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ON SEMIGROUPS OF CONVOLUTION OPERATORS IN HILBERT SPACE

JORGE D. SAMUR

Given an infinitely divisible probability measure on a real separable Hilbert space H and the infinitesimal generator A of the associated semigroup of convolution operators acting on the Banach space of bounded uniformly continuous real functions on H, we describe the action of A on certain classes of differentiable functions.

1. Introduction. For every infinitely divisible probability measure μ on a real separable Banach space E there is an associated strongly continuous semigroup of convolution operators on the Banach space $C_u(E)$, the class of bounded uniformly continuous real-valued functions on E with the norm of uniform convergence. According to the general theory of semigroups of operators, the domain of the infinitesimal generator of every such semigroup is dense in $C_u(E)$. As is well known, one of the central aspects of the study of a specific semigroup of operators is the description of the action of its infinitesimal generator on a class of "smooth" functions which is large enough to characterize the semigroup. In the case when E is finite-dimensional, a result of this kind was obtained by Courrège [3], where the action of all generators of convolution semigroups on a natural class of differentiable functions is described. When E is an infinite-dimensional Banach space, however, the scarcity of differentiable functions (see [*] for a recent discussion) does not allow such a description.

This difficulty can be surmounted in the case when E is a Hilbert space; this is the object of the present paper. We consider the case in which E is a Hilbert space H and describe the action of the generators on certain classes of differentiable functions. We exhibit a natural class of differentiable functions — the class $C_u^{(2)}(H)$, defined below — on which all generators of convolution semigroups can be characterized (Theorem 3.1); our result generalizes the work of Courrège [3]. However, in contrast to the situation in the finite-dimensional case, $C_u^{(2)}(H)$ is not dense in $C_u(H)$ when H is infinite-dimensional.

It is possible to prove a stronger result for convolution semigroups without Gaussian component; in fact, in Theorem 3.6 we describe the action of the generator of any such semigroup on a dense class of differentiable functions. Finally, an extension for the Hilbert case of a result of L. Gross ([**4**]) is stated (Corollary 3.7).

As a by-product, we obtain the well-known Lévy-Khintchine representation in Hilbert space including limit formulas for the terms of the representation which strengthen the known results.

At several points we use ideas and techniques of [**2**].

H will denote a real separable Hilbert space with $\|\cdot\| = \langle \cdot , \cdot \rangle^{1/2}$; the norm of $C_u(H)$ (defined above) is also denoted $\|\cdot\|$. If $f\colon H \to \mathbf{R}$ is a continuous function which is Fréchet differentiable at $x \in H$ we denote by $f'(x)$ the point of H such that $\langle f'(x), \cdot \rangle$ is the derivative of f at x; analogously, $f''(x)$ will be the symmetric bounded operator on H such that $\langle f''(x)(\cdot), \cdot \rangle$ is the second Fréchet derivative of f at x, if it exists. $C_u^{(2)}(H)$ will be the subspace of $C_u(H)$ of those twice Fréchet differentiable functions f such that

$$\|f'\| = \sup_{x\in H} \|f'(x)\| < \infty, \qquad \|f''\| = \sup_{x\in H} \|f''(x)\| < \infty$$

and f'' is uniformly continuous (again, we write $\|\cdot\|$ for different norms).

All measures considered are defined on the Borel σ-algebra of H. We refer to [**6**] for the definition and properties of weak convergence, symbolized here by $\xrightarrow{w}$. N denotes the set of natural numbers greater than zero.

2. Semigroups of measures.

A probability measure μ on a separable Banach space E is *infinitely divisible* if for each $n \in N$ there exists a probability measure $\mu_{1/n}$ on E such that $(\mu_{1/n})^n = \mu$ (ν^n denotes the nth convolution power of a finite measure ν). The characteristic functional $\hat{\mu}$ of μ never vanishes on E' (the topological dual of E) and, consequently (see [**2**], §2), there exists a unique sequentially w^*-continuous function $l\colon E' \to \mathbf{C}$ such that $\hat{\mu} = \exp l$ and $l(0) = 0$. The nth root of μ is unique and $\hat{\mu}_{1/n} = \exp((1/n)l)$.

Recall that a family $\{\mu_t\colon t \geq 0\}$ of probability measures on E is a (weakly continuous) *semigroup of measures* if (1) $\mu_0 = \delta_0$, (2) $\mu_s * \mu_t = \mu_{s+t}$ for $s \geq 0$, $t \geq 0$, and (3) $\mu_t \xrightarrow{w} \mu_0$ as $t \to 0$.

The following fact is well-known; we sketch a proof for the sake of completeness.

PROPOSITION 2.1. *For every infinitely divisible probability measure μ on a separable Banach space E there exists a unique semigroup of measures $\{\mu_t\colon t \geq 0\}$ such that $\mu_1 = \mu$.*

Sketch of proof. We outline only the construction of $\{\mu_t\}$. The existence of l: $E' \to \mathbf{C}$ as above such that $\hat{\mu} = \exp l$ implies that for a positive rational $r = k/n$ $(k, n \in N)$ we may define $\mu_r = (\mu_{1/n})^k$ depending only on r; let $\mu_0 = \delta_0$. We have now a semigroup $\{\mu_r\}$ with non-negative rational parameter such that $\mu_1 = \mu$.

