PACIFIC JOURNAL OF MATHEMATICS
Vol. 108, No. 1, 1983

THE BANACH SPACE JT IS PRIMARY

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It is proved that for every bounded linear operator U on the James' tree space JT there is a subspace $X \subset$ JT, isometric to JT, such that either U or $(I - U)$ acts isomorphically on X and either UX or $(I - U)X$ is complemented in JT. As a consequence, JT is primary.

1. In this paper we prove that the James' tree space, JT, is primary. A Banach space X is primary if whenever $X = Y \oplus Z$, either Y or Z is itself isomorphic to X. Many of the classical Banach spaces are known to be primary [**1**], [**2**], [**3**], [**4**], [**5**], [**9**], [**10**], [**13**].

The space JT was constructed by R. C. James [**8**] as an example of a separable space not containing l_1 yet having non-separable dual. It has also been studied by Lindenstrauss and Stegall [**11**]. Every subspace of JT contains l_2 [**8**], and JT has many subspaces isometric to the quasireflexive Banach space J [**6**], [**7**]. Here we take the norm on J to be

$$\left\| \sum_{n=1}^{\infty} a_n x_n \right\| = \sup_{p_1 < \cdots < p_n} \left\{ \sum_{i=1}^{n-1} \left(\sum_{p_i+1}^{p_{i+1}} a_i \right)^2 \right\}^{1/2}.$$

To show that JT is primary, we prove that for each bounded linear operator U on JT, there exists a subspace X such that U (or $I - U$) acts as an isomorphism on X, X is isometric to JT, and UX (or $(I - U)X$) is complemented in JT. The space X consists of functions supported on a certain subtree of the usual dyadic tree. The first part of the argument is a modification of an idea of Casazza and Lin [**4**]. That is, that if U is a bounded linear operator on a space Y with Schauder basis $\{y_n\}$, then either $\langle y_n^*, Uy_n \rangle \geq \frac{1}{2}$ for infinitely many indices or $\langle y_n^*, (I - U)y_n \rangle \geq \frac{1}{2}$ for infinitely many n. This idea was used also in [**2**].

In §2 we fix the terminology concerning trees and present some elementary propositions about JT and trees. In §3 these are used to construct the subspace X described above. Our notation is standard in Banach space theory, as may be found in [**12**]. If A is a subset of a Banach space, we denote the closed linear span of A by $[A]$. The greatest integer function is denoted by $[\cdot]$. Standard perturbation arguments concerning stability properties of Schauder bases (e.g., Proposition 1.a.9. of [**12**]) are used in several places.

2. In this section we present the definitions and some properties of JT as well as propositions guaranteeing the existence of certain subtrees. We begin with terminology concerning trees.

The standard tree is $\mathfrak{T} = \{(n, i)\colon 0 \le n < \infty, 0 \le i < 2^n\}$. The points (n, i) are called *nodes*. We say that $(n+1, 2i)$ and $(n+1, 2i+1)$ are the *successors* of (n, i). A *segment* is a finite set $S = \{t_1, t_2, \ldots, t_n\}$ of nodes such that for each j, t_{j+1} is a successor of t_j. $\mathfrak{T}$ is partially ordered by the relation $<$, with $t_1 < t_2$ if and only if $t_1 \ne t_2$ and there is a segment S with first element t_1 and last element t_2. If $t_1 < t_2$ we say t_2 is a *follower* of t_1, thus reserving the word "successor" as meaning "immediate follower." The set $\{(n, i)\colon 0 \le i < 2^n\}$ is called the nth *level* of $\mathfrak{T}$. We denote the level of a node t by $\operatorname{lev}(t)$. An *n-branch* is a totally ordered set $\{(m, l_m)\}_{m=n}^{\infty}$, and a branch is a set which is an n-branch for some n. A *tree* is a partially ordered set $\mathfrak{S}$ which is order isomorphic to $\mathfrak{T}$. If $\mathfrak{S}$ and $\mathfrak{S}'$ are trees with $\mathfrak{S}' \subset \mathfrak{S}$, we say $\mathfrak{S}'$ is a *subtree* of $\mathfrak{S}$. If $\mathfrak{S}$ is a tree and $\psi\colon \mathfrak{S} \to \mathfrak{T}$ is an order isomorphism, we may use ψ to carry the above terminology from $\mathfrak{T}$ to $\mathfrak{S}$. In particular, for $s \in \mathfrak{S}$, we define $\operatorname{lev}_{\mathfrak{S}}(s) = \operatorname{lev}(\psi(s))$.

