

INFINITESIMAL RIGIDITY OF ALMOST-CONVEX ORIENTED POLYHEDRA OF ARBITRARY EULER CHARACTERISTIC

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This paper introduces a new method for proving the infinitesimal rigidity of a broad class of polyhedra, the caps-with-collars and their projective (Darboux) transforms, which include, as special cases, the traditional closed convex polyhedra of Cauchy and the refined closed convex and open convex polyhedra of Alexandrov with total curvature 2π . By definition, a cap-with-collars consists of an oriented generalized polyhedral cap with cylindrical polyhedral collars attached to the boundary. The spherical image of the cap (by unit normals coherent to the orientation) must lie in some hemisphere with the collars glued to the boundary of the cap so that their faces are parallel to the polar vector of the hemisphere. Moreover, caps are required to satisfy a local convexity condition, called edge-convexity, which is weaker than traditional convexity. An edge-convex polyhedron need not have a local supporting plane at each point. This allows great topological and morphological variety. A cap-with-collars can have arbitrary Euler characteristic. Among the examples given some are nonconvex; some are surfaces of genus greater than one; some are self-intersecting surfaces; some have branch points and some have pinch points.

1. Introduction. Consider the examples of polyhedral surfaces illustrated in Figure 1.

Figure 1A shows a nonconvex cap-with-collar with an edge-convex cap (shaded faces), and its closed Darboux transform. (The light lines are an Alexandrov refinement of the cap, dividing its plane faces.) Figure 1B shows the closed transform of a nonconvex cap-with-collar with a branchpoint at the central vertex of the cap and one at the conical vertex (i.e., the transform of the point at infinity on the collar). In Figure 1C the polyhedron is an immersion of a closed polyhedron of genus 2 (see Example 3, Section 5). Figure 1D is a pentagram bipyramid, and Figure 1E is a pinched sphere, where the curved lines indicate a convex polyhedral cap. The proofs of their infinitesimal rigidity follow from the basic theorem of the paper and its extensions (Theorem B', Section 5 and Theorem C, Section 6):

THEOREM B. *Let C be a cap with Alexandrov refinement C' , satisfying the hemisphere condition $\mathbf{c} \cdot \mathbf{n} > 0$ and having exactly one*

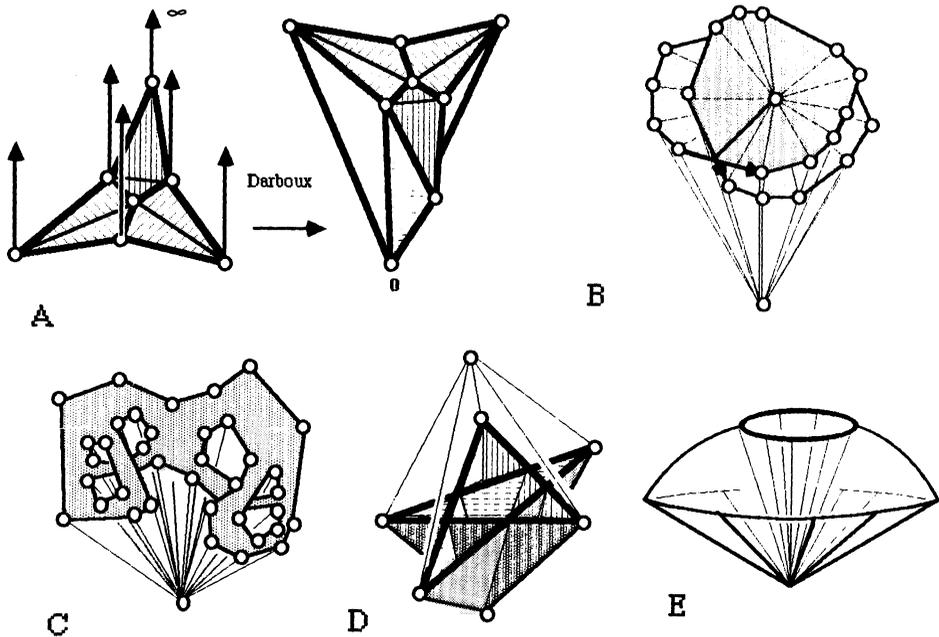


FIGURE 1

boundary component. Attach an infinite cylindrical polyhedral collar with Alexandrov refinement to the boundary component of C with faces parallel to c . The resulting surface is infinitesimally rigid.

Exact definitions of terms used above are to be found in Section 2, and more examples are given in Sections 5 and 6.