In order to define μ_t for real positive t, observe that if $\{r_n\}$ is a bounded sequence of positive rationals the semigroup property gives that $\{\mu_{r_n}\}$ is relatively shift compact (see Def. 2.1 and Th. 2.2 of Ch. III in **[6]**) and recall the following result: if $\{\nu_n\}$ is a sequence of probability measures on E such that (1) $\{\nu_n\}$ is relatively shift compact and (2) $\{\hat{\nu}_n\}$ converges uniformly on the balls of E' to a certain function g, then there exists a probability measure ν on E such that $\nu_n \xrightarrow{w} \nu$ and $\hat{\nu} = g$ (this is proved as Th. 4.5, Ch. VI in **[6]**). □

Given a semigroup of measures $\{\mu_t : t \geq 0\}$ on H, for each $t > 0$ we define the point $x_t \in H$, the bounded symmetric operator T_t on H and the finite positive measure ν_t on H by

$$x_t = \int \frac{x}{1 + \|x\|^2} \left(\frac{1}{t}\right) \mu_t(dx),$$

$$\langle T_t y, y\rangle = \int \frac{\langle x, y\rangle^2}{(1 + \|x\|^2)^2} \left(\frac{1}{t}\right) \mu_t(dx) \qquad (y \in H)$$

and

$$\nu_t(dx) = \frac{\|x\|^2}{1 + \|x\|^2} \left(\frac{1}{t}\right) \mu_t(dx).$$

The following two results may be proved along the lines of Theorem 4.1 of **[2]** (in **[2]** the converse Kolmogorov inequality for Banach spaces is used; here we can use a similar inequality valid for the Hilbert space case due to Varadhan — see **[6]**, Th. 3.3, Ch. VI).

LEMMA 2.2. *Let F be a closed subspace of H and $q(x) = d(x, F)$. If $t > 0$ and $\sigma_t = (1/2)(\mu_t + \bar{\mu}_t)$ then*

$$\int_{\{x:\, q(x) \leq r\}} q^2 \left(\frac{1}{t}\right) d\mu_t < \left(\frac{9}{2}\right) r^2 \left(1 - 12\sigma_t^{[1/t]+1}(\{x : q(x) > r\})\right)^{-1}$$

holds for each $r > 0$ such that $\sigma_t^{[1/t]+1}(\{x : q(x) > r\}) < 1/12$. ($\bar{\mu}_t$ is the measure defined by $\bar{\mu}_t(B) = \mu_t(-B)$ for Borel sets B; $[\cdot]$ denotes the integer part of a real number).

THEOREM 2.3. *$\{\nu_t: t > 0\}$ is relatively (weakly) compact.*

As in [6] (Ch. VI, Def. 2.3) we call *S-operator* a symmetric, positive, bounded operator S on H with finite trace, i.e. $\mathrm{tr}(S) = \sum_{j=1}^{\infty}\langle Se_j, e_j\rangle < \infty$ for some (every) orthonormal basis $\{e_j: j \in N\}$ of H. Recall that a class $\mathscr{A}$ of S-operators is *compact* ([6], Ch. VI, Def. 2.4) if: (I) $\sup_{S\in\mathscr{A}}\mathrm{tr}(S) < \infty$, (II) $\lim_{n\to\infty}\sup_{S\in\mathscr{A}}\sum_{j=n}^{\infty}\langle Se_j, e_j\rangle = 0$ for some orthonormal basis $\{e_j\}$ of H.

Let $(L_{(1)}(H), \|\cdot\|_1)$ be the ideal, in the algebra of all operators on H, of trace class operators endowed with the trace norm and denote by $\mathscr{S}$ the class of all S-operators on H. The following proposition will be useful. We remark in passing that it proves in particular that the notion of compact class does not depend on the choice of orthonormal basis.

PROPOSITION 2.4. *Let $\mathscr{A} \subset \mathscr{S}$. Then $\mathscr{A}$ is a compact class if and only if $\mathscr{A}$ is relatively compact in $(L_{(1)}(H), \|\cdot\|_1)$.*

Proof. For the sufficiency part, fix an (arbitrary) orthonormal basis $\{e_j: j \in N\}$ of H, define for each $n \in N$ the function $\psi_n: \mathscr{S} \to \mathbf{R}$ by $\psi_n(S) = \sum_{j=n}^{\infty}\langle Se_j, e_j\rangle$ and apply Dini's theorem.

Necessity. Let $\{e_j: j \in N\}$ be the orthonormal basis of H for which both conditions of the definition of compact class hold. Take a sequence $\{S_n\} \subset \mathscr{A}$. We write $\|\cdot\|$ ($\|\cdot\|_2$) for the (Hilbert-Schmidt) operator norm.

Observe that for $S \in \mathscr{S}$ its symmetric positive square root $S^{1/2}$ is of Hilbert-Schmidt type and $\|S^{1/2}\|_2^2 = \mathrm{tr}(S)$. The hypothesis implies that $\sup_n\|S_n^{1/2}\|_2 < \infty$; hence by a standard procedure we may find a sequence $\{n_k\}$ and a bounded operator T on H such that $\{S_{n_k}^{1/2}\}$ converges to T in the weak topology of operators.

Then T is symmetric, positive and of Hilbert-Schmidt type:

$$\sum_{j=1}^{\infty}\|Te_j\|^2 = \sum_{j,i}\langle Te_j, e_i\rangle^2 \leq \liminf_k \sum_{j,i}\left\langle S_{n_k}^{1/2}e_j, e_i\right\rangle^2$$

$$= \liminf_k \left\|S_{n_k}^{1/2}\right\|_2^2 \leq \sup_{S\in\mathscr{A}} \mathrm{tr}(S) < \infty.$$

Next we prove that $S_{n_k}^{1/2} \to T$ strongly as $k \to \infty$. For $x \in H$, $m \in N$ we have

$$\left\|\left(S_{n_k}^{1/2} - T\right)x\right\|^2 = \sum_{j=1}^{m}\left\langle\left(S_{n_k}^{1/2} - T\right)x, e_j\right\rangle^2 + \sum_{j=m+1}^{\infty}\left\langle x, \left(S_{n_k}^{1/2} - T\right)e_j\right\rangle^2;$$

since the last term is bounded by

$$2\|x\|^2 \sum_{j=m+1}^{\infty} \left(\langle S_{n_k} e_j, e_j \rangle + \|Te_j\|^2 \right),$$

the weak convergence of $\{S_{n_k}^{1/2}\}$ to T and the compact class property imply that $S_{n_k}^{1/2} x \to Tx$.

