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A HARNACK ESTIMATE FOR REAL NORMAL SURFACE SINGULARITIES

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According to Harnack's theorem the number of topological components of the real part of a nonsingular projective curve X defined over R is at most $g(X)+1$, where $g(X)$ is the genus of X. The purpose of the present paper is to present two estimates which can be considered analogs of Harnack's theorem for normal surface singularities defined over R.

1. Introduction. A simple example will suffice to illustrate the type of result which one may expect. Suppose $A \subseteq \mathbf{P}^2(\mathbf{C})$ is a projective plane curve defined over $\mathbf{R}$ and let $A_{\mathbf{R}}$ be the real part of A. Let $V \subseteq \mathbf{C}^3$ be the cone over A and let $(V_{\mathbf{R}}, 0)$ be the germ at 0 of the real part of V. Then $(V_{\mathbf{R}}, 0)$ is connected, but the punctured variety $(V_{\mathbf{R}} \setminus \{0\}, 0)$ may have two components for each connected component of $A_{\mathbf{R}}$. Thus the number of components of $(V_{\mathbf{R}} \setminus \{0\}, 0)$ is bounded by $2 + 2g(A) = b_0(A) + b_1(A) + b_2(A)$ where $b_i(A)$ is the ith betti number of A. If one resolves the singularity $(V, 0)$, the exceptional curve E is just the curve A, so we conclude that the number of components of $(V_{\mathbf{R}} \setminus \{0\}, 0)$ is bounded by the sum of the betti numbers of the exceptional curve in a resolution of $(V, 0)$. It is in precisely this form that one may obtain a Harnack estimate for an arbitrary normal surface singularity defined over $\mathbf{R}$. Specifically, let (V, p) be a normal surface singularity defined over $\mathbf{R}$ and let $\pi\colon M \to V$ be the minimal normal resolution of V with exceptional curve $E = \pi^{-1}(p)$. Then the following three results will be proved.

1.1. THEOREM. *$\pi\colon M \to V$ is a real resolution, i.e. it is defined over* $\mathbf{R}$.

1.2. THEOREM. $b_0(V_{\mathbf{R}} \setminus \{0\}, 0) \le \sum_{i=0}^{2} b_i(E)$.

1.3. THEOREM. *By Theorem* 1.1, *E is defined over* $\mathbf{R}$ *and there is the estimate $b_0(E_R) \le 1 + p_g(E)$ where $p_g(E)$ is the geometric genus of E.*

After recalling some definitions and preliminary results in §2, Theorem 1.1 is proved in §3, while §4 contains the proofs of the two Harnack estimates.

2. Preliminaries. All complex spaces are assumed to be reduced, second countable and pure dimensional. By *surface* we will mean a complex space of dimension two. Let V be a normal surface and let π: $M \to V$ be a resolution of V, i.e. M is nonsingular, π is proper and π: $M \setminus \pi^{-1}(S(V)) \to V \setminus S(V)$ is biholomorphic, where $S(V)$ denotes the singular set of V. The *minimal resolution* of V is the unique resolution through which all other resolutions factor. This can be obtained from an arbitrary resolution by successively contracting exceptional curves of the first kind (Laufer [**6**] page 87). A *normal* resolution of V is a resolution in which the exceptional curve has nonsingular components which intersect transversely and no three components intersect. There is a unique minimal normal resolution obtained from the minimal resolution by means of quadratic transforms [**6**] page 91.

Let $A = \bigcup_{i=1}^{k} A_i$ be a curve with irreducible components A_i. Associated to A is a graph G, called the *dual graph* of A, formed as follows. The vertices of G are the irreducible components A_i of A, and each point of $A_i \cap A_j$ gives an edge joining the vertices A_i and A_j. If V is a normal surface and π: $M \to V$ is a resolution, then the dual graph of the resolution is the dual graph of the exceptional curve.

The exceptional curves of normal resolutions will be used frequently, so we give them a name. An *N-curve* is a projective curve in which the irreducible components are nonsingular, intersect transversely, and no three components intersect. The topology of an N-curve is completely determined by the topology of the irreducible components and the dual graph, as in the following result, which is easily proved by a Mayer-Vietoris argument (or see Brenton [**2**]). For homology we will always use $\mathbf{Z}_2$ coefficients. Thus $b_i(X) = \dim_{\mathbf{Z}_2} H_i(X, \mathbf{Z}_2)$.