Outline of the method. Let \mathbf{n} denote the unit normal vectors on the faces of a polyhedron with boundary, coherent to the orientation, and \mathbf{c} the polar vector of a fixed hemisphere in which the spherical image lies so that $\mathbf{c} \cdot \mathbf{n} > 0$. An instantaneous velocity field \mathbf{z} , called a motion, is assigned to each point of each face of a polyhedron C such that \mathbf{z} is a rigid motion on each face. That is, at time zero \mathbf{z} has the form $\mathbf{z}(p) = \mathbf{a} + \mathbf{y} \times \mathbf{x}(p)$ at a point p of a face where \mathbf{a} and \mathbf{y} are vectors which are facewise constant and $\mathbf{x} = \mathbf{x}(p)$ is the position vector of p . If \mathbf{a} and \mathbf{y} are constant on the whole polyhedron then \mathbf{z} is called a rigid motion. Velocities on contiguous faces are required to agree on the shared edge. $\mathbf{y} \cdot \mathbf{n}$ is called the spin of \mathbf{z} (Section 3).

In order to study infinitesimal rigidity, we will assign to a point of self-intersection of a polyhedron one velocity vector for each polyhedral face which contains the point. Although this is impossible

for rigid bodies in physical space, mathematically it means that the faces pass through each other without obstruction. In Section 2 we define a polyhedron as an immersion in space of a manifold called an abstract polyhedral network. The velocity field on a polyhedron is then technically a function defined on the abstract polyhedral network although we will speak of it with some abuse as a function on the polyhedron itself (see Section 3).

The method may be divided into three parts, corresponding to Sections 3, 4 and 5:

In Section 3 we show that any motion on the cap satisfying the boundary condition $\mathbf{z} \cdot \mathbf{c} = \text{constant}$ on each boundary component separately (Figure 2A), is equivalent, modulo an infinitesimal rotation with axis \mathbf{c} , to a motion whose spin vanishes identically. This is the Spin Lemma 2.

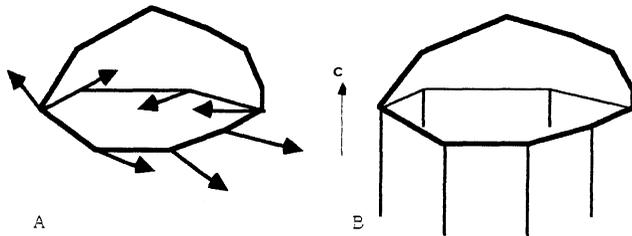


FIGURE 2

In Section 4 we show that any motion with zero spin on a cap is a rigid motion. Theorem A concludes that the original motion, with the boundary condition, is a rigid motion since the boundary condition is preserved under rotations with axis \mathbf{c} .

In Section 5 we show that if an infinite cylindrical collar is glued to a boundary component of a cap (Figure 2B) then any motion \mathbf{z} on the resulting polyhedron is equivalent, modulo an infinitesimal rotation, to a motion \mathbf{z}' satisfying $\mathbf{c} \cdot \mathbf{z}' = \text{constant}$ on the collar and, in particular, on the common boundary of the cap and the cylinder (the Collar Lemma 6). From this it follows that a cap with exactly one collar is infinitesimally rigid (Theorem B).

The projective Darboux transformation (Figure 1A and Section 5) then takes the cap-with-collar to an infinitesimally rigid boundaryless polyhedron which consists of an edge-convex polyhedron with a finite cone attached (Theorem B'). We use this to prove the classical infinitesimal rigidity of closed convex polyhedra with the refinements of Alexandrov (Corollary 10).

In the cases where the cap has more than one boundary component, the transformation takes the cap with collars attached to the boundary components to an edge-convex polyhedron with cones attached whose vertices are coincident (Figure 3). Section 6 shows that if these vertices are constrained to slide independently along a certain straight line or are fastened together then the cap with cones is infinitesimally rigid (Theorem C).

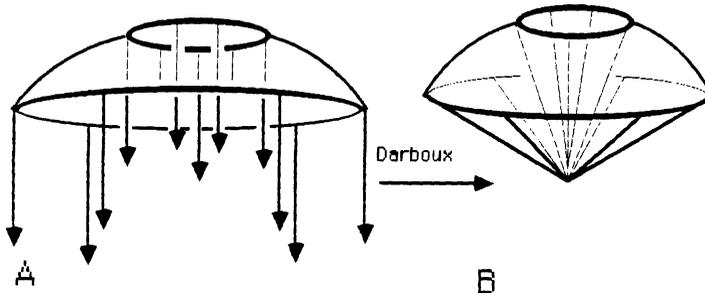


FIGURE 3

As an aside, we note that the projective invariance of infinitesimal rigidity for frameworks and their associated polyhedra was known to workers in statics at least as early as 1863 (Rankine, [Ra]) (see also an historical summary, with additional references in [CW₁]). Differential geometers usually credit this projective invariance to Darboux [Da]. The explicit equations D_1 , D_2 of Section 5, acting simultaneously on the surface and on the velocity vector, appear to have originated with Darboux [B, p. 126], [E, p. 58], [MR, p. 242]. We use his name as a convenient reference to this special projective transformation.