We now define the James' tree space. For each $t \in \mathfrak{T}$, let

$$x_t(s) = \begin{cases} 1, & t = s, \\ 0, & t \ne s. \end{cases}$$

JT is the closed linear span of $\{x_t\}_{t \in \mathfrak{T}}$ with respect to the norm

$$\left\| \sum a_{n,i} x_{n,i} \right\| = \sup_{S_1, \ldots, S_k} \left\{ \sum_{j=1}^{k} \left(\sum_{(n,i) \in S_j} a_{n,i} \right)^2 \right\}^{1/2},$$

where the supremum is taken over all finite collections of mutually disjoint segments $S_1, \ldots, S_k$. The elements $\{x_{n,i}\}$, in the order $x_{0,0}$, $x_{1,0}$, $x_{1,1}$, $x_{2,0}$, $x_{2,1}$, $x_{2,2}$, $x_{2,3}, \ldots$ form a boundedly complete basis for JT. We denote the sequence of biorthogonal functionals by $\{x^*_{n,i}\}$, and shall use the linear functionals and projections defined in the following formulas. Each is easily seen to have norm one. In these definitions, S is a segment, B a branch, t a node, and N an integer.

$$\langle f_S, x \rangle = \sum_{t \in S} \langle x_t^*, x \rangle,$$

$$\langle f_B, x \rangle = \sum_{t \in B} \langle x_t^*, x \rangle,$$

$$P_S x = \sum_{t \in S} \langle x_t^*, x \rangle x_t,$$

$$P_B x = \sum_{t \in B} \langle x_t^*, x \rangle x_t,$$

$$P_N x = \sum_{\substack{t \\ \mathrm{lev}(t) \leq N}} \langle x_t^*, x \rangle x_t,$$

$$P_t x = \sum_{s \geq t} \langle x_s^*, x \rangle x_s.$$

The argument that shows JT to be primary is based on several propositions concerning trees and operators on JT.

Proposition 1. (a) *For any subtree* $\mathfrak{S}$ *of* $\mathfrak{T}$, $[\{x_t\colon t \in \mathfrak{S}\}]$ *is isometric to* JT *and complemented in* JT.

(b) *For any branch* $B \subset \mathfrak{T}$, P_BJT *is isometric to* J.

Proof. Part (b), and the fact that $[\{x_t\colon t \in \mathfrak{S}\}]$ is isometric to JT follow directly from the definition of the norm in JT. Let $\{S_t\}_{t \in \mathfrak{S}}$ be a tree-like collection of disjoint segments of $\mathfrak{T}$ such that $t \in \mathfrak{S} \Rightarrow t \in S_t$ and such that there are no gaps in $\bigcup_{t \in \mathfrak{S}} S_t$. By this we mean that if $t_1, t_2 \in \mathfrak{S}$, and if t_2 is a successor in $\mathfrak{S}$ of t_1, then whenever $t \in \mathfrak{T}$ satisfies $t_1 < t < t_2$, either $t \in S_{t_1}$, or $t \in S_{t_2}$. Then $[\{x_t\colon t \in \mathfrak{S}\}]$ is complemented by the norm one projection