Having proved the strong convergence of $\{S_{n_k}\}$ to T, we can deduce from the inequality

$$\left\|S_{n_k}^{1/2} - T\right\|_2^2 \leq \sum_{j=1}^{m} \left\|\left(S_{n_k}^{1/2} - T\right) e_j\right\|^2 + 2 \sum_{j=m+1}^{\infty} \left(\langle S_{n_k} e_j, e_j \rangle + \|Te_j\|^2 \right)$$

that $\|S_{n_k}^{1/2} - T\|_2 \to 0$ as $k \to \infty$. From this we obtain that $T^2 \in L_{(1)}(H)$ and $\|S_{n_k} - T^2\|_1 \to 0$, because

$$\left\|S_{n_k} - T^2\right\|_1 \leq \left\|S_{n_k}^{1/2}\right\|_2 \cdot \left\|S_{n_k}^{1/2} - T\right\|_2 + \left\|S_{n_k}^{1/2} - T\right\|_2 \cdot \|T\|_2.$$

We have thus proved that $\{S_n\}$ contains a convergent subsequence in $(L_{(1)}(H), \|\cdot\|_1)$. □

REMARK. Let us recall that a probability measure γ on a Banach space E is *Gaussian* if it induces a normal (possibly degenerate) distribution on $\mathbf{R}$ via each $f \in E'$. Consider the set Γ of all Gaussian probability measures on H with the topology of weak convergence and the set $\mathscr{S}$ with the topology induced by $\|\cdot\|_1$. Then the map which associates to every pair (x, S) belonging to the topological product space $H \times \mathscr{S}$ the unique $\gamma \in \Gamma$ such that $\hat{\gamma}(y) = \exp(i\langle x, y\rangle - \frac{1}{2}\langle Sy, y\rangle)$ is a homeomorphism (this follows from well-known facts — see **[6]**, Ch. VI — together with Proposition 2.4).

After these preliminaries, we return to the family of operators $\{T_t\}$ associated to $\{\mu_t\}$.

THEOREM 2.5. *$\{T_t: t > 0\}$ is a compact class of S-operators.*

Proof. Let $\{e_j: j \in N\}$ be an orthonormal basis of H. We must prove:

(I) $$\sup_{t>0} \operatorname{tr}(T_t) < \infty,$$

(II) $$\lim_{n\to\infty} \sup_{t>0} \sum_{j=n+1}^{\infty} \langle T_t e_j, e_j \rangle = 0.$$

Claim (I) follows from Theorem 2.3 since

$$\operatorname{tr}(T_t) = \int \frac{1}{1 + \|x\|^2} \nu_t(dx) \leq \nu_t(H).$$

To prove (II) let F_n be the subspace spanned by $\{e_1,\ldots,e_n\}$ and $q_n(x) = d(x, F_n)$. Observe that

$$\lim_{n\to\infty} \sup_{t\in(0,1]} \sigma_t^{[1/t]+1}(\{x\colon q_n(x) > r\}) = 0, \tag{1}$$

σ_t being as in Lemma 2.2 and $r > 0$. Putting $\lambda_t = \sigma_t^{[1/t]+1}$, we obtain (1) from the relations

$$\lambda_t(\{x\colon q_n(x) > r\}) \leq \lambda_t(\{x\colon q_n(x) \geq r\} \cap K) + \lambda_t(K^c)$$

($t \in (0,1]$), the relative compactness of $\{\lambda_t\colon t \in (0,1]\}$ (see the proof of Theorem 4.1 in [**2**]) and Dini's theorem (note that $q_n \downarrow 0$ pointwise).

Next, Lemma 2.2 and (1) yield: for every $r > 0$ there exists $n_0 \in N$ such that

$$\sup_{t>0} \int_{\{x\colon q_n(x)\leq r\}} q_n^2\left(\frac{1}{t}\right) d\mu_t \leq 9r^2 \quad \text{for } n \geq n_0.$$

This fact, Theorem 2.3, Dini's theorem and the inequalities

$$\sum_{j=n+1}^{\infty} \langle T_t e_j, e_j\rangle \leq \int_{\{x\colon q_n(x)\leq r\}} q_n^2\left(\frac{1}{t}\right) d\mu_t + \int_{\{x\colon q_n(x)\geq r\}\cap K} q_n^2\left(\frac{1}{t}\right) d\mu_t + \nu_t(K^c)$$

($t > 0$) complete the proof through an easy argument. □

Next, we define some auxiliary functions; the proofs of their properties are omitted (use Taylor's formula).

LEMMA 2.6. (a) *Given $f \in C_u^{(2)}(H)$, define for every $x \in H$*

$$B_f(x,y) = \begin{cases} \left[f(x+y) - f(x) - \dfrac{\langle f'(x), y\rangle}{1+\|y\|^2} - \dfrac{1}{2}\dfrac{\langle f''(x)y, y\rangle}{(1+\|y\|^2)^2}\right]\dfrac{1+\|y\|^2}{\|y\|^2}, & y \neq 0, \\ 0, \quad y = 0. \end{cases}$$

Then the family $\{B_f(x,\cdot)\colon x \in H\}$ is uniformly bounded and equicontinuous at each $y \in H$.

(b) *For $y \in H$ let*

$$K(y,x) = \begin{cases} \left[\exp(i\langle x, y\rangle) - 1 - \dfrac{i\langle x, y\rangle}{1+\|x\|^2}\right]\dfrac{1+\|x\|^2}{\|x\|^2}, & x \neq 0, \\ 0, \quad x = 0. \end{cases}$$

Then $K(y,\cdot)$ is bounded and continuous on $H - \{0\}$.

(c) *For* $y \in H$ *let*

$$M(y, x) = K(y, x) + \frac{1}{2} \frac{\langle x, y\rangle^2}{\|x\|^2(1 + \|x\|^2)}$$

for $x \neq 0$, $M(y, 0) = 0$. *Then for every* $r > 0$ *the family* $\{M(y, \cdot): \|y\| \leq r\}$ *is uniformly bounded and equicontinuous at each* $x \in H$.