All of the theorems proved in Sections 3, 4, 5 and 6 hold under the generalized bendings of Alexandrov, in which faces of an edge-convex polyhedron are allowed to crease along diagonals joining vertices, including new vertices which are inserted in edges, with the restriction that no two diagonals meet in the interior of a face.

Section 7 contains remarks about: (1) results which allow new vertices to appear in the interior of faces under certain conditions without destroying infinitesimal rigidity; (2) the infinitesimal rigidity of polar polyhedra; (3) the infinitesimal rigidity of the 1-skeletons (i.e., frameworks) of polyhedra; and (4) second order rigidity and rigidity of some related polyhedra.

In view of the extensive history of the subject of infinitesimal rigidity and the many well-known mathematicians who have worked in this

field perhaps a few words should be said to indicate why the method seems new. (Of course the abundance of new examples is good evidence of this.) The method of Cauchy and methods adapted from it such as Alexandrov's [A, pp. 76, 366], Asimov and Roth's [AR, p. 183] and Whiteley's [W₂] assign +, - or 0 to edges and count the sign changes around a convex vertex whereas our method is concerned with the sign of the spin function on faces.

Pogorelov's very general methods in [P] apply to arbitrary convex surfaces with boundary but the surfaces must have a one-to-one projection onto a plane (or be Darboux transforms of such surfaces). His methods use the fact that the component of the velocity field perpendicular to the plane is a function over the plane whose graph has negative curvature and which therefore takes its extremes on the boundary. Pogorelov also proves the infinitesimal rigidity of classical convex polyhedral caps, based on a lemma of similar nature, under the same restrictions to one-to-one (except possibly on the boundary) projections of the polyhedral surface to a plane [B, p. 121]. (His lemma can be readily reproved using our present techniques.) Dehn [D] and Weyl [E, p. 198] have also given apparently unrelated proofs.

Recently Fogelsanger [F] proved that a "generic" realization in 3-space of any closed, oriented triangulated 2-surface is infinitesimally rigid. However his methods provide no specific infinitesimally rigid geometric realizations. In comparison, our methods provide an open set of infinitesimally rigid realizations for a wide class of surfaces.

Brief background. These investigations grew out of an, as yet unsuccessful, attempt to solve a finite rigidity (or monotopy) problem for a certain class of tubes of negative curvature with fixed boundary, which began when L. Nirenberg gave the author a copy of his paper [N] when the author was a Visiting Member of the Courant Institute at New York University. What resulted instead was a method for proving the infinitesimal and finite rigidity of a general class of smooth caps (not necessarily embedded or simply connected) satisfying the so-called "glidebending" condition [K₁], [K₂], [K₃]. (*Glidebending* means that the displacement on the boundary, whether finite or infinitesimal, is orthogonal to c .) This in turn led to a solution of the infinitesimal glidebending problem for polyhedral caps with the same features (see the remarks following the proof of the Spin Lemma, Section 3).

R. Connelly and W. Whiteley, in discussions with the author at the Special Semester in Structural Rigidity at the Université de Montréal

in February of 1987, discovered that the method applies to a still larger class of topologically interesting polyhedra. Whiteley emphasized that statements about rigidity under glidebending boundary conditions convert to rigidity theorems about closed compact polyhedra via the Darboux transformation if cylindrical collars (with not necessarily convex cross-section) are attached to the boundary. In particular, Connelly created examples which are immersions of 2-manifolds of genus n (see Example 3, Section 5) and Whiteley suggested many others, including the examples with branch points and pinch points, as well as some applications described in Section 7. I am grateful to both of them for these contributions to this paper.

In addition Whiteley has assisted with the terminology, reorganization and rewriting of the paper. Whatever readability it may have is due to him. I would also like to thank the editor H. Samelson for many helpful suggestions.

Future research should attempt to prove the finite rigidity (monotypy in a certain class) of these topologically interesting polyhedra and the finite and infinitesimal rigidity of analogous surfaces formed by gluing pieces of convex surfaces together.

