forall\, T \in H: \\ |\langle T, \psi\rangle| \leq \varepsilon \cdot p_{m+2}(\psi). \end{cases}$$

By (2.4) the space $\mathring{\mathfrak{B}}(\Omega)''$ in the statement $(*)$ may be replaced by $\mathfrak{D}(\Omega)$.

Since the sliding hump proof of the above theorem is rather long and somewhat technical,[1] we first give an outline. We assume that $H \subset \mathring{\mathfrak{B}}(\Omega)'$ is $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$-compact without being $\mathfrak{S}$-equicontinuous. By (3.1) we find a $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$-convergent sequence $(T_k;\ k \in \mathbf{N})$ in H which is not $\mathfrak{S}$-equicontinuous. We now apply (2.5c) to construct a certain sequence $(\varphi_k;\ k \in \mathbf{N})$ in $\mathfrak{D}(\Omega)$. After a careful modification this sequence will be pasted together to yield an element $\psi \in \mathring{\mathfrak{B}}(\Omega)''$ such that $(\langle T_k, \psi\rangle;\ k \in \mathbf{N})$ does not converge. This modification will be achieved by a construction which we now describe.

(3.3) Let $\xi \in \mathfrak{D}(\mathbf{R}^n)$ satisfy $\xi \geq 0$, $\operatorname{supp}(\xi) \subset K[0,1] := \{x \in \mathbf{R}^n;\ |x| \leq 1\}$, $\|\xi\|_1 = 1$, and define $\xi_t(x) := t^{-n}\xi(x/t)$ ($x \in \mathbf{R}^n$, $t > 0$). For

[1] It is a comfort to us that the proof of Theorem (2.6) of Conway [7, p. 478] is likewise complicated.

$s > 0$ we define χ_{2s}: $\mathbf{R}^n \to \mathbf{R}$ by

$$\chi_{2s}(x) := \begin{cases} 1 & \text{for } r(x) \geq 2s, \\ 0 & \text{for } r(x) < 2s \end{cases}$$

and put $\gamma_s := \xi_s * \chi_{2s}$. Then $\gamma_s \in \mathfrak{B}(\mathbf{R}^n)$ and

$$\gamma_s(x) = \begin{cases} 1 & \text{for } r(x) \geq 3s, \\ 0 & \text{for } r(x) < s. \end{cases}$$

Moreover, by differentiation under the integral sign, we obtain the estimate $|\partial^\alpha \gamma_s(x)| \leq s^{-|\alpha|} \|\partial^\alpha \xi\|_1$ ($x \in \mathbf{R}^n$, $\alpha \in \mathbf{N}_0^n$, $s > 0$).

(3.4) LEMMA. *Let the notation be as in* (3.3).

(a) $|\partial^\alpha(\xi_t * \psi)(x)| \leq \|\psi\|_\infty \cdot \|\partial^\alpha \xi_t\|_1$ ($\psi \in L^\infty(\mathbf{R}^n)$, $\alpha \in \mathbf{N}_0^n$, $x \in \mathbf{R}^n$, $t > 0$).

(b) $p_m(\xi_t * \psi - \psi) \leq t \cdot n \cdot p_{m+1}(\psi)$ ($\psi \in \mathfrak{B}^{m+1}(\mathbf{R}^n)$, $m \in \mathbf{N}_0$, $t > 0$).

(c) *Let* $\Omega \neq \mathbf{R}^n$. *For* $m \in \mathbf{N}_0$ *there exists* $e(\xi, m) > 0$ *such that* $p_m(\gamma_s \cdot \varphi - \varphi) \leq s \cdot e(\xi, m) \cdot p_{m+1}(\varphi)$ *holds for all* $\varphi \in \mathfrak{D}(\Omega)$ *and all* $s \in (0, 3^{-1})$.

Proof. (a) follows by differentiation under the integral sign.

(b) is a straightforward estimate which only uses the mean value theorem.

(c) By the Leibniz rule we obtain the estimate

$$\begin{aligned} p_m(\gamma_s\varphi - \varphi) & \\ &\leq \sup\{|\gamma_s(x) - 1| \cdot |\partial^\alpha \varphi(x)|\,;\, x \in \Omega, |\alpha| \leq m\} \\ &\quad + \sup\left\{\sum_{\beta<\alpha} \binom{\alpha}{\beta} |\partial^{\alpha-\beta}\gamma_s(x)| \cdot |\partial^\beta \varphi(x)|\,;\, x \in \Omega, |\alpha| \leq m\right\}. \end{aligned}$$

Now we observe

$$\gamma_s(x) - 1 \neq 0 \Rightarrow 3s > r(x) = \rho(x),$$

$$\beta < \alpha, \partial^{\alpha-\beta}\gamma_s(x) \neq 0 \Rightarrow 3s \geq r(x) = \rho(x) \geq s,$$

and use (1.8b) to estimate

$$
\begin{aligned}
&p_m(\gamma_s\varphi - \varphi)\\
&\quad \le \sup\{\rho(x)\cdot \rho_{-1}(x)\cdot |\partial^\alpha\varphi(x)|\,;\, x\in\Omega,\ \rho(x)<3s,\ |\alpha|\le m\}\\
&\qquad +\sup\Big\{\sum_{\beta<\alpha}\binom{\alpha}{\beta}s^{-|\alpha-\beta|}\|\partial^{\alpha-\beta}\xi\|_1\cdot|\partial^\beta\varphi(x)|\,;\, x\in\Omega,\\
&\qquad\qquad s\le\rho(x)\le 3s,\ |\alpha|\le m\Big\}\\
&\quad \le 3s\cdot\sup\{c(1,\alpha)p_{1+|\alpha|}(\varphi);\ |\alpha|\le m\}\\
&\qquad +\sup\Big\{\sum_{\beta<\alpha}\binom{\alpha}{\beta}\|\partial^{\alpha-\beta}\xi\|_1\cdot 3^{|\alpha-\beta|}\cdot\rho_{-|\alpha-\beta|}(x)\cdot|\partial^\beta\varphi(x)|\,;\\
&\qquad\qquad x\in\Omega,\ s\le\rho(x)\le 3s,\ |\alpha|\le m\Big\}\\
&\quad \le 3s\cdot p_{m+1}(\varphi)\max\{c(1,\alpha);\ |\alpha|\le m\}\\
&\qquad +3s\cdot\sup\Big\{\sum_{\beta<\alpha}\binom{\alpha}{\beta}\|\partial^{\alpha-\beta}\xi\|_1\cdot 3^{|\alpha-\beta|}\cdot c(|\alpha-\beta|+1,\beta)\\
&\qquad\qquad \cdot p_{|\alpha-\beta|+1+|\beta|}(\varphi);\ |\alpha|\le m\Big\}\\
&\quad \le s\cdot p_{m+1}(\varphi)\cdot\Big[3\cdot\max\{c(1,\alpha);\ |\alpha|\le m\}\\
&\qquad +3\max\Big\{\sum_{\beta<\alpha}\binom{\alpha}{\beta}\|\partial^{\alpha-\beta}\xi\|_1\cdot 3^{|\alpha-\beta|}\\
&\qquad\qquad \cdot c(|\alpha-\beta|+1,\beta);\ |\alpha|\le m\Big\}\Big].
\end{aligned}
$$

Thus we may put $e(\xi, m) := [\ldots]$. □

Proof of Theorem (3.2). Let $H\subset\mathring{\mathfrak{B}}(\Omega)'$ be $\sigma(\mathring{\mathfrak{B}}', \mathfrak{B}'')$-compact. Then H is $\sigma(\mathring{\mathfrak{B}}', \mathfrak{B}'')$-bounded, hence $\beta(\mathfrak{B}'', \mathring{\mathfrak{B}}')$-equicontinuous. Thus there exist $m\in\mathbf{N}_0$ and $c>0$ such that

(a) $$|\langle T,\psi\rangle|\le c\cdot p_m(\psi)\qquad (\psi\in\mathfrak{B}(\Omega)'', T\in H)$$

holds. Assume that H is not $\mathfrak{S}$-equicontinuous. From (2.5c) we then obtain