The proof of the next theorem requires the following uniqueness result (see **[6]**, Ch. IV, Th. 8.1):

LEMMA 2.7. *If* $x_0, x_0' \in H$, S, S' *are S-operators*, ν, ν' *are positive finite measures on* H *such that* $\nu(\{0\}) = \nu'(\{0\}) = 0$ *and*

$$\exp\left[i\langle x_0, y\rangle - \frac{1}{2}\langle Sy, y\rangle + \int K(y, \cdot)\, d\nu\right] = \exp\left[i\langle x_0', y\rangle - \frac{1}{2}\langle S'y, y\rangle + \int K(y, \cdot)\, d\nu'\right]$$

holds for all $y \in H$, *then* $x_0 = x_0'$, $S = S'$, $\nu = \nu'$ (K *is as in* (b) *of the preceding lemma*).

THEOREM 2.8. *There exist* $x_0 \in H$, *an S-operator* T_0 *and a positive finite measure* ν_0 *on* H *such that* $x_t \to x_0$, $T_t \to T_0$ *in* $\|\cdot\|_1$ *and* $\nu_t \xrightarrow{w} \nu_0$ *as* $t \to 0$.

Proof. Given $t > 0$, $y \in H$ and taking $M(y, \cdot)$ as in Lemma 2.6(c) we have

$$(1) \qquad \left(\frac{1}{t}\right)(\hat{\mu}_t(y) - 1) = i\langle x_t, y\rangle - \left(\frac{1}{2}\right)\langle T_t y, y\rangle + \int M(y, \cdot)\, d\nu_t.$$

Take a positive sequence $\{t_n\}$ such that $t_n \to 0$. Theorems 2.3, 2.5 and Proposition 2.4 imply that there exist a positive finite measure ν_0 on H and an S-operator T_0 such that

$$(2) \qquad T_{t_{n_k}} \to T_0 \quad \text{in } \|\cdot\|_1 \qquad \text{and} \qquad \nu_{t_{n_k}} \xrightarrow{w} \nu_0 \qquad (k \to \infty).$$

On the other hand, if $l\colon H \to \mathbf{C}$ is the sequentially w^*-continuous function such that $\hat{\mu}_1 = \exp l$ and $l(0) = 0$, the inequality

$$|(1/t)(\hat{\mu}_t(y) - 1) - l(y)| \leq t \cdot \exp|l(y)|$$

holds for $t \in (0, 1]$, which shows that the left member of (1) converges to l uniformly on the balls of H. Moreover $\{\int M(y, \cdot)\, d\nu_{t_{n_k}}\}$ converges in the same manner to $\int M(y, \cdot)\, d\nu_0$, by Lemma 2.6(c) and a well known result of R. Ranga Rao (see [6], Ch. II, Th. 6.8). The convergence of $\{\langle T_{t_{n_k}} y, y\rangle\}$ to $\langle T_0 y, y\rangle$ is also uniform on the balls of H.

Then (1) gives that $\{\langle x_{t_{n_k}}, \cdot\rangle\}$ converges uniformly on the balls of H and, consequently, there exists $x_0 \in H$ such that $x_{t_{n_k}} \to x_0$ in H. Hence

$$\hat{\mu}_1(y) = \exp\left[i\langle x_0, y\rangle - \left(\frac{1}{2}\right)\langle Sy, y\rangle + \int K(y, \cdot)\, d(\nu_0 - \nu_0(\{0\})\delta_0)\right]$$

for all $y \in H$ ($K(y, \cdot)$ is as in Lemma 2.6(b)) with $S = T_0 - U$, U being the symmetric operator defined by

$$\langle Uy, y\rangle = \int_{\{0\}^c} \frac{\langle x, y\rangle^2}{\|x\|^2(1 + \|x\|^2)}\, \nu_0(dx).$$

We prove now that S is an S-operator. Observing that U is positive, it suffices to show that S is positive; to this end, fix $y \in H$ and define u: $H \to \mathbf{R}$ by

$$u(x) = \begin{cases} \dfrac{\langle x, y\rangle^2}{\|x\|^2(1 + \|x\|^2)}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This function is lower semicontinuous and bounded; then

$$\begin{aligned} \langle Uy, y\rangle = \int u\, d\nu_0 &\le \liminf_k \int u\, d\nu_{t_{n_k}} \\ &= \liminf_k \left\langle T_{t_{n_k}} y, y\right\rangle = \langle T_0 y, y\rangle \end{aligned}$$

since $\nu_{t_{n_k}} \xrightarrow{w} \nu_0$. This gives the positiveness of S.

Finally, we obtain the theorem by a standard argument involving subsequences, the uniqueness Lemma 2.7 and the following remark: if ν_0 and S are obtained as above (from a sequence $\{t_n\}$) then $\nu_0(\{0\}) = \operatorname{tr}(S)$; in fact, putting $v(x) = (1 + \|x\|^2)^{-1}$, (2) implies

$$\begin{aligned} \nu_0(\{0\}) &= \int v\, d\nu_0 - \int_{\{0\}^c} v\, d\nu_0 = \lim_k \int v\, d\nu_{t_{n_k}} - \operatorname{tr}(U) \\ &= \lim_k \operatorname{tr}\left(T_{t_{n_k}}\right) - \operatorname{tr}(U) = \operatorname{tr}(T_0) - \operatorname{tr}(U) = \operatorname{tr}(S). \qquad \square \end{aligned}$$

COROLLARY 2.9. (a) (*Lévy-Khintchine representation*) *Let μ be a probability measure on H. Then μ is infinitely divisible if and only if there exist $x_0 \in H$, an S-operator S and a positive finite measure ν on H such that*

$\nu(\{0\}) = 0$ *which satisfy*

$$\hat{\mu}(y) = \exp\left[i\langle x_0, y\rangle - \frac{1}{2}\langle Sy, y\rangle + \int K(y, \cdot)\, d\nu\right]$$

for all $y \in H$.

The objects x_0, S *and* ν *are uniquely determined by* μ. (*We will say that* $[x_0, S, \nu]$ *is the Lévy-Khintchine representation of* μ.)