2. Basic concepts and definitions. A *polyhedron* is a mapping of an abstract polyhedral network to E^3 whose restriction to each face is one-to-one and preserves Euclidean distance (a more precise definition will be given below). In order to define such a network we need some preliminary definitions.

A *finite polygonal region*, or simply *polygonal region*, is a compact set in the Euclidean plane whose boundary is a simple closed polygonal curve (i.e., the boundary is a polyhedral 1-sphere [\mathbf{M}_2 , p. 16]). The edges and vertices of the boundary polygon are called the *edges and vertices of the polygonal region* (Figure 4).

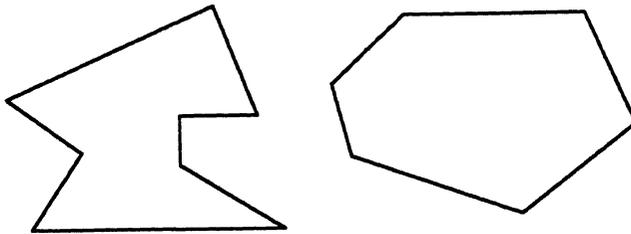


FIGURE 4

We will also use infinite regions to construct collars (Section 5) but the caps themselves will be unions of these finite polygonal regions.

Other, more general polygonal regions will be discussed in Section 6.

An *abstract polyhedral network* is a finite nonempty collection K of polygonal regions together with their edges and vertices, with edges and vertices identified in a special way, subject to the following rules (the word *gluing* will also be used to describe identification, see [AZ], [B], [E]):

1. Any two vertices may be glued. Any two edges having the same length may be glued but must be glued together along their entire lengths. Points are identified which correspond under an isometry of the edges.

2. If two polygonal regions R and S are glued at vertex p then either they are glued along an edge having p as endpoint or they are connected by a finite sequence of polygonal regions glued along edges having p as an endpoint beginning at an edge of R and ending at an edge of S (see Figure 5).

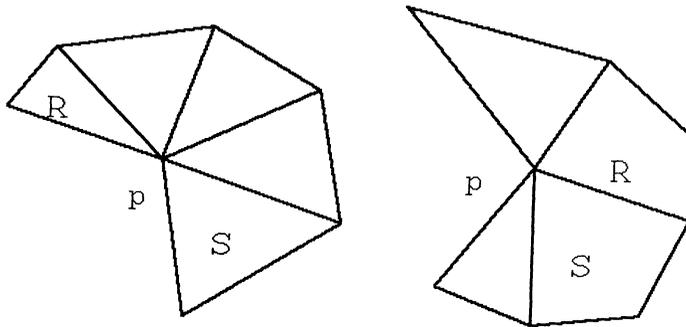


FIGURE 5

3. No edge is glued to more than one other edge.

4. K is *edge-connected*: any two polygonal regions are joined by a chain of polygonal regions which are glued along edges.

5. K is *oriented*: each face has an orientation determined by fixing a circuit (called a positive orientation) of its boundary and the circuits belonging to two faces having a common edge “cancel” on the common edge in the usual way, (see e.g., [L]).

An edge which is not glued to another edge is called a *boundary edge* of K . The points of a boundary edge are *boundary points* of K and the set of all boundary points is the *boundary* of K . $|K|$ denotes the union of the elements of K (with, of course, identified points counting as one point). $|K|$ is called an *abstract polyhedron with subdivision* K .

Property 4 implies that two points a and b in the interiors of two polygonal regions are joined by a path in $|K|$ which contains no vertex.

The polygonal regions in K are the *faces* of K . The edges and vertices of the faces of K are called *edges* and *vertices* of K . The set of edges is called the 1-*skeleton* of K . A face with at least one boundary edge of K is called a *boundary face* of K . Two faces of K are *contiguous* if they are glued together at a common edge.

Neighborhoods of a point of $|K|$ (following [AZ, p. 13]). The polygonal regions R_j are 2-dimensional manifolds with boundary in a natural way. If a point p belonging to $|K|$ was not involved in the identification then we take as a neighborhood of p in $|K|$ any neighborhood of p considered as a point of R_j . If p is an interior point of two identified edges of R_j and R_k then we take as a neighborhood of p in $|K|$ the union of two neighborhoods homeomorphic to half planes which p had in R_j and R_k before identification. The neighborhood of p is separated by an open segment containing p . If p is the identification of several vertices a neighborhood of p in $|K|$ is likewise the union of neighborhoods which p had in each of the polygonal regions to which it belonged before gluing. Because of property 2 above $|K|$ is then a manifold with boundary (which may be empty).