(b) $$\begin{cases}\exists\,\varepsilon_0\in(0,1)\ \forall K\Subset\Omega\ \exists\,T\in H\ \exists\,\varphi\in\mathfrak{D}(\Omega),\ K\cap\operatorname{supp}(\varphi)=\varnothing,\\ p_{m+2}(\varphi)=1, \text{ such that } |\langle T,\varphi\rangle|>17\varepsilon_0.\end{cases}$$

Since Ω has a fundamental sequence of compact subsets, we may use (b) to find a sequence $(T_j; j \in \mathbf{N})$ in H which has the following property:

$$\text{(c)} \quad \begin{cases} \forall K \Subset \Omega, \forall i \in \mathbf{N}\, \exists\, j \geq i\, \exists\, \varphi \in \mathfrak{D}(\Omega), K \cap \operatorname{supp}(\varphi) = \varnothing, \\ p_{m+2}(\varphi) = 1 \text{ such that } |\langle T_j, \varphi \rangle| > 17\varepsilon_0. \end{cases}$$

H is sequentially compact by (3.1). Therefore a subsequence of $(T_j; j \in \mathbf{N})$ is $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$-convergent to some $T_0 \in H$. Since this subsequence also satisfies (c) from above, we may assume

$$\text{(d)} \qquad T_j \to T_0 \in H \qquad (j \to \infty) \quad \text{with respect to } \sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'').$$

From (a) and (2.4) we obtain

$$\text{(e)} \quad \begin{cases} \forall j \in \mathbf{N} \cup \{0\}\, \forall \varepsilon > 0\, \exists K \Subset \Omega\, \forall \psi \in \mathring{\mathfrak{B}}(\Omega)'', \\ K \cap \operatorname{supp}(\psi) = \varnothing : |\langle T_j, \psi \rangle| \leq \varepsilon \cdot p_m(\psi). \end{cases}$$

For $k \in \mathbf{N}$ we put $K(k) := \{x \in \Omega; |x| \leq k, r(x) \geq k^{-1}\}$.

We now construct inductively

— a sequence $(\varphi_j; j \in \mathbf{N})$ in $\mathfrak{D}(\Omega)$, and

— two strictly monotone maps k, l: $\mathbf{N} \to \mathbf{N}$

such that for all $j \in \mathbf{N}$ the following conditions are satisfied:

(C.1) $p_{m+2}(\varphi_j) = 1$.

(C.2) $\operatorname{supp}(\varphi_i) \subset K(l(j) - 1)$ $(i \leq j)$,
$\operatorname{supp}(\varphi_i) \cap K(l(j) + 1) = \varnothing$ $(i > j)$.

(C.2) $|\langle T_{k(j)}, \varphi_j \rangle| > 17\varepsilon_0$.

(C.4) $|\langle T_0, \varphi_j \rangle| \leq \varepsilon_0$.

(C.5) $|\langle T_{k(j)} - T_0, \Sigma_{i \in J} \varphi_i \rangle| \leq \varepsilon_0$ for all subsets $J \subset \{1, \ldots, j-1\}$.

(C.6) $|\langle T_{k(j)} - T_0, \psi \rangle| \leq 2 \cdot \varepsilon_0 \cdot p_m(\psi)$ for all $\psi \in \mathring{\mathfrak{B}}(\Omega)''$ satisfying $K(l(j)) \cap \operatorname{supp}(\psi) = \varnothing$.

Before we begin the induction we use (e) to find $K_0 \Subset \Omega$ such that $|\langle T_0, \psi \rangle| \leq \varepsilon_0 \cdot p_m(\psi)$ holds for all $\psi \in \mathring{\mathfrak{B}}(\Omega)''$ satisfying $K_0 \cap \operatorname{supp}(\psi) = \varnothing$. We then choose $l_0 \in \mathbf{N}$ such that $K_0 \subset K(l_0 - 1)$ holds.

To begin the induction we use (c) to find $k(1) \in \mathbf{N}$ and $\varphi_1 \in \mathfrak{D}(\Omega)$, $K(l_0) \cap \operatorname{supp}(\varphi_1) = \varnothing$, $p_{m+2}(\varphi_1) = 1$ such that $|\langle T_{k(1)}, \varphi_1 \rangle| > 17\varepsilon_0$. By (e) there exists $K' \Subset \Omega$ such that $|\langle T_{k(1)}, \psi \rangle| \leq \varepsilon_0 \cdot p_m(\psi)$ holds for all $\psi \in \mathring{\mathfrak{B}}(\Omega)''$ satisfying $K' \cap \operatorname{supp}(\psi) = \varnothing$. We now define $l(1) \in \mathbf{N}$ to be the smallest integer $l \in \mathbf{N}$ such that $K(l_0) \cup K' \cup \operatorname{supp}(\varphi_1) \subset K(l-1)$. Then (C.1),...,(C.5) are satisfied for $j = 1$. (C.6) is also satisfied because of $K_0 \subset K(l(1))$.

Now suppose that $\varphi_1, \ldots, \varphi_j$, $k(1) < \cdots < k(j)$, and $l(1) < \cdots < l(j)$ are already constructed such that (C.1),...,(C.6) are satisfied.

We first use (d) to find $k_0 \in \mathbf{N}$ such that $|\langle T_k - T_0, \Sigma_{i \in J} \varphi_i \rangle| \leq \varepsilon_0$ holds for all $k \geq k_0$ and all subsets $J \subset \{1, \ldots, j\}$.

According to (c) there exists $k(j+1) > \max\{k(j), k_0\}$ and $\varphi_{j+1} \in \mathfrak{D}(\Omega)$ satisfying $K(l(j)+1) \cap \operatorname{supp}(\varphi_{j+1}) = \varnothing$ and $p_{m+2}(\varphi_{j+1}) = 1$ such that $|\langle T_{k(j+1)}, \varphi_{j+1} \rangle| > 17\varepsilon_0$ holds.

By (e) there exists $K' \Subset \Omega$ such that $|\langle T_{k(j+1)}, \psi \rangle| \leq \varepsilon_0 \cdot p_m(\psi)$ holds for all $\psi \in \mathring{\mathfrak{B}}(\Omega)''$ satisfying $K' \cap \operatorname{supp}(\psi) = \varnothing$. We now define $l(j+1)$ to be the smallest integer $l \in \mathbf{N}$ such that $K(l(j)) \cup K' \cup \operatorname{supp}(\varphi_{j+1}) \subset K(l-1)$. Then (C.1),...,(C.6) are satisfied for $1, \ldots, j+1$. Thus the construction of the sequence $(\varphi_j; j \in \mathbf{N})$ and of the two maps $k, l \colon \mathbf{N} \to \mathbf{N}$ is finished.

Now a careful modification of the sequence $(\varphi_j; j \in \mathbf{N})$ will provide us with an element $\psi \in \mathring{\mathfrak{B}}(\Omega)''$ such that $(\langle T_{k(j)}, \psi \rangle; j \in \mathbf{N})$ does not converge.

Let us first consider the case $\Omega = \mathbf{R}^n$ which is less complicated. We fix $t \in (0, 1/2)$ such that $t \cdot n \leq \varepsilon_0 \cdot \min\{1, c^{-1}\}$ where c is taken from (a). By Lemma (3.4b) we then obtain $p_m(\xi_t * \varphi_j - \varphi_j) \leq t \cdot n \cdot p_{m+1}(\varphi_j) \leq t \cdot n \leq \varepsilon_0 \cdot \min\{1, c^{-1}\}$ for all $j \in \mathbf{N}$. Because of $\operatorname{supp}(\xi_t * \varphi) \subset \operatorname{supp}(\varphi) + K[0, 1/2]$ $(\varphi \in \mathfrak{D}(\mathbf{R}^n))$, the functions $\psi_j := \xi_t * \varphi_j$ $(j \in \mathbf{N})$ satisfy $\operatorname{supp}(\psi_i) \subset K(l(j))$ $(i \leq j)$ and $\operatorname{supp}(\psi_i) \cap K(l(j)) = \varnothing$ $(i > j)$ by (C.2). In particular, the sets $\operatorname{supp}(\psi_j)$ $(j \in \mathbf{N})$ are pairwise disjoint, and from Lemma (3.4a) we infer that the function $\psi := \Sigma_{j \in \mathbf{N}} \psi_j$ belongs to $\mathfrak{B}(\mathbf{R}^n) = \mathring{\mathfrak{B}}(\mathbf{R}^n)''$. Moreover, the sets $\operatorname{supp}(\varphi_j - \psi_j)$ $(j \in \mathbf{N})$ are pairwise disjoint.