$$Px = \sum_{t \in \mathfrak{S}} \langle f_{S_t}, x \rangle x_t.$$

Proposition 2. *Let* U: JT $\to$ JT *be a bounded linear operator*, $\varepsilon > 0$, N *an integer*, $\mathfrak{S}$ *a subtree of* $\mathfrak{T}$ *and* $t_0 \in \mathfrak{S}$. *Then there exists* $t_1 \in \mathfrak{S}$, $t_1 > t_0$, *such that*

$$\|P_N U x_{t_1}\| < \varepsilon.$$

Proof. If no such t_1 exists, then for any follower $t \in \mathfrak{S}$ of t_0, there exists t', $\mathrm{lev}(t') \leq N$ with

$$|\langle x_{t'}^*, P_N U x_t \rangle| \geq \varepsilon / K, \tag{2}$$

where $K = 2^{N+1} - 1$. Thus, for any L and any collection $\{t_l\}_{l=1}^{L}$ of followers in $\mathfrak{S}$ of t_0, $[L/K]$ of the t_l satisfy (2) for the same node t'. Hence there is a choice of signs $\{\theta_l = \pm 1\}$ such that

$$\left\| \sum_{l=1}^{L} P_N U(\theta_l x_{t_l}) \right\| \geq \left\langle x_{t'}^*, \sum_{l=1}^{L} P_N U(\theta_l x_{t_l}) \right\rangle \geq \frac{\varepsilon}{K} \left[\frac{L}{K} \right]. \tag{3}$$

However, we may choose $\{t_l\}_{l=1}^{L}$ to be mutually non-comparable with respect to the order on $\mathfrak{T}$, in which case it follows from (1) that

$$\left\| \sum_{l=1}^{L} P_N U(\theta_l x_{t_l}) \right\| \leq \|U\| \left\| \sum \theta_l x_{t_l} \right\| = \|U\| L^{1/2} \tag{4}$$

Since (3) and (4) are contradictory for large L, the proposition is proven.

PROPOSITION 3. *Let* U: JT $\to$ JT *be a bounded linear operator,* $\varepsilon > 0$, N *an integer,* $\mathfrak{S}$ *a subtree of* $\mathfrak{T}$ *and* $t_0, t_1, \ldots, t_k$ *mutually incomparable nodes of* $\mathfrak{S}$. *Then there exists* $t > t_0$, $t \in \mathfrak{S}$, $M \in N$, *and segments* $S_0, S_1, \ldots, S_k$ *of* $\mathfrak{T}$ *such that*

(a) $\|P_N U x_t\| < \varepsilon$,

(b) $\|(I - P_M) U x_t\| < \varepsilon$,

(c) *For each* i, $t_i \in S_i$, S_i *ends at level* $M + 1$ *of* $\mathfrak{T}$, *and there exists* $t_i' \in \mathfrak{S}$ *with* $t_i' > s$ *for all* $s \in S_i$,

(d) *For each* i, $\|P_{S_i} U x_t\| < \varepsilon$.

Proof. Let K satisfy $2^{-K/2}\|U\| < \varepsilon/2$, and choose $N_1 \geq \max(N, \mathrm{lev}(t_i))$ so that for each i, there are 2^K branches of $\mathfrak{S}$ which pass through t_i and through distinct nodes on the N_1th level of $\mathfrak{T}$. By Proposition 2, there exists $t > t_0$, $t \in \mathfrak{S}$ with $\|P_{N_1} U x_t\| < \varepsilon/2$. Thus (a) is satisfied. Select $M > N_1$ so that (b) holds, and for each i, let $S_i^1, S_i^2, \ldots, S_i^{2^K}$ be segments of $\mathfrak{T}$ containing t_i, passing through distinct nodes of the N_1th level of $\mathfrak{T}$ and satisfying (c).For each fixed i, we claim there exists j so that