We repeat the definition of a polyhedron more precisely: A *polyhedron in E^3* is a mapping $C: |K| \rightarrow E^3$ of an abstract polyhedron with subdivision K which is facewise isometric (that is, C is one-to-one on each face and preserves Euclidean distance). We will also refer to the image $C(|K|)$ as the *polyhedron C* . Terms and notations referring to an object in K as defined above will be used also to refer to their images under C . For example, the image of a face of K will be called a *face of C* ; two faces of C will be called *contiguous* if their antecedents are contiguous in K ; the *boundary* of C is the image of the boundary of K . (Since the mapping C is not necessarily one-to-one in the large, a point may lie in the boundary of C and also in the relative interior of C according to this definition.) We will usually be thinking of C but use the underlying topology of K . Occasionally we speak of K and C interchangeably when no harm is caused.

The orientation of K induces an orientation of the faces C in a natural way. We choose a unit vector \mathbf{n} normal to the plane of each face of C , coherent with the orientation. A polyhedron C is *edge-convex* if one of the two possible orientations has the property that for each face f all faces contiguous with it lie, except for the glued edges

themselves, in the negative open half-space determined by the plane of f and the normal \mathbf{n} . In what follows we will assume, for definiteness, that this orientation has been chosen on caps-with-collars and we will speak of \mathbf{n} as pointing “out” of such surfaces. Edge-convexity implies that the nonboundary faces of a cap are convex polygonal regions. Boundary faces may be nonconvex.

An oriented, compact, edge-convex polyhedron C is a *cap* if there is a constant unit vector \mathbf{c} such that:

$$\mathbf{c} \cdot \mathbf{n} > 0 \text{ on every face of } C.$$

This stipulation is called the *hemisphere condition*.

REMARK 1. The usual definitions of a cap assume that a cap has nonempty boundary but it is not hard to prove this from our definition of a cap.

Proof. Assume there is no boundary face. For simplicity, we rotate the polyhedron (or \mathbf{c}) so that a face (one for which $\mathbf{c} \cdot \mathbf{n}$ is a minimum) satisfies $\mathbf{c} \cdot \mathbf{n} = 0$, while the remaining faces satisfy $\mathbf{c} \cdot \mathbf{n} \geq 0$. We now prove that this face is a boundary face of the cap.

Suppose f is a nonboundary face with $\mathbf{c} \cdot \mathbf{n} = 0$. Denote the positively oriented edge vectors of f by \mathbf{e}_i and the faces sharing the corresponding edges of f by f_i , $i = 1, 2, \dots, m$. There exist m scalars λ_i such that $\mathbf{e}_i = \lambda_i \mathbf{n} \times \mathbf{n}_i$. The λ_i are positive by edge-convexity and by the orientation. Then (summing from 1 to m) $\mathbf{n} \times \sum \lambda_i \mathbf{n}_i = \sum \mathbf{e}_i = \mathbf{0}$. Therefore there exists a scalar β such that $\sum \lambda_i \mathbf{n}_i = \beta \mathbf{n}$. Hence $\sum \lambda_i \mathbf{c} \cdot \mathbf{n}_i = \beta \mathbf{c} \cdot \mathbf{n} = 0$. It follows that $\mathbf{c} \cdot \mathbf{n}_i = 0$, for all of the normals to the faces contiguous with f . This is a contradiction to the edge-convexity at f and the fact that the face cannot be collinear. (If a face f_i contiguous with f satisfied $\mathbf{c} \cdot \mathbf{n}_i = 0$ then the shared edge would be parallel to \mathbf{c} by edge-convexity and so all edges of f would be parallel to \mathbf{c} .)

Alexandrov [A, Chapter X] (see also [B, p. 121]) refined theorems on infinitesimal rigidity by subdividing the faces of a polyhedron. Explicitly, C is refined by a finite number of applications of the two steps:

1. Introduce a new vertex in the interior of an edge of a face of C .
2. Introduce a new edge in a face of C (called a *conditional* edge or a *diagonal*) by joining two vertices, new or old, of the face by segments which lie, except for their endpoints, in the interior of the face, such that two diagonals do not intersect in the interior of the face.

This divides the face into a finite number of *conditional faces*. Two conditional faces sharing an edge are again called *contiguous*. We call C , with the new edges, vertices and conditional faces distinguished, an *Alexandrov refinement* of C and denote it by C' . (Adding new vertices in the interior of a face may or may not destroy infinitesimal rigidity—see Section 7.)