(b) *Given the semigroup* $\{\mu_t: t \geq 0\}$, *if* $[x_0, S, \nu]$ *is the Lévy-Khintchine representation of* μ_1, *we have*:

(i) $x_t \to x_0$;

(ii) $T_t \to S + U$ *in* $\|\cdot\|_1$, U *being the S-operator defined by*

$$\langle Uy, y\rangle = \int_{\{0\}^c} \frac{\langle x, y\rangle^2}{\|x\|^2(1 + \|x\|^2)}\nu(dx);$$

(ii′) *if the S-operators* $V_{t,\delta}$ *are defined by*

$$\langle V_{t,\delta}y, y\rangle = \int_{\|x\| \leq \delta} \langle x, y\rangle^2\left(\frac{1}{t}\right)\mu_t(dx),$$

then

$$\lim_{\delta\to 0}\ \limsup_{t\to 0}\|V_{t,\delta} - S\|_1 = 0;$$

(iii) $\nu_t \xrightarrow{w} \nu + \operatorname{tr}(S)\cdot\delta_0$ (δ_0 *is the unit point mass at* 0).

Proof of (ii′). Put $B_\delta = \{x \in H: \|x\| \leq \delta\}$. Let T_0, ν_0, U as in the proof of Theorem 2.8 and for $t > 0$, $\delta > 0$ consider the S-operators defined by:

$$\langle T_{t,\delta}y, y\rangle = \int_{B_\delta} \frac{\langle x, y\rangle^2}{(1 + \|x\|^2)^2}\left(\frac{1}{t}\right)\mu_t(dx), \qquad T_t^\delta = T_t - T_{t,\delta},$$

$$\langle U_\delta y, y\rangle = \int_{B_\delta - \{0\}} \frac{\langle x, y\rangle^2}{\|x\|^2(1 + \|x\|^2)^2}\nu_0(dx), \qquad U^\delta = U - U_\delta.$$

Fix $\delta > 0$ such that $\nu_0(\{x: \|x\| = \delta\}) = 0$. Since $S = T_0 - U$, for every $t > 0$ we have

$$V_{t,\delta} - S = (V_{t,\delta} - T_{t,\delta}) + (U - T_t^\delta) + (T_t - T_0).$$

We know that $\lim_{t\to 0}\|T_t - T_0\|_1 = 0$.

For each $t > 0$, since $V_{t,\delta} - T_{t,\delta}$ is positive, one has

$$\|V_{t,\delta} - T_{t,\delta}\|_1 = \operatorname{tr}(V_{t,\delta} - T_{t,\delta}) = \int_{B_\delta} \|x\|^2\left(\frac{2 + \|x\|^2}{1 + \|x\|^2}\right)\nu_t(dx);$$

hence the choice made of δ gives

$$\lim_{t\to 0}\|V_{t,\delta} - T_{t,\delta}\|_1 = \int_{B_\delta} \|x\|^2\left(\frac{2+\|x\|^2}{1+\|x\|^2}\right)\nu_0(dx).$$

For the remaining term, observe first that $\{T_t^\delta : t > 0\}$ is relatively $\|\cdot\|_1$-compact. Next, consider a $\|\cdot\|_1$-limit T_0^δ of a sequence $\{T_{t_n}^\delta\}$ with $t_n \to 0$; again by the choice of δ,

$$\langle T_0^\delta y, y\rangle = \lim_{n\to\infty}\int_{B_\delta^c} \frac{\langle x, y\rangle^2}{\|x\|^2(1+\|x\|^2)}\nu_{t_n}(dx) = \langle U^\delta y, y\rangle.$$

This shows that $T_t^\delta \to U^\delta$ in $\|\cdot\|_1$ as $t \to 0$. Then

$$\lim_{t\to 0}\|U - T_t^\delta\|_1 = \|U_\delta\|_1.$$

The preceding argument proves that if $\delta > 0$ satisfies

$$\nu_0(\{x : \|x\| = \delta\}) = 0$$

one has

$$\limsup_{t\to 0}\|V_{t,\delta} - S\|_1 \le \int_{B_\delta} \|x\|^2\left(\frac{2+\|x\|^2}{1+\|x\|^2}\right)\nu_0(dx) + \int_{B_\delta - \{0\}} \frac{1}{1+\|x\|^2}\nu_0(dx).$$

The finiteness of ν_0 implies the result. □

REMARK. Part (a) is due to Varadhan (see [**6**]). (b) may be compared with the statement of the Lévy-Khintchine representation in Banach spaces in [**1**] (Corollary 1.11); in particular, (ii′) strengthens the limit formula (2) in [**1**].

3. Semigroups of convolution operators. Given a probability measure μ on H, we define the *convolution operator* P_μ on $C_u(H)$ by $P_\mu f(x) = \int f(x+y)\,\mu(dy)$ ($f \in C_u(H)$, $x \in H$) and to every semigroup of measures $\{\mu_t : t \ge 0\}$ on H we associate the family $\{P_t : t \ge 0\}$ such that $P_t = P_{\mu_t}$ for each t. It is a *strongly continuous semigroup of operators*; in other words: (1) $P_0 = I$ (the identity operator), (2) $P_s P_t = P_{s+t}$ for $s \ge 0$, $t \ge 0$, and (3) $\lim_{t\to 0}\|P_t f - f\| = 0$ for each $f \in C_u(H)$. Let us remark that every strongly continuous semigroup of convolution operators on $C_u(H)$ may be obtained from a semigroup of measures as above.

Given such a semigroup $\{P_t : t \ge 0\}$ define $A_t = (1/t)(P_t - I)$ for $t > 0$ and the operator A by $Af = \lim_{t\to 0} A_t f$ on the linear subspace D of those $f \in C_u(H)$ for which the limit exists in $C_u(H)$. A is called the *infinitesimal generator* of $\{P_t\}$ and D its *domain*. Throughout this section $\{\mu_t\}$, $\{P_t\}$, D and A are related in this manner and $[x_0, S, \nu]$ is the Lévy-Khintchine representation of μ_1.