$$\left\| P_{S_i^j}(I - P_{N_1}) U x_t \right\| < \varepsilon/2.$$

Indeed, if this is not the case, then

$$\frac{\varepsilon^2}{4} 2^K \leq \sum_{j=1}^{2^K} \left\| P_{S_i^j}(I - P_{N_1}) U x_t \right\|^2 \leq \left\| (I - P_{N_1}) U x_t \right\|^2 \leq \|U\|^2 \leq \frac{\varepsilon^2}{4} 2^K,$$

a contradiction. Denoting this S_i^j by S_i, we obtain

$$\begin{aligned} \left\| P_{S_i} U x_t \right\| &\leq \left\| P_{S_i}(I - P_{N_1}) U x_t \right\| + \left\| P_{S_i} P_{N_1} U x_t \right\| \\ &\leq \varepsilon/2 + \left\| P_{N_1} U x_t \right\| < \varepsilon. \end{aligned}$$

We omit the proofs of the next two propositions. Proposition 5 may be proved inductively, using Proposition 4 repeatedly.

PROPOSITION 4. *Let* $\mathfrak{S}$ *be a tree and* A *a subset of* $\mathfrak{S}$. *Then there exists a subtree* $\mathfrak{S}' \subset \mathfrak{S}$ *such that either* $\mathfrak{S}' \subset A$ *or* $\mathfrak{S}' \subset \tilde{A}$, *the complement of* A.

PROPOSITION 5. *Let f be a bounded real valued function defined on a tree $\mathbb{S}$. Then for any $\varepsilon > 0$, there exists a subtree S' such that for any branch B of $\mathbb{S}'$*

(a) $\lim_{t\to\infty; t\in B} f(t) = L_B$ *exists, and*

(b) $\Sigma_{t\in B} |f(t) - L_B| < \varepsilon$.

3. In this section we apply the results of §2 to prove

THEOREM 6. *Let U be a bounded linear operator on* JT. *Then there exists a subspace X of* JT *such that*

(a) *X is isometric to* JT,

(b) *$U|_X$ (or $(I - U)|_X$) is an isomorphism,*

(c) *UX (or $(I - U)X$) is complemented in* JT.

Proof. We will construct a subtree $\mathbb{S} \subset \mathfrak{T}$ such that either $\{Ux_t\}_{t\in\mathbb{S}}$ or $\{(I - U)x_t\}_{t\in\mathbb{S}}$ is equivalent to $\{x_t\}$ and has complemented span. The desired subspace is then $X = [\{x_t\}_{t\in\mathbb{S}}]$.

Let $V = I - U$ and $0 < \gamma < \frac{1}{2}$. For each $t \in \mathfrak{T}$, let B_t be a 0-branch containing t. Then

$$1 = \langle f_{B_t}, Ux_t \rangle + \langle f_{B_t}, Vx_t \rangle,$$

so either $\langle f_{B_t}, Ux_t\rangle \geq \frac{1}{2}$ or $\langle f_{B_t}, Vx_t\rangle \geq \frac{1}{2}$. By standard perturbation arguments we may assume Ux_t and Vx_t are finitely supported, say that $P_{N_t}Ux_t = Ux_t$ and $P_{N_t}Vx_t = Vx_t$. Denoting by S_t the segment $B_t \cap \{s\colon \mathrm{lev}(s) \leq N_t\}$, we may assume that for each t, either $\langle f_{S_t}, Ux_t\rangle > \gamma$ or $\langle f_{S_t}, Vx_t\rangle > \gamma$. Denote the last element of S_t by $l(t)$.

We construct a subtree $\mathbb{S}_1 \subset \mathfrak{T}$ inductively. Let $(0,0) \in \mathbb{S}_1$, and assume the nth level of $\mathbb{S}_1$ is already constructed. The $(n + 1)$st level of $\mathbb{S}_1$ consists of all nodes in $\mathfrak{T}$ which are successors of nodes $l(t)$ where t belongs to the nth level of $\mathbb{S}_1$.