We now obtain the following estimates:

$$
\begin{aligned}
|\langle T_{k(j)}, \psi_j \rangle| &\geq |\langle T_{k(j)}, \varphi_j \rangle| - |\langle T_{k(j)}, \varphi_j - \psi_j \rangle| \\
&\geq 17\varepsilon_0 - c \cdot p_m(\varphi_j - \psi_j) \geq 17\varepsilon_0 - \varepsilon_0 \qquad (j \in \mathbf{N})
\end{aligned}
$$

by (C.3) and (a).

$$
\begin{aligned}
|\langle T_0, \psi_j \rangle| &\leq |\langle T_0, \varphi_j \rangle| + |\langle T_0, \varphi_j - \psi_j \rangle| \\
&\leq \varepsilon_0 + c \cdot p_m(\varphi_j - \psi_j) \leq 2\varepsilon_0 \qquad (j \in \mathbf{N})
\end{aligned}
$$

by (C.4) and (a).

$$\begin{aligned}\Big|\Big\langle T_{k(j)} - T_0, \sum_{i<j} \psi_i \Big\rangle\Big| &\leq \Big|\Big\langle T_{k(j)} - T_0, \sum_{i<j} \varphi_j \Big\rangle\Big| + \Big|\Big\langle T_{k(j)} - T_0, \sum_{i<j} (\varphi_i - \psi_i)\Big\rangle\Big| \\ &\leq \varepsilon_0 + 2 \cdot c \cdot \max\{p_m(\varphi_i - \psi_i);\, i<j\} \leq 3\varepsilon_0 \qquad (j \in \mathbf{N})\end{aligned}$$

by (C.5) and (a).

$$\begin{aligned}\Big|\Big\langle T_{k(j)} - T_0, \sum_{i>j} \psi_i \Big\rangle\Big| &\leq 2 \cdot \varepsilon_0 \cdot p_m\Big(\sum_{i>j} \psi_i\Big) \\ &\leq 2 \cdot \varepsilon_0 \cdot \sup\{p_m(\varphi_i) + p_m(\varphi_i - \psi_i);\, i \in \mathbf{N}\} \leq 4\varepsilon_0 \qquad (j \in \mathbf{N})\end{aligned}$$

by (C.6).

Taking these estimates together we obtain

$$\begin{aligned}|\langle T_{k(j)} - T_0, \psi \rangle| &\geq |\langle T_{k(j)} - T_0, \psi_j \rangle| - \Big|\Big\langle T_{k(j)} - T_0, \sum_{i<j} \psi_i \Big\rangle\Big| - \Big|\Big\langle T_{k(j)} - T_0, \sum_{i>j} \psi_i \Big\rangle\Big| \\ &\geq (17\varepsilon_0 - \varepsilon_0) - 2\varepsilon_0 - 3\varepsilon_0 - 4\varepsilon_0 \geq \varepsilon_0 \qquad (j \in \mathbf{N}).\end{aligned}$$

This contradiction finishes the proof of Theorem (3.2) for $\Omega = \mathbf{R}^n$. Now let $\Omega \neq \mathbf{R}^n$. Then $\{\operatorname{dist}(\operatorname{supp}(\varphi_i), \operatorname{supp}(\varphi_j));\, i, j \in \mathbf{N},\, i \neq j\}$ need not be bounded away from zero. We first fix $s \in (0, 1/3)$ such that $2 \cdot s \cdot e(\xi, m+1) \leq \varepsilon_0 \cdot \min\{1, c^{-1}\}$. By Lemma (3.4c) we then obtain

$$p_{m+1}(\gamma_{2s}\varphi_j - \varphi_j) \leq 2 \cdot s \cdot e(\xi, m+1) \cdot p_{m+2}(\varphi_j) \leq \varepsilon_0 \cdot \min\{1, c^{-1}\}$$

for all $j \in \mathbf{N}$. Now we choose $j_0 \in \mathbf{N}$ such that $(l(j_0) - 1)^{-1} \leq s$. From (C.2) we then obtain for all $j \geq j_0$

$$\operatorname{supp}(\gamma_{2s}\varphi_i) \subset \{x \in \Omega;\, |x| \leq l(j) - 1,\, \rho(x) \geq 2s\} \qquad (i \leq j),$$

$$\operatorname{supp}(\gamma_{2s}\varphi_i) \subset \{x \in \Omega;\, |x| > l(j) + 1,\, \rho(x) \geq 2s\} \qquad (i > j).$$

Now we fix $t \in (0, s)$ such that $2 \cdot t \cdot n \leq \varepsilon_0 \cdot \min\{1, c^{-1}\}$. By Lemma (3.4b) we then obtain

$$\begin{aligned}p_m(\xi_t * (\gamma_{2s}\varphi_j) - \varphi_j) &\leq p_m(\xi_t * (\gamma_{2s}\varphi_j) - \gamma_{2s}\varphi_j) + p_m(\gamma_{2s}\varphi_j - \varphi_j) \\ &\leq t \cdot n \cdot p_{m+1}(\gamma_{2s}\varphi_j) + \varepsilon_0 \cdot \min\{1, c^{-1}\} \\ &\leq t \cdot n \cdot \big(p_{m+1}(\varphi_j) + p_{m+1}(\varphi_j - \gamma_{2s}\varphi_j)\big) + \varepsilon_0 \cdot \min\{1, c^{-1}\} \\ &\leq t \cdot n \cdot (1 + 1) + \varepsilon_0 \cdot \min\{1, c^{-1}\} \\ &\leq 2 \cdot \varepsilon_0 \cdot \min\{1, c^{-1}\} \quad \text{for all } j \in \mathbf{N}.\end{aligned}$$

Because of $\operatorname{supp}(\xi_t * \varphi) \subset \operatorname{supp}(\varphi) + K[0, s]$ $(\varphi \in \mathfrak{D}(\Omega))$, the functions $\psi_j := \xi_t * (\gamma_{2s}\varphi_j)$ $(j \geq j_0)$ satisfy

$$\operatorname{supp}(\psi_i) \subset \{x \in \Omega;\ |x| \leq l(j), \rho(x) \geq s\} \qquad (j_0 \leq i \leq j),$$
$$\operatorname{supp}(\psi_i) \subset \{x \in \Omega;\ |x| > l(j), \rho(x) \geq s\} \qquad (i > j \geq j_0).$$

With the help of Lemma (3.4a) we thus infer that the function $\psi := \sum_{i \geq j_0} \psi_i$ belongs to $\check{\mathfrak{B}}(\Omega) = \mathring{\mathfrak{B}}(\Omega)''$. Moreover, the above relations and (C.2) imply $\operatorname{supp}(\varphi_i - \psi_i) \cap \operatorname{supp}(\varphi_j - \psi_j) = \varnothing$ for all $i > j \geq j_0$.