1.2. PROPOSITION. *Let $A = \bigcup_{i=1}^{k} A_i$ be an N-curve with dual graph G. Then*

$$(2.1.1) \qquad b_1(A) = \sum_{i=1}^{k} b_1(A_i) + b_1(G),$$

$$(2.1.2) \qquad b_2(A) = k,$$

$$(2.1.3) \qquad p_g(A) = \dim_{\mathbf{C}} H^1(A, \mathcal{O}_A) = \sum_{i=1}^{k} g(A_i) + b_1(G),$$

where $g(A_i)$ denotes the genus of the nonsingular curve A_i.

The basic estimate we shall use in our proofs is the following "Smith theory" inequality.

2.2. THEOREM. *Let X be a finite cell complex and T: $X \to X$ a continuous involution with fixed point set F. Then*

$$\dim H_*(F) \leq \dim H_*(X).$$

The symbol $\dim H_*(\ )$ refers to the sum of the betti numbers. For the proof of this result see Wilson [9] page 72.

3. Real resolutions. A *complex space with conjugation* is a complex space X together with an antiholomorphic involution σ: $X \to X$. The fixed point set of σ is called the real part of X and will be denoted $X_{\mathbf{R}}$. If (X, σ) and (Y, τ) are complex spaces with conjugations, then a holomorphic map f: $X \to Y$ is said to be *real* if $\tau \circ f = f \circ \sigma$. Thus $f(X_{\mathbf{R}}) \subseteq Y_{\mathbf{R}}$.

3.1. THEOREM. *Let (V, σ) be a normal surface with conjugation and let π: $M \to V$ be the minimal resolution of V. Then M has a conjugation τ such that π is a real map.*

Proof. It will first be proved that there is some real resolution of V. According to a classical theorem of Zariski (see Lipman [7]) a resolution of each singular point of V can be obtained by means of a finite sequence of quadratic transformations at singular points, followed by normalizations. Each of these two operations will be considered separately.

3.2. LEMMA. *Let (W, σ) be a reduced complex space with conjugation and let θ: $W' \to W$ be the normalization. Then there is a conjugation τ on W' with respect to which θ is a real map.*

Proof. θ: $W' \setminus \theta^{-1}(S(W)) \to W \setminus S(W)$ is an analytic isomorphism so define τ on $W' \setminus \theta^{-1}(S(W))$ by $\tau = \theta^{-1} \circ \sigma \circ \theta$. If $p \in S(W)$ then $\theta^{-1}(p)$ is in one-to-one correspondence with the irreducible components of the germ (W, p). Since σ must give a bijection between the irreducible components of (W, p) and the irreducible components of $(W, \sigma(p))$, use this bijection to define τ: $\theta^{-1}(p) \to \theta^{-1}(\sigma(p))$.

Now consider conjugations under quadratic transforms. Thus let (W, σ) be a (normal) complex space with conjugation. Then $S(W)$ is invariant under σ. Let $p \in S(W)$. Then $\sigma(p) \in S(W)$ and there are two cases which will be considered separately.

3.3. *Case* I. $p \in W_{\mathbf{R}}$, i.e. $\sigma(p) = p$.

In this case a holomorphic imbedding $(W, p) \subseteq (\mathbf{C}^n, 0)$ may be chosen which is conjugation invariant. Recall that if $\Gamma \subseteq (\mathbf{C}^n, 0) \times \mathbf{P}^{n-1}(\mathbf{C})$ is defined by

$$\Gamma = \left\{((z_1,\ldots,z_n), [w_1,\ldots,w_n]): z_i w_j = z_j w_i \quad \text{for } 1 \leq i, j \leq n\right\}$$

then $\mathbf{C}^n \setminus \{0\} \subseteq \Gamma$ and the quadratic transform of (W, p) is the closure of $W \setminus p$ in Γ. Since Γ is defined by real equations and $W \setminus p$ is conjugation invariant, it follows that the strict transform of (W, p) is also conjugation invariant and this gives an extension of σ to the quadratic transform of W at p.

3.4. *Case* II. $\sigma(p) \neq p$.