Let $A = \{t \in \mathbb{S}_1\colon \langle f_{S_t}, Ux_t\rangle > \gamma\}$. By Proposition 4 there is a subtree $\mathbb{S}_2$ of $\mathbb{S}_1$ such that either $\mathbb{S}_2 \subset A$ or $\mathbb{S}_2 \subset \tilde{A}$. We shall assume $\mathbb{S}_2 \subset A$, and hence shall discuss the operator U, rather than $I - U$. For each $t \in \mathbb{S}_2$, let $\gamma_t = \langle f_{S_t}, Ux_t\rangle$. Then $\gamma \leq \gamma_t \leq \|U\|$, so by Proposition 5 we may assume that for each branch B of $\mathbb{S}_2$

$$\lim_{\substack{t\to\infty \\ B}} \gamma_t = \gamma_B \text{ exists,} \tag{5}$$

and

$$\sum_{t\in B} |\gamma_t - \gamma_B| < \gamma/2. \tag{6}$$

Condition (6) ensures that the multiplier operator T on J defined by $Tx_t = (\gamma_t/\gamma_B)x_t$ satisfies $\|I - T\| < \frac{1}{2}$. Hence T is invertible and $\|T^{-1}\| < 2$.

The desired subtree $\mathbb{S} = \{t(n, i)\} \subset \mathbb{S}_2$ is constructed inductively using Proposition 3. We will not reproduce the full details, but will indicate the first step. Parts (a), (b) and (d) of Proposition 3 are "gliding hump" conclusions, and allow us to compute norms. Part (c) guarantees that the inductive construction may be continued in $\mathbb{S}_2$.

Let $\varepsilon > 0$, and $\{\varepsilon_t > 0\}$ a sequence such that $\Sigma\, \varepsilon_i < \varepsilon$. Let $t(0, 0)$ be the initial node of $\mathbb{S}_2$, place $t(0, 0) \in \mathbb{S}$, and let N be an integer such that

$$(I - P_N)Ux_{t(0,0)} = 0.$$

By Proposition 3, there exists $t(1, 0) > l(t(0, 0))$, $t(1, 0) \in \mathbb{S}_2$, an integer M and a segment S_0, with $l(t(0, 0)) \in S_0$ satisfying

$$\|P_N Ux_{t(1,0)}\| < \varepsilon_1,$$

$$\|(I - P_M)Ux_{t(1,0)}\| < \varepsilon_2,$$

and

$$\|P_{S_0} Ux_{t(1,0)}\| < \varepsilon_3.$$

Let $t(1, 0) \in \mathbb{S}$. Now let t_0 be a node of $\mathbb{S}_2$ following S_0, and t_1 a node of $\mathbb{S}_2$ following $S_{t(1,0)}$. Again by Proposition 3, there exists $t(1, 1) \in \mathbb{S}_2$, $t(1, 1) > t_0$, an integer M' and a segment S_1 following t_1 so that

$$\|P_M Ux_{t(1,1)}\| < \varepsilon_4,$$

$$\|(I - P_{M'})Ux_{t(1,1)}\| < \varepsilon_5,$$

and

$$\|P_{S_1} Ux_{t(1,1)}\| < \varepsilon_6.$$

The first level of $\mathbb{S}$ is completed by placing $t(1, 1) \in \mathbb{S}$.

Proceeding in this fashion, after standard perturbation arguments, we may assume that for each $t \in \mathbb{S}$,

$$\text{(7)} \qquad \left\langle f_{S_{t(n+1,i)}} - f_{S_{t(n,[i/2])}}, Ux_t \right\rangle = \begin{cases} \gamma_{t(n+1,i)}, & t = t(n + 1, i), \\ 0 & \text{otherwise.} \end{cases}$$