We obtain the following estimates

$$\begin{aligned} |\langle T_{k(j)}, \psi_j\rangle| &\geq |\langle T_{k(j)}, \varphi_j\rangle| - |\langle T_{k(j)}, \varphi_j - \psi_j\rangle| \\ &\geq 17\varepsilon_0 - c \cdot p_m(\varphi_j - \psi_j) \\ &\geq 17\varepsilon_0 - 2\varepsilon_0 \qquad (j \geq j_0) \end{aligned}$$

by (C.3) and (a).

$$\begin{aligned} |\langle T_0, \psi_j\rangle| &\leq |\langle T_0, \varphi_j\rangle| + |\langle T_0, \varphi_j - \psi_j\rangle| \\ &\leq \varepsilon_0 + c \cdot p_m(\varphi_j - \psi_j) \leq 3\varepsilon_0 \qquad (j \geq j_0) \end{aligned}$$

by (C.4) and (a).

$$\begin{aligned} &\left|\left\langle T_{k(j)} - T_0, \sum_{j_0 \leq i < j} \psi_i \right\rangle\right| \\ &\quad \leq \left|\left\langle T_{k(j)} - T_0, \sum_{j_0 \leq i < j} \varphi_i \right\rangle\right| + \left|\left\langle T_{k(j)} - T_0, \sum_{j_0 \leq i < j} (\varphi_i - \psi_i) \right\rangle\right| \\ &\quad \leq \varepsilon_0 + 2 \cdot c \cdot \max\{p_m(\varphi_i - \psi_i);\ i < j\} \\ &\quad \leq 5\varepsilon_0 \qquad (j \geq j_0) \end{aligned}$$

by (C.5) and (a).

$$\begin{aligned} \left|\left\langle T_{k(j)} - T_0, \sum_{i > j} \psi_i \right\rangle\right| &\leq 2 \cdot \varepsilon_0 \cdot p_m\Big(\sum_{i > j} \psi_i\Big) \\ &\leq 2 \cdot \varepsilon_0 \cdot \sup\{p_m(\varphi_i) + p_m(\varphi_i - \psi_i);\ i \in \mathbf{N}\} \\ &\leq 6\varepsilon_0\,. \qquad (j \geq j_0) \end{aligned}$$

by (C.6).

Taking these estimates together we obtain

$$
\begin{aligned}
&|\langle T_{k(j)} - T_0, \psi \rangle| \\
&\geq |\langle T_{k(j)} - T_0, \psi_j \rangle| - \left| \left\langle T_{k(j)} - T_0, \sum_{j_0 \leq i < j} \psi_i \right\rangle \right| - \left| \left\langle T_{k(j)} - T_0, \sum_{i > j} \psi_i \right\rangle \right| \\
&\geq (17\varepsilon_0 - 2\varepsilon_0) - 3\varepsilon_0 - 5\varepsilon_0 - 6\varepsilon_0 = \varepsilon_0 \qquad (j \geq j_0).
\end{aligned}
$$

This contradiction finishes the proof of Theorem (3.2). Whew! (cf. Agmon [**1**, p. 58]). □

(3.5) COROLLARY. (a) $\mathfrak{S} = \mathfrak{T}_\sigma = \mathfrak{T}_{\beta c} = \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$.

(b) $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$ *and* $\beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$ *have the same compact sets, hence the same convergent sequences.*

(c) $(\mathring{\mathfrak{B}}(\Omega)', \sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}''))$ *is sequentially complete.*

(d) $(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})$ *is complete.*

Proof. (a) follows from Theorem (3.2) and (2.9).

(b) follows from $\mathfrak{T}_{\beta c} = \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ since — by the completeness of $\tau(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'') = \beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}})$ — the systems $\{B \subset \mathring{\mathfrak{B}}(\Omega)'; B$ is relatively $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$-compact$\}$ and $\{B \subset \mathring{\mathfrak{B}}(\Omega)'; B$ is relatively $\beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$-compact$\}$ are both saturated (cf. Köthe [**21**, §21.1 (4), p. 256]).

(c) follows from (b) since any two linear topologies having the same convergent sequences also have the same Cauchy-sequences (Webb [**31**, Prop. 1.4, p. 343]).

(d) As the Mackey-dual of the ultrabornological space $(\mathring{\mathfrak{B}}(\Omega)', \beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}))$ (cf. Theorem (1.9c)), the space $(\mathring{\mathfrak{B}}(\Omega)'', \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$ is complete (Köthe [**21**, §28.5 (1), p. 385]). We note that alternatively the proof of Theorem 1 (ii) of Buck [**6**, p. 98] could be modified to yield directly the completeness of $(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})$. □

We now prove that the LB-space

$$
(\mathring{\mathfrak{B}}(\Omega)', \beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}})) = \bigcup_{m \in \mathbf{N}_0} (\mathring{\mathfrak{B}}^m(\Omega), p_m)'
$$

is sequentially retractive.

(3.6) PROPOSITION. *Let* $(T_k;\ k \in \mathbf{N})$ *be a* $\sigma(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}'')$*-convergent sequence in* $\mathring{\mathfrak{B}}(\Omega)'$, $T_k \to T_0 \in \mathring{\mathfrak{B}}(\Omega)'$ $(k \to \infty)$. *Then there exists* $m \in \mathbf{N}_0$ *such that* $T_k \in \mathring{\mathfrak{B}}^m(\Omega)'$ $(k \in \mathbf{N} \cup \{0\})$, *and* $(T_k;\ k \in \mathbf{N})$ *converges to* T_0 *with respect to* $\beta((\mathring{\mathfrak{B}}^m)', \mathring{\mathfrak{B}}^m)$.

Proof. According to Theorem (3.2) there exist $m \in \mathbf{N}_0$ and $c > 0$ such that

(a) $|\langle T_k, \psi\rangle| \leq c \cdot p_m(\psi)$ $(\psi \in \mathring{\mathfrak{B}}(\Omega)'', k \in \mathbf{N} \cup \{0\})$.

(b) $\forall\, \varepsilon > 0\ \exists\, K \Subset \Omega\ \forall\, \psi \in \mathring{\mathfrak{B}}(\Omega)'', K \cap \operatorname{supp}(\psi) = \varnothing\ \forall\, k \in \mathbf{N} \cup \{0\}: |\langle T_k, \psi\rangle| \leq \varepsilon \cdot p_{m+2}(\psi)$.

According to (1.6) the space $\mathring{\mathfrak{B}}(\Omega)$ is dense in $(\mathring{\mathfrak{B}}^{m+2}(\Omega), \mathfrak{T}_0^{m+2})$, hence $\beta((\mathring{\mathfrak{B}}^{m+2})', \mathring{\mathfrak{B}}^{m+2}) = \beta((\mathring{\mathfrak{B}}^{m+2})', \mathring{\mathfrak{B}})$. It is therefore sufficient to prove that $(T_k;\ k \in \mathbf{N})$ converges uniformly on all $\mathfrak{T}_0^{m+2}$-bounded subsets B of $\mathring{\mathfrak{B}}(\Omega)$. Let $B \subset \mathring{\mathfrak{B}}(\Omega)$ be $\mathfrak{T}_0^{m+2}$-bounded and let $\varepsilon > 0$ be given. We use the constant $d(m+2) > 0$ from (2.3a) to define

$$M := 3 \cdot (1 + d(m+2)) \cdot \sup\{p_{m+2}(\varphi);\ \varphi \in B\} + 1.$$