In this case one may choose an imbedding of (W, p) in $(\mathbf{C}^n, (i, 0,\ldots,0))$ via holomorphic coordinates $\xi_1,\ldots,\xi_n$. Then $\overline{\xi_i \circ \sigma}$ $(1 \leq i \leq n)$ are holomorphic coordinates on $(\sigma W, \sigma(p))$ which give an imbedding of $(\sigma W, \sigma(p))$ into $(\mathbf{C}^n, (-i, 0,\ldots,0))$. Thus there is a commutative diagram

$$\begin{array}{ccc} (W, p) & \subseteq & (\mathbf{C}^n, (i,0,\ldots,0)) \\ \downarrow \sigma & & \downarrow \text{conjugation} \\ (\sigma W, \sigma(p)) & \subseteq & (\mathbf{C}^n, (-i,0,\ldots,0)) \end{array}$$

Now perform simultaneous quadratic transforms at $(i, 0,\ldots,0)$ and $(-i, 0,\ldots,0)$. It is then clear from the construction that the strict transform of (W, p) is taken via conjugation to the strict transform of $(\sigma W, \sigma(p))$. Hence σ extends to a conjugation on the space obtained by doing simultaneous quadratic transforms at p and $\sigma(p)$.

We now return to the proof of Theorem 3.1. By Zariski's theorem some resolution of V will be obtained if one alternately does quadratic transformations and normalizations. If, in addition, one is careful to simultaneously do quadratic transformations at both p and $\sigma(p)$, then (3.2)–(3.4) show that a real resolution π': $(M', \tau') \to (V, \sigma)$ is obtained. By Theorem 5.9 (page 87) of Laufer [**6**], the minimal resolution of V is obtained from M' by successively collapsing exceptional curves of the first kind in M'. But the condition for a curve to be exceptional of the first kind is purely topological (genus 0 and self intersection -1). Thus if A is exceptional of the first kind, then $\tau'(A)$ is also exceptional of the first kind, and if one simultaneously collapses A and $\tau'(A)$, (this is possible because $A \cap \tau'(A) = \varnothing$ by negative-definiteness of exceptional sets) a new surface is obtained which also has a conjugation map. Since one eventually arrives at the minimal resolution of V by this process, the proof of 3.1 is complete.

3.5. REMARKS. (1) Further applications of (3.3) and (3.4) show that the minimal normal resolution of V also supports a conjugation with respect to which the resolution map is real.

(2) If $(V, 0) \subseteq (\mathbf{C}^n, 0)$ is a conjugation invariant variety and $\pi\colon M \to V$ is a real resolution, then the exceptional curve $\pi^{-1}(0)$ may have no real points. A necessary and sufficient condition for $\pi^{-1}(0)_{\mathbf{R}}$ to be nonempty is that $(V_{\mathbf{R}}, 0) \not\subseteq (S(V), 0)$. See **[5]** for an algebraic version of this result. In the analytic case it is an easy consequence of the properness of π: $M \to V$. For example, the cone $z_1^2 + z_2^2 + z_3^2 = 0$ is resolved by a single quadratic transformation at 0 and the exceptional curve is the rational curve $w_1^2 + w_2^2 + w_3^2 = 0$ in $\mathbf{P}^2(\mathbf{C})$ which is conjugation invariant, but which has no real points.

4. Harnack estimates. If (X, p) is the germ of a topological space at p then $b_i(X, p)$ denotes the ith betti number of a sufficiently small representative of the germ (X, p) near p.

4.1. THEOREM. *Let (V, p) be a normal surface singularity with conjugation and let $\pi\colon M \to V$ be a real resolution of V with exceptional curve E. Then*

$$b_0(V_{\mathbf{R}} \setminus \{p\}, p) \leq \sum_{i=0}^{2} b_i(E).$$

Proof. If $V_{\mathbf{R}} = \{p\}$ the inequality is trivially satisfied since the left hand side is 0. Thus assume that $V_{\mathbf{R}} \neq \{p\}$. Let $X = M_{\mathbf{R}}$ and $A = E_{\mathbf{R}}$. Then $V_{\mathbf{R}} \setminus \{p\} \simeq X \setminus A$ so it suffices to compute $b_0(X \setminus A)$. First note that $H_0(X, X \setminus A) = 0$. This is because A is one dimensional and every connected component of X has dimension 2. Thus every connected component of X intersects $X \setminus A$. Also $H_1(X, X \setminus A) \simeq H^1(A)$ by Alexander duality (Spanier **[8]**, page 296). (All homology and cohomology is computed with $\mathbf{Z}_2$ coefficients.)