With each $t \in \mathfrak{S}$, we associate a segment S'_t passing through t and the support of Ux_t. The $S'_{t(n,j)}$ are constructed in pairs as follows. Let t_1 be the last node of $\mathfrak{T}$ belonging to $S_{t(n,2i)} \cap S_{t(n,2i+1)}$, t_2 the last node of $\mathfrak{T}$ in $S_{t(n+1,4i)} \cap S_{t(n+1,4i+1)}$, and t_3 the last node of $\mathfrak{T}$ in $S_{t(n+1,4i+2)} \cap S_{t(n+1,4i+3)}$. Let $S'_{t(n,2i)}$ be the maximal segment with last element t_2 and not containing t_1, and let $S'_{t(n,2i+1)}$ be the maximal segment having last element t_3 and not containing t_1. Then there are no gaps (in $\mathfrak{T}$) between the S'_t, and by (7)

$$\left\langle f_{S'_{t(n,i)}}, Ux_t \right\rangle = \begin{cases} \gamma_{t(n,i)}, & t = t(n,i), \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

To show that $\{Ux_t\}_{t\in\mathfrak{S}}$ is equivalent to $\{x_t\}_{t\in S}$, let $\{a_{n,i}\}$ be a finite set of scalars. There exist segments $S_1, \dots, S_k$ such that

$$\left\| \sum a_{n,i} x_{t(n,i)} \right\| = \left\{ \sum_{j=1}^{k} \left(\sum_{S_j} a_{n,i} \right)^2 \right\}^{1/2},$$

and we may assume that each S_j is a union of segments S'_t. Furthermore, there exist disjoint branches $B_1, \dots, B_l$ such that each S_j is a subset of some B_i. Then

$$\begin{aligned} \left\| \sum a_{n,i} x_{t(n,i)} \right\| &= \left\{ \sum_{j=1}^{l} \left(\sum_{S_i \subset B_j} \left(\sum_{S_i} a_{n,i} \right)^2 \right) \right\}^{1/2} \\ &= \left\{ \sum_j \left\| P_{B_j} \left(\sum a_{n,i} x_{t(n,i)} \right) \right\|_j^2 \right\}^{1/2}, \end{aligned}$$

by Proposition 1,

$$\begin{aligned} &\le \frac{1}{\gamma} \left\{ \sum_j \left\| P_{B_j} \left(\sum \gamma_{B_j} a_{n,i} x_{t(n,i)} \right) \right\|_J^2 \right\}^{1/2}, \\ &\le \frac{2}{\gamma} \left\{ \sum_j \left\| P_{B_j} \left(\sum a_{n,i} \gamma_{t(n,i)} x_{t(n,i)} \right) \right\|_J^2 \right\}^{1/2}, \end{aligned}$$

by the remark following (6),

$$= \frac{2}{\gamma} \left\{ \sum_j \sum_{S''_i \subset B_j} \left\{ \sum_{S''_i} a_{n,i} \gamma_{t(n,i)} \right\}^2 \right\}^{1/2},$$

for some choice of disjoint segment S_i'' containing the S_t',

$$\leq \frac{2}{\gamma} \left\| \sum a_{n,i} U x_{t(n,i)} \right\|, \quad \text{by (7)}.$$

Thus for any scalar sequence $\{a_{n,i}\}$, we have

$$\begin{aligned} \left\| \sum a_{n,i} x_{t(n,i)} \right\| &\leq \frac{2}{\gamma} \left\| \sum a_{n,i} U x_{t(n,i)} \right\| \\ &\leq \frac{2\|U\|}{\gamma} \left\| \sum a_{n,i} x_{t(n,i)} \right\|, \end{aligned}$$

so that $\{x_{t(n,i)}\}$ and $\{Ux_{t(n,i)}\}$ are equivalent. Thus U acts as an isomorphism on $X = [\{x_t\}_{t \in \mathbb{S}}]$ and X is isometric to JT be Proposition 1.