According to (b) from above there exists $K \Subset \Omega$ such that

$$\sup\{|\langle T_k, \psi\rangle|\ ;\ k \in \mathbf{N} \cup \{0\}\} \leq M^{-1} \cdot \varepsilon \cdot p_{m+2}(\psi)$$

holds for all $\psi \in \mathring{\mathfrak{B}}(\Omega)''$ satisfying $K \cap \operatorname{supp}(\psi) = \varnothing$. Now we choose $j \in \mathbf{N}$ such that $\eta_j(x) = 1$ holds in a neighbourhood of K. Since B is bounded in $(\mathring{\mathfrak{B}}^{m+2}(\Omega), \mathfrak{T}_0^{m+2})$ and the inclusion

$$\left(\mathfrak{E}^{m+1}(\Omega), \beta\left(\mathfrak{E}^{m+1}, (\mathfrak{E}^{m+1})'\right)\right) \hookrightarrow \left(\mathfrak{E}^{m}(\Omega), \beta\left(\mathfrak{E}^{m}, (\mathfrak{E}^{m+1})'\right)\right)$$

is compact (Horváth [**18**, p. 239, Example 3, p. 241, Prop. 11]), the set $\eta_j B$ is relatively compact in $(\mathring{\mathfrak{B}}^m(\Omega), \mathfrak{T}_0^m)$. According to (a) from above the set $\{T_k;\ k \in \mathbf{N} \cup \{0\}\}$ is equicontinuous on $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0^m \cap \mathring{\mathfrak{B}}(\Omega))$. Thus the sequence $(T_k;\ k \in \mathbf{N})$ which converges pointwise on $\eta_j B$ also converges uniformly on $\eta_j B$ (Bourbaki [**4**, §2, n° 4, Thm. 1, p. 29]). Thus there exists $k_0 \in \mathbf{N}$ such that $|\langle T_k - T_0, \eta_j\varphi\rangle| \leq \varepsilon/3$ holds for all $\varphi \in B$, $k \geq k_0$. Now we use (2.3a) to estimate

$$\begin{aligned}
|\langle T_k - T_0, \varphi\rangle| &\leq |\langle T_k - T_0, \eta_j\varphi\rangle| + |\langle T_k - T_0, (1 - \eta_j)\varphi\rangle| \\
&\leq \varepsilon/3 + 2 \cdot M^{-1} \cdot \varepsilon \cdot p_{m+2}\left((1 - \eta_j)\varphi\right) \\
&\leq \varepsilon/3 + 2 \cdot M^{-1} \cdot \varepsilon \cdot (1 + d(m+2)) \cdot p_{m+2}(\varphi) \leq \varepsilon
\end{aligned}$$

for all $\varphi \in B$ and all $k \geq k_0$. □

(3.7) COROLLARY. $(\mathring{\mathfrak{B}}(\Omega)', \beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}})) = \bigcup_{m \in \mathbf{N}_0} (\mathring{\mathfrak{B}}^m(\Omega), \mathfrak{T}_0^m)'$ *is strongly boundedly retractive* (*in the sense of Bierstedt, Meise* [**3**, *p.* 100]), *hence* $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ *is quasi-normable.*

Proof. According to Neus [**22**, p. 138, Satz 1] the notions "strongly boundedly retractive", "boundedly retractive", and "sequentially retractive" coincide for countable inductive limits of normed spaces. The fact that $\bigcup_{m\in\mathbf{N}_0}(\mathring{\mathfrak{B}}^m(\Omega), p_m)'$ is boundedly retractive yields that $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ is quasi-normable (Grothendieck [**15**, p. 106, Def. 4]). □

Finally we prove a nuclearity criterion for $(\mathring{\mathfrak{B}}(\Omega)'', \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$.

(3.8) PROPOSITION. *Let* $\Omega = \mathring{\Omega} \subset \mathbf{R}^n$.
(a) *If* $\Lambda = \mathring{\Lambda} \subset \Omega$ *is quasi-bounded, then*

$$\tau\big(\mathring{\mathfrak{B}}(\Omega)'', \mathring{\mathfrak{B}}(\Omega)'\big) \cap \mathring{\mathfrak{B}}(\Lambda)'' = \mathfrak{T}_0(\Lambda) = \beta\big(\mathring{\mathfrak{B}}(\Lambda)'', \mathring{\mathfrak{B}}(\Lambda)'\big).$$

(b) *The following statements are equivalent*:
(b.1) $(\mathring{\mathfrak{B}}(\Omega)'', \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$ *is nuclear.*
(b.2) $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ *is nuclear.*
(b.3) *There exists* $p \in (0, \infty)$ *such that* $r(\cdot) \in L^p(\Omega)$.

Proof. (a) Since Λ is quasi-bounded we obtain $\check{\mathfrak{T}}(\Lambda) = \beta(\mathring{\mathfrak{B}}(\Lambda)'', \mathring{\mathfrak{B}}(\Lambda)') = \mathfrak{S}(\Lambda)$ by (2.8d). Because of $C_0(\Lambda) \subset C_0(\Omega)$ and $\tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') = \mathfrak{S}$, Thm. (1.8b) yields the inclusions

$$\begin{aligned}\check{\mathfrak{T}}(\Lambda) = \mathfrak{S}(\Lambda) \subset \mathfrak{S}(\Omega) \cap \mathring{\mathfrak{B}}(\Lambda)'' &= \tau\big(\mathring{\mathfrak{B}}(\Omega)'', \mathring{\mathfrak{B}}(\Omega)'\big) \cap \mathring{\mathfrak{B}}(\Lambda)'' \\ &\subset \beta\big(\mathring{\mathfrak{B}}(\Omega)'', \mathring{\mathfrak{B}}(\Omega)'\big) \cap \mathring{\mathfrak{B}}(\Lambda)'' = \check{\mathfrak{T}}(\Omega) \cap \mathring{\mathfrak{B}}(\Lambda)'' = \check{\mathfrak{T}}(\Lambda).\end{aligned}$$

Thus

$$\big(\mathring{\mathfrak{B}}(\Lambda)'', \beta\big(\mathring{\mathfrak{B}}(\Lambda)'', \mathring{\mathfrak{B}}(\Lambda)'\big)\big) = \big(\mathring{\mathfrak{B}}(\Lambda), \mathfrak{T}_0\big)$$

is a topological subspace of $(\mathring{\mathfrak{B}}(\Omega)'', \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$.

(b) The equivalence (b.2) $\Leftrightarrow$ (b.3) was proved in [**12**, Thm. (4.12)]. If Ω is quasi-bounded, the equivalence (b.1) $\Leftrightarrow$ (b.2) is an immediate consequence of [**12**, Thm. (4.11)]. If Ω is not quasi-bounded, there exist $\varepsilon > 0$ and a sequence $(x_j;\ j \in \mathbf{N})$ in Ω such that the closed balls $K[x_j, \varepsilon]$ $(j \in \mathbf{N})$ are pairwise disjoint and contained in Ω. For every sequence $t = (t_j;\ j \in \mathbf{N})$ in $\mathbf{R}^+$ such that $t_j \in (0, \varepsilon)$ $(j \in \mathbf{N})$ we define $\Lambda = \Lambda(t) := \bigcup_{j\in\mathbf{N}} K(x_j, t_j)$ and $r_\Lambda(x) := \operatorname{dist}(x, \complement\Lambda)$ $(x \in \mathbf{R}^n)$. An easy calculation shows $r_\Lambda \in C_0(\Lambda) \Leftrightarrow (t_j;\ j \in \mathbf{N}) \in c_0(\mathbf{N})$ and $r_\Lambda \in L^p(\Lambda) \Leftrightarrow (t_j;\ j \in \mathbf{N}) \in l^{p+n}(\mathbf{N})$. We now choose a sequence $(t_j;\ j \in \mathbf{N}) \in c_0(\mathbf{N}) \setminus \bigcup_{0<p<\infty} l^p(\mathbf{N})$ and obtain a quasi-bounded subset $\Lambda = \mathring{\Lambda} \subset \Omega$ such that $r_\Lambda \notin L^p(\Lambda)$ for all $p \in (0, \infty)$. By [**12**, Thm. (4.12)] and (a) from above, none of the spaces $(\mathring{\mathfrak{B}}(\Omega)'', \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}'))$ and $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ is nuclear. □

(3.9) REMARK. Recently Valdivia [**29**] and Vogt [**30**] independently proved a representation of $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ as a sequence space: $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ is topologically isomorphic to a complemented subspace of $s \mathbin{\hat{\otimes}_\pi} c_0$, where s denotes the space of rapidly decreasing sequences. $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ is topologically isomorphic to all of $s \mathbin{\hat{\otimes}_\pi} c_0$ if and only if Ω is not quasi-bounded (i.e. $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ is not reflexive, cf. [**12**, Thm. (4.11)]). If $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0)$ is nuclear, then it is topologically isomorphic to s (cf. also [**12**, Rem. (4.13)]). With the help of Proposition 13b of Grothendieck [**17**, Ch. II, p. 76] we thus obtain another proof of Corollary (3.7). Moreover, since $(\mathring{\mathfrak{B}}(\Omega)', \beta(\mathring{\mathfrak{B}}', \mathring{\mathfrak{B}}))$ is topologically isomorphic to a complemented subspace of $s' \mathbin{\hat{\otimes}_\pi} l^1$, we may apply Proposition 24.3 of Grothendieck [**17**, Ch. I, p. 116] to obtain another proof of Corollary (3.5b). (By Theorem 6 of Grothendieck [**17**, Ch. II, p. 34] the space $s' \mathbin{\hat{\otimes}_\pi} l^1$ is topologically isomorphic to $L_e(s_\tau, l^1)$.)