3. The spin of an infinitesimal motion. An *infinitesimal motion* of a polyhedron C with Alexandrov refinement C' (or briefly *motion on C'*) is a vector field defined at each point p of C which has the form

$$(1) \quad \mathbf{z}(p) = \mathbf{a} + \mathbf{y} \times \mathbf{x},$$

where \mathbf{a} and \mathbf{y} are facewise constant (on the faces of C') vectors and $\mathbf{x} = \mathbf{x}(p)$ is the position vector of p . If \mathbf{a} and \mathbf{y} are constant on the polyhedron the infinitesimal motion is called a *rigid motion*. The vectors \mathbf{y} are called the *rotation vectors* (or *angular velocity vectors*) of the motion. If $\mathbf{z} \equiv \text{constant}$ on C , the motion is a *translation*.

Strictly speaking \mathbf{z} is a function on the abstract polyhedron $|K|$; a point of self-intersection of $C(|K|)$ is copied once for each of its preimages under the mapping C and each copy has a well-defined value of the velocity \mathbf{z} . If this is kept in mind, the above notation (which has an intuitive appeal since one thinks of the points of intersection of faces as moving through each other), although a technical abuse, should cause no difficulties. Thus an infinitesimal motion may be interpreted intuitively as giving the instantaneous velocity of points of the polyhedron when the polyhedron moves so that each face moves as a rigid body but dihedral angles between contiguous faces are allowed to change.

A polyhedron C with Alexandrov refinement C' is *infinitesimally rigid* if it admits only rigid motions. In this case the velocity field \mathbf{z} coincides with the velocity field of a rigid motion of space. A polyhedron C with Alexandrov refinement C' , which admits only rigid motions under boundary conditions on the motions, is called *infinitesimally rigid under the boundary conditions* (or simply *infinitesimally rigid* if there is no danger of ambiguity).

We adopt the *convention* that a symbol with a bar and the same symbol without a bar denote corresponding objects on two contiguous faces.

Let \mathbf{e} be the oriented edge vector of an oriented edge e separating two contiguous faces of a polyhedron. The change in velocity $\Delta\mathbf{z}$ along e in the direction of \mathbf{e} may be computed, using (1), for each

face. For the directed edge $\mathbf{e} = \mathbf{x}_1 - \mathbf{x}_2$

$$(2) \quad \Delta \mathbf{z} = (\mathbf{a} + \mathbf{y} \times \mathbf{x}_1) - (\mathbf{a} + \mathbf{y} \times \mathbf{x}_2) = \mathbf{y} \times (\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{y} \times \mathbf{e}$$

and, likewise, $\Delta \mathbf{z} = \bar{\mathbf{y}} \times \mathbf{e}$.

Subtracting the equations for the two faces at e we obtain $(\mathbf{y} - \bar{\mathbf{y}}) \times \mathbf{e} = \mathbf{0}$. Hence, for some scalar α ,

$$(3) \quad (\mathbf{y} - \bar{\mathbf{y}}) = \alpha \mathbf{e}$$

for any edge separating two faces. In other words, the rotation of a face relative to an adjacent face is parallel to the common edge. (3) implies the existence of an ω -bending, in the terminology of Gluck [G], or a *static stress*, in the terminology of Whiteley [W₁, W₂].

If \mathbf{n} and $\bar{\mathbf{n}}$ are the unit normal vectors of the faces, chosen to be coherent with the orientation of the polyhedron (Section 2), we have $\mathbf{e} \cdot \mathbf{n} = 0 = \mathbf{e} \cdot \bar{\mathbf{n}}$. Equation (3) now yields, after taking the dot products with \mathbf{n} and $\bar{\mathbf{n}}$ and transposing:

$$(4) \quad \mathbf{y} \cdot \mathbf{n} = \bar{\mathbf{y}} \cdot \mathbf{n} \quad \text{and} \quad \mathbf{y} \cdot \bar{\mathbf{n}} = \bar{\mathbf{y}} \cdot \bar{\mathbf{n}}.$$

These equations are also used by H. Weyl in his treatment of the infinitesimal rigidity of polyhedra [E, p. 198].

We call $\mathbf{y} \cdot \mathbf{n}$ the *spin* of \mathbf{z} on a face. A non-rotating observer perched on the tip of \mathbf{n} , looking down at the face, will see the face rotate counterclockwise, clockwise, or not at all according as the spin is positive, negative, or zero. It is immediate that

$$(5) \quad \text{the spin is constant on every original face, (the union of the conditional faces of an Alexandrov refinement).}$$

Two infinitesimal motions are *equivalent modulo a rigid motion* \mathbf{z} if their difference is equal to \mathbf{z} . (This is an equivalence relation on the set of infinitesimal motions on a polyhedron.)