THEOREM 3.1. *$C_u^{(2)}(H) \subset D$ and for every $f \in C_u^{(2)}(H)$ and each $x \in H$*

$$Af(x) = \langle x_0, f'(x) \rangle + \frac{1}{2}\operatorname{tr}(Sf''(x))$$
$$+ \int \left[f(x+y) - f(x) - \frac{\langle f'(x), y\rangle}{1+\|y\|^2} \right] \frac{1+\|y\|^2}{\|y\|^2} \nu(dy).$$

Proof. Fix $f \in C_u^{(2)}(H)$; for $t > 0$ and $x \in H$ we have

$$(1) \qquad A_t f(x) = \langle x_t, f'(x)\rangle + \frac{1}{2}\int \frac{\langle f''(x)y, y\rangle}{(1+\|y\|^2)^2}\left(\frac{1}{t}\right)\mu_t(dy)$$
$$+ \int B_f(x, \cdot)\, d\nu_t$$

(B_f as in Lemma 2.6(a)).

First, we prove that for every symmetric operator Δ on H

$$(2) \qquad \int \frac{\langle \Delta y, y\rangle}{(1+\|y\|^2)^2}\left(\frac{1}{t}\right)\mu_t(dy) = \operatorname{tr}(T_t\Delta).$$

If Δ is an orthogonal projection P, taking orthonormal bases $\{e_i\colon i \in I\}$ and $\{e_j\colon j \in J\}$ of $P(H)$ and $P(H)^\perp$, respectively (with $I \cap J = \varnothing$), one has: $\{e_i\colon i \in I \cup J\}$ is an orthonormal basis of H and for every $y \in H$

$$\langle Py, e_i\rangle = \begin{cases} \langle y, e_i\rangle, & i \in I, \\ 0, & i \in J, \end{cases}$$
$$\langle Py, y\rangle = \|Py\|^2 = \sum_{i \in I\cup J} \langle y, Pe_i\rangle\langle y, e_i\rangle.$$

Hence

$$\int \frac{\langle Py, y\rangle}{(1+\|y\|^2)^2}\left(\frac{1}{t}\right)\mu_t(dy) = \sum_{i\in I\cup J}\langle T_t Pe_i, e_i\rangle = \operatorname{tr}(T_tP).$$

Given a symmetric operator Δ, the spectral theorem implies that there exists a sequence $\{\Delta_n\}$ of operators which are finite linear combinations of orthogonal projections such that $\|\Delta_n - \Delta\| \to 0$ ($n \to \infty$). Since (2) is true for each Δ_n, we may apply Lebesgue's dominated convergence theorem to the integrals and the inequality $|\mathrm{tr}(T_t\Delta_n) - \mathrm{tr}(T_t\Delta)| \leq \|T_t\|_1 \cdot \|\Delta_n - \Delta\|$ in order to conclude that (2) holds for Δ.

Consequently, we can rewrite (1) as

$$A_t f(x) = \langle x_t, f'(x)\rangle + \frac{1}{2}\,\mathrm{tr}(T_t f''(x)) + \int B_f(x,\cdot)\,d\nu_t.$$

Let $T_0 = S + U$ (U as in Corollary 2.9(b)) and $\nu_0 = \nu + \mathrm{tr}(S)\cdot\delta_0$. For every $t > 0$ and $x \in H$ we have

$$|\langle x_t, f'(x)\rangle - \langle x_0, f'(x)\rangle| \leq \|x_t - x_0\| \cdot \|f'\|,$$
$$|\mathrm{tr}(T_t f''(x)) - \mathrm{tr}(T_0 f''(x))| \leq \|T_t - T_0\|_1 \cdot \|f''\|;$$

moreover $\{B_f(x,\cdot)\colon x \in H\}$ is uniformly bounded and equicontinuous (Lemma 2.6 (a)). Then Corollary 2.9(b) and the result cited in the proof of Theorem 2.8 imply that

$$A_t f(x) \to \langle x_0, f'(x)\rangle + \frac{1}{2}\,\mathrm{tr}(T_0 f''(x)) + \int B_f(x,\cdot)\,d\nu_0$$

uniformly in $x \in H$ as $t \to 0$. This shows that $f \in D$.

But

$$\int B_f(x,\cdot)\,d\nu_0 = \int \left[f(x+y) - f(x) - \frac{\langle f'(x), y\rangle}{1+\|y\|^2}\right]\frac{1+\|y\|^2}{\|y\|^2}\nu(dy)$$
$$-\frac{1}{2}\int \frac{\langle f''(x)y, y\rangle}{\|y\|^2(1+\|y\|^2)^2}\nu(dy)$$

and the last integral equals $\mathrm{tr}(Uf''(x))$ (argue as in the proof of (2)). This gives the announced expression for Af. □

Let us mention now without proof two corollaries of Theorem 3.1 which are generalizations of results in **[3]** (Lemmas 8 and 6).

COROLLARY 3.2. *$\{P_t\colon t \geq 0\}$ is uniformly continuous on $C_u(H)$ (i.e. P_t tends to I in the operator norm as $t \to 0$) if and only if there exists a positive finite measure λ on H such that* $\mu_t = \exp[t(\lambda - \|\lambda\|\delta_0)]$. *(For a finite measure α,* $\exp(\alpha)$ *is defined by the usual expansion, converging in the total variation norm).*

COROLLARY 3.3. *The following conditions are equivalent*:

(1) *A is of local character on D (i.e. $Af(x) = 0$ when $f \in D$ vanishes in a neighborhood of $x \in H$).*

(2) μ_1 *is Gaussian.*

(3) $\lim_{t \to 0}(1/t)\mu_t(V^c) = 0$ *for every neighborhood* V *of* 0.

REMARK. Theorem 3.1 includes results of Courrège for the finite-dimensional case ([3], Théorèmes 1 and 2). In that situation $C_u^{(2)}(H)$ is dense in $C_u(H)$, but this is no longer true when H has infinite dimension. This fact was proved by D. Herrero (personal communication) who, using arguments of [7] (§5, Th. 1), showed that $C_u^{(2)}(l^2)$ is not dense in $C_u(l^2)$ (explicitly: if $A = \{x \in l^2 \colon x_\iota \leq 0, \|x\| \leq 1\}$ and $f(x) = \min\{1, d(x, A)\}$ then for all $g \in C_u^{(2)}(l^2)$ one has $\|f - g\| \geq 1/2$).[1]

In view of this negative result it seems of interest to show that for a special class of semigroups — namely, semigroups without Gaussian part — there exists a dense subspace of $C_u(H)$ which is contained in the domain of every generator. Again, the density result depends on [7].