According to Conway [**7**, p. 478, Cor. 2.5] the topology $\tau(l^\infty, l^1)$ is a strict topology. We therefore expect that in the above sequence space representation the strict topology $\mathfrak{S}$ on $\mathring{\mathfrak{B}}(\mathbf{R}^n)'' = \mathfrak{B}(\mathbf{R}^n)$ corresponds to $\tau(s, s') \mathbin{\hat{\otimes}_\pi} \tau(l^\infty, l^1)$.

4. $(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})$ is an Orlicz-Pettis-space.

(4.1) Let $(E, \mathfrak{R})$ be a locally convex Hausdorff space, $(x_\iota;\ \iota \in I)$ a family in E, and let $\mathfrak{F}(I)$ denote the set of all finite subsets of I; $\mathfrak{F}(I)$ is directed by inclusion.

(a) $(x_\iota;\ \iota \in I)$ satisfies the *Cauchy-condition* (is *summable*) if the net $(\Sigma_{\iota \in J} x_\iota;\ J \in \mathfrak{F}(I))$ is a Cauchy-net (is convergent).

(b) $(x_\iota;\ \iota \in I)$ is *subfamily-summable* (or SF-summable) if $(x_\iota;\ \iota \in L)$ is summable for all $L \subset I$.

(c) A linear map A from $(E, \mathfrak{R})$ into a locally convex space $(F, \mathfrak{S})$ is called Σ-*continuous* if

$$A\Big(\lim_J \sum_{k \in J} x_k\Big) = \lim_J \sum_{k \in J} A(x_k)$$

holds for all $\mathfrak{R}$-SF-summable sequences $(x_k;\ k \in \mathbf{N})$ ([**10**, p. 81, 4.]).

(4.2) The *Orlicz-Pettis-topology* $\mathrm{OP}(\mathfrak{R})$ *associated to* $\mathfrak{R}$ is defined to be the finest locally convex topology on E which has the same SF-summable sequences as $\mathfrak{R}$ (cf. [**10**, p. 75, 3.]). $(E, \mathfrak{R})$ is called an *Orlicz-Pettis-space* (or OP-space) if $\mathfrak{R} = \mathrm{OP}(\mathfrak{R})$. A linear map A: $(E, \mathfrak{R}) \to (F, \mathfrak{S})$ is Σ-continuous if and only if A: $(E, \mathrm{OP}(\mathfrak{R})) \to (F, \mathfrak{S})$ is continuous ([**10**, p. 81, Prop. (4.1)]). If R: $(E, \mathfrak{R}) \to (F, \mathfrak{S})$ is linear and continuous, then R: $(E, \mathrm{OP}(\mathfrak{R})) \to (F, \mathrm{OP}(\mathfrak{S}))$ is continuous ([**10**, p. 75, Thm. (3.1)]).

(4.3) The classical Orlicz-Pettis theorem (cf. [**9**, p. 74]) implies OP($\mathfrak{R}$) $\supset \tau(E, E')$. Therefore $(E, \mathfrak{R})$ is an Orlicz-Pettis-space if and only if the following conditions are satisfied:

(a) $\mathfrak{R} = \tau(E, E')$.

(b) Every Σ-continuous linear functional on $(E, \mathfrak{R})$ is continuous (i.e. $(E, \mathrm{OP}(\mathfrak{R}))' \subset E'$).

(4.4) Every ultrabornological space $(E, \mathfrak{R})$ is an OP-space ([**10**, p. 79, Prop. (3.17)]). The space $(m_0, \tau(m_0, l^1))$, where $m_0 := \{x\colon \mathbf{N} \to \mathbf{K};\ x(\mathbf{N})$ is finite$\}$, was a first example of an OP-space which is not ultrabornological (cf. [**2**]). We will show that $(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})$ is another example of that type, if Ω is not quasi-bounded (cf. (2.8d)).

The following generalization of the measurable-graph theorem of Schwartz [**27**, p. 160] was proved by H. Pfister.

(4.5) THEOREM (*Pfister* [**23**, p. 169, *Satz* 5.3 b) (γ)]). *Let $(E, \mathfrak{R})$ be an Orlicz-Pettis-space and $(F, \mathfrak{S})$ a Suslin locally convex space. If $A\colon (E, \mathfrak{R}) \to (F, \mathfrak{S})$ is a linear map whose graph is a Borel set in $(E \times F, \mathfrak{R} \times \mathfrak{S})$ (or is sequentially closed in $(E \times F, \mathfrak{R} \times \mathfrak{S})$), then $A\colon (E, \mathfrak{R}) \to (F, \mathfrak{S})$ is continuous.*

We note that the notation of Pfister [**23**] is different from that of [**10**].

The following theorem shows that Theorem (4.5) applies to $(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})$.

(4.6) THEOREM. $(\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})$ *is an Orlicz-Pettis-space.*

Proof. Because of $\mathfrak{S} = \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ and (4.3) it is sufficient to show that every OP($\mathfrak{S}$)-continuous linear functional $\Phi\colon \mathring{\mathfrak{B}}(\Omega)'' \to \mathbf{K}$ is $\mathfrak{S}$-continuous. Let $\Phi\colon (\mathring{\mathfrak{B}}(\Omega)'', \mathrm{OP}(\mathfrak{S})) \to \mathbf{K}$ be linear and continuous. The continuity of the inclusion $(\mathring{\mathfrak{B}}(\Omega), \mathfrak{T}_0) \hookrightarrow (\mathring{\mathfrak{B}}(\Omega)'', \mathfrak{S})$ and the functorial property of OP($\cdot$) (cf. (4.2)) show that $\Phi \mid \mathring{\mathfrak{B}}(\Omega)\colon (\mathring{\mathfrak{B}}(\Omega), \mathrm{OP}(\mathfrak{T}_0)) \to \mathbf{K}$ is continuous. Since $\mathfrak{T}_0$ is ultrabornological, we have $\mathrm{OP}(\mathfrak{T}_0) = \mathfrak{T}_0$ by (4.4). Thus there exists $T \in \mathring{\mathfrak{B}}(\Omega)'$ such that $\Phi \mid \mathring{\mathfrak{B}}(\Omega) = T$. According to (2.3b) every $\psi \in \mathring{\mathfrak{B}}(\Omega)''$ is the sum of the $\mathfrak{S}$-SF-summable sequence $(\theta_k\psi;\ k \in \mathbf{N})$, hence $\mathring{\mathfrak{B}}(\Omega)$ is dense in $(\mathring{\mathfrak{B}}(\Omega)'', \mathrm{OP}(\mathfrak{S}))$ and we have

$$\Phi(\psi) = \lim_J \sum_{k \in J} \Phi(\theta_k\psi) = \lim_J \sum_{k \in J} \langle T, \theta_k\psi \rangle$$

for every $\psi \in \mathring{\mathfrak{B}}(\Omega)''$. Now Theorem (1.9b) shows that Φ coincides with the unique $\sigma(\mathring{\mathfrak{B}}'', \mathfrak{B}')$-continuous extension of T to all of $\mathring{\mathfrak{B}}(\Omega)''$. Thus $\Phi \in \mathring{\mathfrak{B}}(\Omega)'$. □

(4.7) It follows from (4.3), that $\mathrm{OP}(\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')) = \mathrm{OP}(\mathfrak{S}) = \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ holds, and it was shown in Theorem (4.2) of [**10**, p. 82] that for an arbitrary locally convex space $(E, \mathfrak{R})$ the topology $\mathrm{OP}(\mathfrak{R})$ is the finest locally convex topology $\mathfrak{S}$ on E satisfying $\mathfrak{S} \cap K = \mathfrak{R} \cap K$ for all sets $K \subset E$ of the form $K = \{\Sigma_{k \in L} x_k;\ L \subset \mathbf{N}\}$, where $(x_k;\ k \in \mathbf{N})$ is an $\mathfrak{R}$-SF-summable sequence in E. Since for those sets K the space $(K, \mathfrak{R} \cap K)$ is compact and metrizable (cf. [**10**, p. 74]), the following corollary is an amelioration of the result $\mathfrak{T}_\sigma = \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ of (3.5a).