The first step in proving infinitesimal rigidity consists in showing that any motion satisfying a certain boundary condition is equivalent, modulo an infinitesimal rotation with axis \mathbf{c} , to a motion whose spin vanishes identically; the second in showing that any motion with zero spin is a rigid motion if various conditions hold on the polyhedron. The original motion must then be a rigid motion. The second part will be treated in the next section. The first is embodied in

THE SPIN LEMMA 2. *Let C be a cap satisfying the hemisphere condition $\mathbf{c} \cdot \mathbf{n} > 0$. Let C' be an Alexandrov refinement of C . If \mathbf{z} is a velocity field on C' satisfying the boundary condition $\mathbf{z} \cdot \mathbf{c} = \text{constant}$*

on each component of the boundary then \mathbf{z} is equivalent to a motion with zero spin.

Proof. Pick any face g_0 of C' . Let subscript $_0$ denote objects defined on g_0 . $\mathbf{y}' = \mathbf{y} - (\mathbf{y}_0 \cdot \mathbf{n}_0 / \mathbf{c} \cdot \mathbf{n}_0) \mathbf{c}$ is the rotation vector of the motion $\mathbf{z}' = \mathbf{z} - (\mathbf{y}_0 \cdot \mathbf{n}_0 / \mathbf{c} \cdot \mathbf{n}_0) \mathbf{c} \times \mathbf{x}$ which is equivalent to \mathbf{z} . Note that $\mathbf{y}'_0 \cdot \mathbf{n}_0 = 0$ so that \mathbf{z}' has zero spin on g_0 . Also, the changes in velocity of \mathbf{z}' and \mathbf{z} along any boundary edge satisfy $\mathbf{c} \cdot \Delta \mathbf{z} = \mathbf{c} \cdot \Delta \mathbf{z}' = 0$. Therefore to prove the theorem it suffices to show the simpler lemma.

LEMMA 3. *If a motion \mathbf{z} satisfies the condition that $\mathbf{c} \cdot \mathbf{z}$ is constant on each component of the boundary separately and has zero spin on a face g_0 of C' then it has spin identically equal to zero.*

Proof. Let \mathcal{K} be the nonempty union of the faces of C' on which $\mathbf{y} \cdot \mathbf{n} = 0$. The boundary edges of \mathcal{K} are original edges (that is, edges of C) by (5). We will show that $\mathcal{K} = C'$.

Consider the edge-connected component \mathcal{K}_0 of \mathcal{K} containing g_0 . If all the boundary edges of \mathcal{K}_0 are boundary edges of C' then $\mathcal{K}_0 = \mathcal{K} = C'$, since C' is edge-connected, and the conclusion holds. Therefore we may suppose that there is some face f_1 of \mathcal{K}_0 which has a boundary edge which is not a boundary edge of C' . Then $\mathbf{c} \cdot \mathbf{n} > 0$ and $\mathbf{y} \cdot \mathbf{n} = 0$ on f_1 with $\bar{\mathbf{y}} \cdot \bar{\mathbf{n}} \neq 0$ on some contiguous face \bar{f}_1 (see Figure 6). Suppose for example that $\bar{\mathbf{y}} \cdot \bar{\mathbf{n}} > 0$ on \bar{f}_1 (the case $\bar{\mathbf{y}} \cdot \bar{\mathbf{n}} < 0$ is essentially the same).

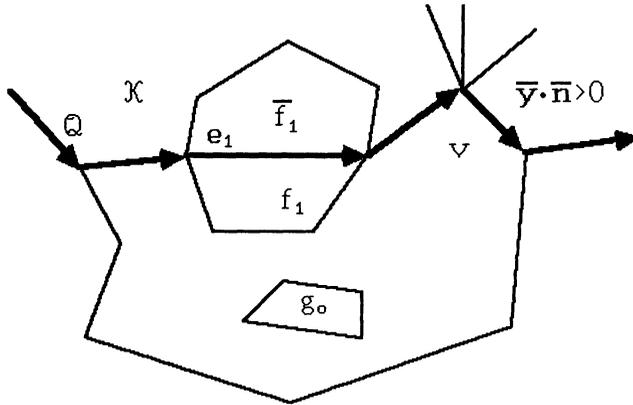


FIGURE 6

Let \mathcal{K}^+ denote the maximal edge-connected subset of C' containing \bar{f}_1 such that the spin is positive on each face of \mathcal{K}^+ . (\mathcal{K}^+ can be

obtained by successively adjoining faces along edges, by the definition of edge-connected, Section 2.) \mathcal{K}^+ is, of course, connected. Let e_1 denote the edge separating f_1 and \bar{f}_1 and \mathbf{e}_1 the edge vector of e_1 .