The exact homology sequence of the pair $(X, X \setminus A)$ contains the segment $H_1(X, X \setminus A) \to H_0(X \setminus A) \to H_0(X) \to 0$. Thus $b_0(X \setminus A) \leq b_0(X) + b_1(A)$. But it is easy to see that if one chooses a sufficiently small neighborhood of p, then the resulting X will satisfy $b_0(X) = b_0(A)$. (Simply triangulate M so that X, E, and A are all subcomplexes and then take a sufficiently fine barycentric subdivision.) Since $A = E_{\mathbf{R}}$ an application of Theorem 2.2 gives

$$b_0(X \setminus A) \leq b_0(A) + b_1(A) \leq \sum_{i=0}^{2} b_i(E).$$

4.2. REMARK. The example presented in the introduction shows that the estimate in Theorem 4.1 is probably the best that can be obtained. For a concrete example, the cone $V = \{z^2 = x^2 + y^2\} \subseteq \mathbf{C}^3$ will have $b_0(V_{\mathbf{R}} \setminus \{0\}) = 2$ while E is the projective line so the sum of the betti numbers of E will also be 2.

Let (V, p) be a real normal surface singularity and let $\pi\colon M \to V$ be a real resolution. Theorem 4.1 gives an estimate of the number of topological components of $V_{\mathbf{R}} \setminus \{p\}$. A second natural question is to ask for the number of topological components of the real part $E_{\mathbf{R}}$ of the exceptional curve E of (V, p). The next result gives such an estimate. It is essentially an extension of Harnack's theorem to curves which are not necessarily irreducible. The specific curves to be considered are the N-curves introduced in section 2.

4.3. THEOREM. *Let $A = \bigcup_{i=1}^{k} A_i$ be a connected N-curve with a conjugation σ. Then $b_0(A_{\mathbf{R}}) \leq 1 + p_g(A)$.*

4.4. REMARK. If G is the dual graph of A, recall from Proposition 2.1 that the geometric genus $p_g(A) = \sum_{i=1}^{k} g(A_i) + b_1(G)$.

Proof. (of 4.3) The conjugation σ determines an involution of the dual graph G of the curve A by sending the vertex of G corresponding to the irreducible component A_i of A to the vertex corresponding to the irreducible component $\sigma(A_i)$. Consider first the special case in which σ induces the identity on G, i.e. $\sigma(A_i) = A_i$ for $1 \leq i \leq k$. By the Smith theory inequality and Proposition 2.1,

$$\begin{aligned} (4.1) \qquad b_0(A_{\mathbf{R}}) + b_1(A_{\mathbf{R}}) &\leq 1 + b_1(A) + b_2(A) \\ &= 1 + 2\sum_{i=1}^{k} g(A_i) + b_1(G) + k \\ &= 2 + 2\sum_{i=1}^{k} g(A_i) + b_1(G) + (k-1). \end{aligned}$$

Claim. $b_1(A_{\mathbf{R}}) - b_0(A_{\mathbf{R}}) \geq (k-1) - b_1(G)$.

Substituting this inequality into formula (4.1) gives Theorem 4.3 in the special case in which σ induces the identity on G. The claim will be verified by induction on k, the number of irreducible components of the curve A. If $k = 1$ then $A_{\mathbf{R}}$ consists of a disjoint collection of circles so $b_1(A_{\mathbf{R}}) = b_0(A_{\mathbf{R}})$ and the claim is satisfied in this case. Now let A' be an N-curve with $k - 1$ irreducible components and suppose $A = A' \cup A_k$. Let G' be the dual graph of A' and consider separately two cases.

Case 1. $b_1(G) = b_1(G')$.

In this case $A_k \cap A'$ must consist of a single point p and $p \in A_{\mathbf{R}}$. Thus a circle of $(A_k)_{\mathbf{R}}$ and a circle of $A'_{\mathbf{R}}$ are connected at the point p, so that $b_0(A_{\mathbf{R}}) = b_0(A'_{\mathbf{R}}) + b_0((A_k)_{\mathbf{R}}) - 1$ and $b_1(A_{\mathbf{R}}) = b_1(A'_{\mathbf{R}}) + b_1((A_k)_{\mathbf{R}})$. Hence

$$b_1(A_{\mathbf{R}}) - b_0(A_{\mathbf{R}}) = b_1(A'_{\mathbf{R}}) - b_0(A'_{\mathbf{R}}) + 1 \geq (k - 1) - b_1(G).$$

Case 2. $b_1(G) > b_1(G')$.