(4.8) Corollary. *$\tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$ is the finest locally convex topology $\mathfrak{R}$ on $\mathring{\mathfrak{B}}(\Omega)''$ which satisfies $\mathfrak{R} \cap K = \sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}') \cap K = \beta(\mathfrak{E}, \mathfrak{E}') \cap K$ for all (compact and metrizable) subsets K of the form $K = \{\Sigma_{k \in L} \psi_k;\ L \subset \mathbf{N}\}$, where $(\psi_k;\ k \in \mathbf{N})$ is an arbitrary $\sigma(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')$-SF-summable sequence in $\mathring{\mathfrak{B}}(\Omega)''$.*

5. Appendix: Another system of semi-norms for the strict topology on $\mathring{\mathfrak{B}}(\Omega)''$. In this appendix we will first establish the following result

(5.1) Proposition. *Let $f \in C_0(\Omega)$ and $m \in \mathbf{N}_0$. Then there exist $h(m) > 0$ and $g \in C_0(\Omega)$ such that*

$$\| f \cdot \rho_{-m} \cdot \psi \|_\infty \le h(m) \cdot \max\{\| g \partial^\alpha \psi \|_\infty;\ |\alpha| \le m + 2\}$$

holds for all $\psi \in \mathring{\mathfrak{B}}(\Omega)''$.

Proof. Since the inequality is obvious for $\Omega = \mathbf{R}^n$ or $m = 0$ we assume $\Omega \neq \mathbf{R}^n$ and $m \ge 1$. In the following estimates we tacitly use the fact that the functions f, $\partial^\alpha \psi$ ($\psi \in \mathring{\mathfrak{B}}(\Omega)''$, $\alpha \in \mathbf{N}_0^n$) may be continuously extended by 0 to all of $\mathbf{R}^n$.

Let $x \in \Omega$ satisfy $\rho(x) < 1$. We choose $z \in \partial\Omega$ such that $\rho(x) = r(x) = |x - z|$. Using $\partial^\beta \psi(z) = 0$ ($\beta \in \mathbf{N}_0^n$) we obtain from Taylor's formula

$$\psi(x) = m \sum_{|\alpha| = m} \frac{1}{\alpha!} (x - z)^\alpha \cdot \int_0^1 (1 - t)^{m-1} \partial^\alpha \psi(z + t(x - z))\, dt.$$

This implies the estimate

$$\begin{aligned}
&| f(x)\rho_{-m}(x)\psi(x) | \\
&\qquad \leq \left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \right) | f(x) | \cdot \sup\{ | \partial^\alpha \psi(z + t(x-z)) | ; \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad | \alpha | = m, t \in [0,1] \}.
\end{aligned}$$

For $\alpha \in \mathbf{N}_0^n$, $| \alpha | = m$, we put $\xi_\alpha(t) := \partial^\alpha \psi(z + t(x - z))$ $(t \in [0, 1])$. Now we use integration by parts and $\xi_\alpha^{(k)}(0) = 0$ $(k = 0, 1)$ to obtain the estimate

$$| \xi_\alpha(t) | \leq | t \cdot \xi'_\alpha(t) | + \sup\{ | s \cdot \xi''_\alpha(s) | ; s \in [0,1] \} \qquad (t \in [0,1]).$$

Observing $\rho(z + t(x - z)) = t | x - z |$ and $| x - z | < 1$ we obtain

$$\begin{aligned}
| t \cdot \xi_\alpha^{(k)}(t) | &\leq \sum_{|\beta|=k} \frac{k!}{\beta!} t \cdot | x - z |^{|\beta|} \cdot | (\partial^{\alpha+\beta}\psi)(z + t(x-z)) | \\
&\leq \sum_{|\beta|=k} \frac{k!}{\beta!} \rho(z + t(x-z)) \cdot | (\partial^{\alpha+\beta}\psi)(z + t(x-z)) | \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k = 1, 2, t \in [0,1]).
\end{aligned}$$

We define the function $f_1 \colon \Omega \to \mathbf{R}$ by

$$f_1(u) := \sup\{ | f(x) | ; x \in \Omega, | x - u | \leq 1 \} \qquad (u \in \Omega).$$

Because of $\rho(x) < 1$ we have $| f(x) | \leq f_1(z + t(x - z))$ $(t \in [0, 1])$. Putting

$$h(m) := 2 \cdot \left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \right) \cdot \max\left\{ \sum_{|\beta|=k} \frac{k!}{\beta!} ; k = 1, 2 \right\} + 1 \qquad (m \in \mathbf{N}_0)$$

we thus obtain

$$\begin{aligned}
&| f(x) \cdot \rho_{-m}(x)\psi(x) | \\
&\quad \leq 2 \cdot \left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \right) \cdot \max\left\{ \sum_{|\beta|=k} \frac{k!}{\beta!} ; k = 1, 2 \right\} \\
&\qquad \cdot \sup\{ f_1(z + t(x-z)) \cdot \rho(z + t(x-z)) \cdot | \partial^\gamma \psi(z + t(x-z)) | ; \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,1], | \gamma | \leq m + 2 \} \\
&\quad \leq h(m) \cdot \max\{ \| f_1 \rho \partial \ \ \psi \|_\infty ; | \gamma | \leq m + 2 \}.
\end{aligned}$$

With the help of the partition of unity from (1.3) we define

$$a_k := \sup\{f_1(x)\cdot\rho(x);\, x \in \operatorname{supp}(\theta_k)\} \qquad (k \in \mathbf{N}).$$

Since $(\operatorname{supp}(\theta_k);\, k \in \mathbf{N})$ is locally finite in Ω and $f \in C_0(\Omega)$ we obtain $(a_k;\, k \in \mathbf{N}) \in c_0$. Therefore the function

$$g(x) := \sum_{k\in\mathbf{N}} a_k \cdot \theta_k(x) \qquad (x \in \Omega)$$

belongs to $C_0(\Omega)$ and satisfies $g(x) \geq f_1(x)\cdot\rho(x)$ $(x \in \Omega)$. Thus we obtain

$$|f(x)\cdot\rho_{-m}(x)\cdot\psi(x)| \leq h(m)\cdot\max\{\|g\partial^\alpha\psi\|_\infty;\, |\alpha| \leq m+2\}$$

for all $x \in \Omega$ satisfying $\rho(x) < 1$. Because of $h(m) \geq 1$ $(m \in \mathbf{N}_0)$ and $g(x) \geq |f(x)|\cdot\rho(x)$ $(x \in \Omega)$ the above inequality holds for all $x \in \Omega$. $\square$

Since $\mathring{\mathfrak{B}}(\Omega)''$ is closed under differentiation, Proposition (5.1) shows that the strict topology $\mathfrak{S}$ on $\mathring{\mathfrak{B}}(\Omega)''$ is also generated by the semi-norms

$$\bar{q}_{\alpha,f}(\psi) := \|f\cdot\partial^\alpha\psi\|_\infty \qquad (\alpha \in \mathbf{N}_0^n, f \in C_0(\Omega)).$$

As we remarked in (2.2b), the semi-norms $\bar{q}_{\alpha,f}$ $(\alpha \in \mathbf{N}_0^n,\, f \in C_0(\Omega))$ are defined on all of $\mathfrak{B}(\Omega)$. We denote by $\overline{\mathfrak{S}}$ the locally convex topology on $\mathfrak{B}(\Omega)$ which is generated by the semi-norms $(\bar{q}_{\alpha,f};\, \alpha \in \mathbf{N}_0^n, f \in C_0(\Omega))$.