Let us call $C_{u,L}^{(1)}(H)$ the space of those functions $f \in C_u(H)$ which are Fréchet differentiable with a derivative f' that satisfies: (i) $\|f'\| = \sup_{x \in H}\|f'(x)\| < \infty$, (ii) for some $\delta > 0$ and $M > 0$ it holds $\|f'(x) - f'(y)\| \leq M\|x - y\|$ when $\|x - y\| \leq \delta$.

PROPOSITION 3.4. $C_{u,L}^{(1)}(H)$ *is dense in* $C_u(H)$.

Proof. It is proved in [7], §4, Corollary 4 that given closed subsets A, B of H at positive distance, there exists a continuous function $f \colon H \to [0,1]$ with Lipschitz derivative such that $f|_A = 0$ and $f|_B = 1$. For such a pair of sets with $d(A, B) = 4\delta > 0$ we may prove, applying the result to $A' = \{x \colon d(x, A) \leq \delta\}$ and $B' = \{x \colon d(x, A) \geq 2\delta\}$, that in fact f can be chosen with $\|f'\| < \infty$ and then $f \in C_{u,L}^{(1)}(H)$.

The proof is completed by using the following modification of a well known fact whose proof we omit (see [5], Ch. 7, Problem P): let M be a metric space and L a linear subspace of $C_u(M)$. Suppose that for each pair of closed subsets A, B of M at positive distance and for each real interval $[a, b]$ there exists $f \in L$ such that $a \leq f \leq b, f|_A = a$ and $f|_B = b$. Then L is dense in $C_u(M)$. □

LEMMA 3.5. *Let* $f \in C_{u,L}^{(1)}(H)$ *and define*

$$L_f(x, y) = \begin{cases} \left[f(x+y) - f(x) - \dfrac{\langle f'(x), y \rangle}{1 + \|y\|^2} \right] \dfrac{1 + \|y\|^2}{\|y\|^2}, & y \neq 0, \\ 0, \quad y = 0. \end{cases}$$

Then $\{L_f(x, \cdot) \colon x \in H\}$ *is equicontinuous at every* $y \neq 0$ *and uniformly bounded.*

[1] We thank D. Herrero for many helpful conversations on this point and J. Eells, who suggested that [7] might be relevant in a letter to A. de Acosta.

THEOREM 3.6. *Suppose that* $S = 0$. *Then* $C_{u,L}^{(1)}(H) \subset D$ *and for every* $f \in C_{u,L}^{(1)}(H)$ *and each* $x \in H$

$$Af(x) = \langle x_0, f'(x) \rangle + \int \left[f(x+y) - f(x) - \frac{\langle f'(x), y \rangle}{1 + \|y\|^2} \right] \frac{1 + \|y\|^2}{\|y\|^2} \nu(dy).$$

Proof. Let $f \in C_{u,L}^{(1)}(H)$; for $t > 0$ and $x \in H$ one has

$$A_t f(x) = \langle x_t, f'(x) \rangle + \int L_f(x, \cdot)\, d\nu_t$$

(L_f as in Lemma 3.5).

By the preceding lemma and the fact that $\nu_t \xrightarrow{w} \nu$ and $\nu(\{0\}) = 0$ (Corollary 2.9(b)) we may deduce that $\int L_f(x, \cdot)\, d\nu_t \to \int L_f(x, \cdot)\, d\nu$ uniformly in x as $t \to 0$, which gives the theorem. The proof requires a slight generalization of a result previously used: let M be a complete separable metric space and $\{\mu_\alpha\}$ a relatively compact net of positive finite measures on (the Borel sets of) M, weakly convergent to μ. Then if $\mathscr{F}$ is a set of Borel functions from M to $\mathbf{R}$ which is uniformly bounded and equicontinuous at each $x \in \Lambda^c$, Λ being a closed set with $\mu(\Lambda) = 0$, one has

$$\lim_\alpha \sup_{f \in \mathscr{F}} \left| \int f\, d\mu_\alpha - \int f\, d\mu \right| = 0. \qquad \square$$

For our final statement, let us recall the construction of the Hilbert space of a centered Gaussian measure γ on a separable Banach space E (see [**1**], §5). Let $L^2(\gamma)$ be the Hilbert space of (classes of) square γ-integrable functions on E and $\widehat{E'}$ be the closure in $L^2(\gamma)$ of E' endowed with the $L^2(\gamma)$-norm, denoted $\|\cdot\|_{L^2} = \langle \cdot , \cdot \rangle_{L^2}^{1/2}$. Define H_γ as the linear subspace of E of those h such that $f \mapsto f(h)$ is L^2-continuous on E'. For $h \in H_\gamma$ the Riesz representation gives a unique $\phi(h) \in \widehat{E'}$ such that $f(h) = \langle \phi(h), f \rangle_{L^2}$ for all $f \in E'$; it is verified that $\phi\colon H_\gamma \to \widehat{E'}$ is an algebraic isomorphism and H_γ becomes a Hilbert space isomorphic to $\widehat{E'}$ by defining $\langle h, k \rangle_\gamma = \langle \phi(h), \phi(k) \rangle_{L^2}$ ($h, k, \in H_\gamma$). The norm $\|\cdot\|_\gamma$ is stronger than $\|\cdot\|$ (the norm of E) on H_γ and $\overline{H}_\gamma^{\|\cdot\|}$ coincides with the support of γ.