It will always be true that $b_0(A_{\mathbf{R}}) \leq b_0((A_k)_{\mathbf{R}}) + b_0(A'_{\mathbf{R}})$ and $b_1(A_{\mathbf{R}}) \geq b_1((A_k)_{\mathbf{R}}) + b_1(A'_{\mathbf{R}})$. Thus

$$\begin{aligned} b_1(A_{\mathbf{R}}) - b_0(A_{\mathbf{R}}) &\geq b_1(A'_{\mathbf{R}}) - b_0(A'_{\mathbf{R}}) \geq (k - 2) - b_1(G') \\ &\geq (k - 1) - b_1(G) \end{aligned}$$

since $b_1(G) > b_1(G')$.

Thus the claim is verified and hence Theorem 4.3 is proved in the case in which every irreducible component of A is conjugation invariant.

Now consider a second special case. In this case A will consist of two irreducible components A_1 and A_2 which are interchanged by the conjugation map σ. Then the fixed point set $A_{\mathbf{R}}$ consists of finitely many points which are contained in $A_1 \cap A_2$. The dual graph G of A consists of 2 vertices joined by $e = \#(A_1 \cap A_2)$ edges. Thus $b_0(A_{\mathbf{R}}) \leq e = 1 + b_1(G)$.

We now proceed with the general case. Thus let A be a connected N-curve with conjugation σ and with dual graph G. The involution σ on A induces an involution T on G. Extending T to be a simplicial map on the topological space G, Theorem 2.2 may be applied to conclude that

$$b_0(F) + b_1(F) \leq 1 + b_1(G) \tag{4.2}$$

where F is the fixed point set of T. The fixed points of T are of two distinct types. Type I are the vertices of G fixed by T (i.e. the irreducible components of A which are invariant under the conjugation σ) together with the edges joining fixed vertices. The fixed points of type II are the centers of the edges joining two adjacent vertices which are interchanged by T.

Let $C_1, \ldots, C_r$ be the connected components of F of type I and let $D_1, \ldots, D_s$ be the pairs of adjacent vertices of G which are interchanged by T. Then corresponding to each C_j is a connected curve $A_{C_j} \subseteq A$ whose irreducible components are the vertices of C_j. Each irreducible component of A_{C_j} is conjugation invariant. Similarly, for each D_i there is a connected curve $A_{D_i} \subseteq A$ consisting of two irreducible components which are interchanged by σ. Then

$$A_{\mathbf{R}} = \bigcup_{j=1}^{r} \left(A_{C_j}\right)_{\mathbf{R}} \cup \bigcup_{i=1}^{s} \left(A_{D_i}\right)_{\mathbf{R}}.$$

By the two special cases done above,

$$\begin{aligned} b_0(A_{\mathbf{R}}) &\leq r + \sum_{j=1}^{r} p_g\left(A_{C_j}\right) + s + \sum_{i=1}^{s} b_1(D_i) \\ &\leq \sum_{j=1}^{k} g(A_j) + b_0(F) + b_1(F) \\ &\leq \sum_{j=1}^{k} g(A_j) + 1 + b_1(G) = 1 + p_g(A). \end{aligned}$$

The third inequality comes from formula (4.2) while the second inequality comes from Proposition 2.1 and the fact that each D_i contributes exactly $1 + b_1(D_i)$ isolated fixed points to F since that is exactly the number of edges joining the two vertices of D_i. Thus the proof of Theorem 4.3 is complete.

4.5. Remark. A special case is worth mentioning. Suppose that A is a connected N-curve with a conjugation and assume that $H_1(A, \mathbf{R}) = 0$. Then $p_g(A) = 0$ by Proposition 2.1 so in this case the theorem says that $A_{\mathbf{R}}$ is connected. This occurs for example if A is the exceptional set in the resolution of a rational singularity (Artin [**1**], Brieskorn [**4**]). Furthermore, from the explicit formulas in Brieskorn [**3**], which are formulas with real coefficients, one sees that all of the rational double points admit conjugations.

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