(5.2) PROPOSITION. *$(\mathfrak{B}(\Omega), \overline{\mathfrak{S}})$ is a Schwartz-space. In particular, $(\mathring{\mathfrak{B}}(\Omega)'', \tau(\mathring{\mathfrak{B}}'', \mathring{\mathfrak{B}}')) = (\mathring{\mathfrak{B}}(\Omega)'', \overline{\mathfrak{S}} \cap \mathring{\mathfrak{B}}'')$ is a Schwartz-space.*

Proof. For $m \in \mathbf{N}_0$ and $f \in C_0(\Omega)$, $0 < f(x) < 1$ $(x \in \Omega)$, we provide the space $\mathfrak{B}_f^m(\Omega) := \{\varphi \in \mathfrak{E}^m(\Omega);\, f\cdot\partial^\alpha\varphi \in C_0(\Omega)\ (|\alpha| \leq m)\}$ with the norm $\bar{q}_{m,f}(\varphi) := \max\{\|f\cdot\partial^\alpha\varphi\|_\infty;\, |\alpha| \leq m\}$ $(\varphi \in \mathfrak{B}_f^m(\Omega))$. Evidently, the space $(\mathfrak{B}(\Omega), \overline{\mathfrak{S}})$ is the projective limit of the Banach-spaces $(\mathfrak{B}_f^m(\Omega), \bar{q}_{m,f})$ $(m \in \mathbf{N}_0, f \in C_0(\Omega), 0 < f(x) < 1\ (x \in \Omega))$,

$$(\mathfrak{B}(\Omega), \overline{\mathfrak{S}}) = \bigcap \Big\{\big(\mathfrak{B}_f^m(\Omega), \bar{q}_{m,f}\big);\, m \in \mathbf{N}_0, f \in C_0(\Omega),$$
$$0 < f(x) < 1\ (x \in \Omega)\Big\}.$$

Given $m \in \mathbf{N}_0$, $f \in C_0(\Omega)$, $0 < f(x) < 1$ $(x \in \Omega)$, we define $k := m+1$, $g(x) := \sqrt{f(x)}$ $(x \in \Omega)$. We show that the inclusion

$$\big(\mathfrak{B}_g^k(\Omega), \bar{q}_{k,g}\big) \hookrightarrow \big(\mathfrak{B}_f^m(\Omega), \bar{q}_{m,f}\big)$$

is compact. Let H be a $\bar{q}_{k,g}$-bounded subset of $\mathfrak{B}_g^k(\Omega)$ and put $M_{k,g}(H) := \sup\{\bar{q}_{k,g}(\varphi);\ \varphi \in H\}$. For $x \in \Omega$ there is $\varepsilon > 0$ such that $K[x, \varepsilon] := \{y \in \mathbf{R}^n;\ |x - y| \le \varepsilon\} \subset \Omega$. We put

$$m(x, g, \varepsilon) := \inf\{g(y);\ y \in K[x, \varepsilon]\}.$$

Using the mean value theorem, we obtain the estimate

$$\begin{aligned} &|f(x)\cdot\partial^\alpha\varphi(x) - f(y)\cdot\partial^\alpha\varphi(y)| \\ &\quad \le |f(x) - f(y)| \cdot M_{k,g}(H)/g(x) + |x - y| \cdot n \cdot M_{k,g}(H)/m(x, g, \varepsilon) \\ &\qquad\qquad\qquad (\varphi \in H,\ y \in K[x, \varepsilon],\ |\alpha| \le m), \end{aligned}$$

which together with the continuity of f shows that $\{f\cdot\partial^\alpha\varphi;\ \varphi \in H,\ |\alpha| \le m\}$ is equicontinuous at $x \in \Omega$. The estimate

$$\begin{aligned} |f(x)\cdot\partial^\alpha\varphi(x)| &= g(x)\cdot|g(x)\cdot\partial^\alpha\varphi(x)| \\ &\le g(x)\cdot M_{k,g}(H) \qquad (x \in \Omega,\ \varphi \in H,\ |\alpha| \le m) \end{aligned}$$

shows that $\{f\cdot\partial^\alpha\varphi;\ \varphi \in H,\ |\alpha| \le m\}$ is also equicontinuous at the point ∞ of the Alexandroff-compactification of Ω. Since the $\bar{q}_{m,f}$-boundedness of H is evident, the theorem of Arzela-Ascoli (applied to the Alexandroff-compactification of Ω) implies that H is relatively compact in $(\mathfrak{B}_f^m(\Omega), \bar{q}_{m,f})$. An appeal to Proposition 9 of Horváth [**18**, p. 282] now finishes the proof. □

Added in proof. A considerable simplification of the proof of Proposition (5.1) was communicated to the authors by Prof. Dr. D. Vogt (Wuppertal). His proof even works in the case that $f \in C_0(\Omega)$ is replaced by $f \in C_0(\mathbf{R}^n)$ and $|\alpha| \le m + 2$ is replaced by $|\alpha| \le m + 1$. Besides the fact that an extension of f is now unnecessary, the first six lines of our proof remain unchanged. Now the new proof runs as follows:

$$\begin{aligned} \psi(x) = (m+1)\cdot \sum_{|\alpha| = m+1} \frac{1}{\alpha!}(x - z)^\alpha \\ \cdot \int_0^1 (1 - t)^m \cdot \partial^\alpha\psi(z + t(x - z))\, dt. \end{aligned}$$

Using $\rho(x) = |x - z| < 1$ and $\rho(z + t(x - z)) = t|x - z|$ $(t \in [0, 1])$, this implies the estimate

$$|f(x) \cdot \rho_{-m}(x)\psi(x)|$$

$$\le (m+1) \sum_{|\alpha|=m+1} \frac{1}{\alpha!} \int_0^1 |x - z|^{1/2} (1-t)^m |f(x)| \cdot |\partial^\alpha \psi(z + t(x - z))| \, dt$$

$$\le (m+1)\left(\sum_{|\alpha|=m+1} \frac{1}{\alpha!}\right) \cdot \int_0^1 t^{-1/2}(1-t)^m \, dt \cdot |f(x)|$$

$$\times \sup\{\rho^{1/2}(z + t(x - z)) \, |\partial^\alpha \psi(z + t(x - z))|; \; t \in [0, 1], |\alpha| = m + 1\}.$$

The function $f_1(x) := \sup\{|f(\xi)|; \; |x - \xi| \le 1\}$ $(x \in \mathbf{R}^n)$ belongs to $C_0(\mathbf{R}^n)$ and satisfies $f_1(y) \ge |f(x)|$ for all $y \in \mathbf{R}^n$ such that $|x - y| \le 1$. Putting

$$h(m) := \max\left\{1, (m+1)\left(\sum_{|\alpha|=m+1} \frac{1}{\alpha!}\right) \cdot \int_0^1 t^{-1/2}(1-t)^m \, dt\right\}$$

and

$$g(x) := \rho^{1/2}(x) \cdot f_1(x) \qquad (x \in \Omega)$$

we obtain $g \in C_0(\Omega)$ and the estimate

$$|f(x)\rho_{-m}(x)\psi(x)| \le h(m) \cdot \max\{\|g \cdot \partial^\alpha \psi\|_\infty; \; |\alpha| \le m + 1\}$$

for all $x \in \Omega$ satisfying $\rho(x) < 1$. Because of $h(m) \ge 1$ and $g(x) \ge |f(x)| \cdot \rho^{1/2}(x)$ $(x \in \Omega)$, the above inequality holds for all $x \in \Omega$.

This new version of Proposition (5.1) can also be used to simplify the proof of Theorem (3.2).

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