When $E = H$ is a Hilbert space with inner product $\langle \cdot , \cdot \rangle$, one has $\langle \langle \cdot , y \rangle, \langle \cdot , y' \rangle \rangle_{L^2} = \langle Sy, y' \rangle$ ($y, y', \in H$), where S is the covariance operator of γ; $h \in H_\gamma$ iff $y \mapsto \langle x, y \rangle$ is continuous on H with respect to the bilinear form given by S and $\phi\colon H_\gamma \to \widehat{H'}$ is characterized by $\langle h, y\rangle = \langle \phi(h), \langle \cdot, y\rangle\rangle_{L^2}$ for $h \in H_\gamma$, $y \in H$. Moreover $S(H) = \phi^{-1}(H')$ and $\langle Sx, Sx'\rangle_\gamma = \langle S^{1/2}x, S^{1/2}x'\rangle$ for $x, x', \in H$; by the density of $S(H)$ in H_γ it follows that

$$(*) \qquad \langle Sx, h\rangle_\gamma = \langle x, h\rangle \quad \text{for } x \in H, h \in H_\gamma.$$

Given a real function f on the Banach space E, we say (as in **[4]**) that f is *twice H_γ-differentiable at* $x \in E$ if the function $g(h) = f(x + h)$, $g\colon H_\gamma \to \mathbf{R}$ is twice $\|\cdot\|_\gamma$-Fréchet differentiable at 0; in this case, we denote by $D^2f(x)$ the operator on H_γ associated with the second derivative of g at 0. This notion is weaker than $\|\cdot\|$-Fréchet differentiability. If $T \in L_{(1)}(H_\gamma)$ we denote its trace by $\operatorname{tr}_\gamma(T)$.

Now we can extend in the Hilbert space case Corollary 3.2 of **[4]**.

COROLLARY 3.7. *Let γ be the centered Gaussian measure with covariance operator S. The class of functions f which satisfy the following conditions is dense in $C_u(H)$.*

(i) $f \in C_{u,L}^{(1)}(H)$,

(ii) *for each $x \in H$, f is twice H_γ-differentiable at x and* $D^2f(x) \in L_{(1)}(H_\gamma)$,

(iii) $D^2f\colon H \to L_{(1)}(H_\gamma)$ *is bounded and uniformly continuous,*

(iv) $f \in D$ *and*

$$Af(x) = \langle x_0, f'(x)\rangle + \frac{1}{2}\operatorname{tr}_\gamma(D^2f(x))$$
$$+ \int \left[f(x+y) - f(x) - \frac{\langle f'(x), y\rangle}{1 + \|y\|^2} \right] \frac{1 + \|y\|^2}{\|y\|^2} \nu(dy)$$

for every $x \in H$.

Proof. The proof of Corollary 3.2 in **[4]** and Theorem 3.6 of the present paper show that the set of functions $f(x) = \int_0^\infty e^{-t}(\Gamma_t g)(x)\,dt$, with $g \in C_{u,L}^{(1)}(H)$ and $\{\Gamma_t\}$ the semigroup of operators associated to γ, is dense in $C_u(H)$ and satisfies properties (i)–(iv) (the results of **[4]** needed are proved there only when H_γ is dense in H, but the proofs remain valid in general). □

REMARK. The class of functions defined in Corollary 3.7 contains $C_u^{(2)}(H)$ and the expression for Af coincides with that of Theorem 3.1. In fact, the following statement holds: if $f\colon H \to \mathbf{R}$ is a continuous function with second $\|\cdot\|$-Fréchet derivative $f''(x)$ at some point $x \in H$ then f is twice H_γ-differentiable at x, $D^2f(x) = Sf''(x)|_{H_\gamma} \in L_{(1)}(H_\gamma)$ and $\mathrm{tr}_\gamma(D^2f(x)) = \mathrm{tr}(Sf''(x))$.

To prove this, observe that we have the maps

$$H_\gamma \xrightarrow{\iota} H, \quad H \xrightarrow{\sigma} H' \xrightarrow{\kappa} (H_\gamma)' \xrightarrow{\tau^{-1}} H_\gamma,$$

σ and τ being the Riesz representations for H and H_γ, ι the inclusion map, which satisfies $\|\iota(x)\| \le c\|x\|_\gamma$ with $c^2 = \int \|y\|^2 \gamma(dy)$, κ the injection which to every $\psi \in H'$ associates its restriction to H_γ. Using $(*)$ we may check that $\tau^{-1}\kappa\sigma = S$.

If f is twice $\|\cdot\|$-Fréchet differentiable at $x \in H$ then it is twice H_γ-differentiable at x and in fact $D^2f(x) = \tau^{-1}\kappa\sigma f''(x)\iota = Sf''(x)\iota$. Take now an orthonormal basis $\{e_n : n \in N\}$ of H such that $Se_n = \lambda_n e_n$ with $\lambda_n \ge 0$ and put $u_n = \lambda_n^{(1/2)} e_n$ for $n \in N$ such that $\lambda_n > 0$; $(*)$ gives that $\{u_n : n \in N \text{ with } \lambda_n > 0\}$ is an orthonormal basis of H_γ. In order to conclude that $D^2f(x)$ is of trace class on H_γ note that $S: H \to H_\gamma$ is the (Hilbertian) adjoint of $\iota: H_\gamma \to H$, which is of Hilbert-Schmidt type (this is well known: $\|\iota(u_n)\|^2 = \lambda_n$).

To complete the proof, observe that by $(*)$ we have

$$\begin{aligned}\langle D^2f(x)u_n, u_n\rangle_\gamma &= \langle f''(x)\iota u_n, u_n\rangle = \lambda_n\langle f''(x)e_n, e_n\rangle \\ &= \langle Sf''(x)e_n, e_n\rangle\end{aligned}$$

for every $n \in N$ such that $\lambda_n > 0$.

Thus we have for each semigroup $\{P_t\}$ a dense subspace of $C_u(H)$, depending on the Gaussian part of μ_1, on which we can describe the infinitesimal generator of $\{P_t\}$. It is not known to us if in the infinite-dimensional case it is true that the intersection of the domains of the generators of all such semigroups is dense in $C_u(H)$ (the finite-dimensional case is solved by the cited result of Courrège, i.e. Theorem 3.1 for $H = \mathbf{R}^n$). In view of Theorem 3.6, it seems that the Gaussian case must provide the answer.

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Departamento de Matemática
Universidad Nacional de La Plata
Casilla de Correo 172
1900 La Plata
Argentina