To see that $[\{Ux_t\}_{t\in\mathbb{S}}]$ is complemented, let P be the projection onto $[\{x_t\}_{t\in\mathbb{S}}]$ defined in the proof of Proposition 1, using the segments S_t'. The argument that shows that $\{x_t\}_{t\in\mathbb{S}}$ and $\{Ux_t\}_{t\in\mathbb{S}}$ are equivalent also shows that $S = P \mid [\{Ux_t\}_{t\in\mathbb{S}}]$ is invertible. Then $[\{Ux_t\}_{t\in\mathbb{S}}]$ is complemented by $S^{-1}P$.

THEOREM 7. *The James tree space* JT *is primary.*

Proof. We use the Pelczynski decomposition method. Observe that with $B = \{(n,0): n = 0, 1, 2, \dots\}$, we have

$$\mathrm{JT} = P_B \mathrm{JT} \oplus \left(\sum_{n=1}^{\infty} \oplus P_{n,1} \mathrm{JT} \right)_{l_2} \approx J \oplus \left(\sum_{n=1}^{\infty} \oplus \mathrm{JT} \right)_{l_2}.$$

From this it follows that JT is isomorphic to its square, since

$$\begin{aligned} \mathrm{JT} &\approx J \oplus \left(\sum \oplus \mathrm{JT} \right)_{l_2} \approx J \oplus \mathrm{JT} \oplus \left(\sum \oplus \mathrm{JT} \right)_{l_2} \\ &\approx \mathrm{JT} \oplus J \oplus \left(\sum \oplus \mathrm{JT} \right)_{l_2} \approx \mathrm{JT} \oplus \mathrm{JT}. \end{aligned}$$

Now, if $\mathrm{JT} = Y \oplus Z$, by Theorem 6 we may assume $Y \approx W \oplus \mathrm{JT}$. Then

$$\begin{aligned} Y &\approx W \oplus \mathrm{JT} \approx W \oplus \mathrm{JT} \oplus \mathrm{JT} \approx Y \oplus \mathrm{JT} \\ &\approx Y \oplus \left(\sum \oplus \mathrm{JT} \right)_{l_2} \oplus J \\ &\approx Y \oplus \left(\sum \oplus Y \oplus Z \right)_{l_2} \oplus J \approx \mathrm{JT}. \end{aligned}$$

REFERENCES

[1] D. Alspach and Y. Benyamini, *Primariness of spaces of continuous functions on ordinals*, Israel J. Math., **27** (1977), 64–92.

[2] D. Alspach, P. Enflo, and E. Odell, *On the structure of separable $\mathcal{L}_p$ spaces* ($1 < p < \infty$), Studia Math., **60** (1977), 79–90.

[3] P. G. Casazza, *James' quasi-reflexive space is primary*, Israel J. Math., **26** (1977), 294–305.

[4] P. G. Casazza and B. L. Lin, *Projections on Banach spaces with symmetric bases*, Studia Math., **52** (1974), 189–193.

[5] P. G. Casazza, C. A. Kottman, and B. L. Lin, *On some classes of primary Banach spaces*, Canad. J. Math., **29** (1977), 856–873.

[6] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. Math., **52** (1951), 518–527.

[7] ———, *A non-reflexive Banach space isometric with its second conjugate space*, Proc. Nat. Acad. Sci. USA, **37** (1951), 174–177.

[8] ———, *A separable somewhat reflexive Banach space with non-separable dual*, Bull. Amer. Math. Soc., **80** (1974), 738–743.

[9] J. Lindenstrauss, *On complemented subspaces of m*, Israel J. Math., **5** (1967), 153–156.

[10] J. Lindenstrauss and A. Pelczynski, *Contributions to the theory of classical Banach spaces*, J. Functional Anal., **8** (1971), 225–249.

[11] J. Lindenstrauss and C. Stegall, *Examples of separable spaces which do not contain l_1 and whose duals are non-separable*, Studia Math., **54** (1975), 81–105.

[12] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, New York, 1977.

[13] A. Pelczynski, *Projections in certain Banach spaces*, Studia Math., **19** (1960), 209–228.

Received December 14, 1981 and in revised form April 9, 1982.

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