Let \mathcal{Q} be the component of the boundary of \mathcal{K}^+ which contains e_1 . \mathcal{Q} is an oriented cycle of edges, with its orientation coherent to that of C' . In other words an observer with feet on the “positive” side of C' walking around \mathcal{Q} in the “positive” direction will see the faces of \mathcal{K}^+ on his left. Thus faces with positive spin will be on the left and, if the edge is a nonboundary edge, faces on the right will have nonpositive spin, with zero spin on f_1 . (These statements depend on the topology of the abstract polyhedron, and are not affected by self-intersection in the cap.)

Our observer, coming to a vertex v , turning to his left into \mathcal{K}^+ , and walking around the vertex v from face to face across the edges having v as endpoint with v on his right must come to an oriented boundary edge of \mathcal{K}^+ that goes away from v . Continuing in this manner, after passing a finite number of vertices he must return to his starting point (see Figure 6).

Notice that the edges in \mathcal{Q} must be original edges of C by (5) since the spin is not constant across these edges. Using (2) and edge-convexity, Figure 7 shows that:

$$(6) \quad \mathbf{c} \cdot \Delta \mathbf{z} \leq 0 \text{ on the edges } e \text{ of } \mathcal{Q} \text{ with strict inequality} \\ \text{holding on all nonboundary edges.}$$

Thus \mathcal{Q} may be called a *falling path*.

This leads immediately to a contradiction since the total change in \mathbf{z} on a closed path is zero: $\sum \mathbf{c} \cdot \Delta \mathbf{z} = \mathbf{c} \cdot \sum \Delta \mathbf{z} = 0$. We will give an algebraic proof of (6) below, but first we give a more intuitive, geometric proof.

Geometric Proof of (6). Figure 7 shows the section of the polyhedron by a plane perpendicular to a general nonboundary edge e in \mathcal{Q} , with its edge vector \mathbf{e} directed into the page. Since $\mathbf{c} \cdot \mathbf{n} > 0$ and $\mathbf{c} \cdot \bar{\mathbf{n}} > 0$, the perpendicular projection of \mathbf{c} into this plane must lie inside the indicated shaded dihedral angle. Since $(\mathbf{y} - \bar{\mathbf{y}}) \times \mathbf{e} = \mathbf{0}$, the projections of \mathbf{y} and $\bar{\mathbf{y}}$ into this plane will coincide. Since $\bar{\mathbf{y}} \cdot \mathbf{n} = \mathbf{y} \cdot \mathbf{n} \leq 0$ and $\bar{\mathbf{y}} \cdot \bar{\mathbf{n}} > 0$, the projection $\bar{\mathbf{y}}$ must lie within the second shaded dihedral angle. The reader can check that these conditions in the diagram, and the edge-convexity at \mathbf{e} , show that $\Delta \mathbf{z} \cdot \mathbf{c} = \bar{\mathbf{y}} \times \mathbf{e} \cdot \mathbf{c} < 0$.

Algebraic Proof of (6). Let e be an edge in \mathcal{Q} which is a nonboundary edge of C . Since \mathbf{n} and $\bar{\mathbf{n}}$ span a plane perpendicular to e we

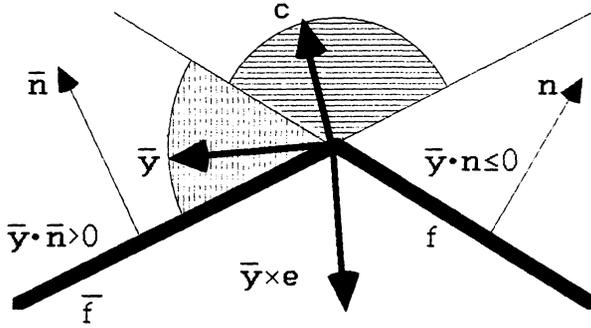


FIGURE 7

can write $\Delta z = \bar{y} \times e = \alpha n + \beta \bar{n}$ for suitable scalars α, β . Since $c \cdot n > 0$ and $c \cdot \bar{n} > 0$, $\Delta z \cdot c = \bar{y} \times e \cdot c = (\alpha n + \beta \bar{n}) \cdot c < 0$ if $\alpha < 0$ and $\beta \leq 0$.

Taking the dot product of $\bar{y} \times e$ with \bar{y} , we obtain $\alpha \bar{y} \cdot n + \beta \bar{y} \cdot \bar{n} = 0$. Since $\bar{y} \cdot \bar{n} > 0$ and $\bar{y} \cdot n = y \cdot n \leq 0$, α and β , if both are not zero, must have the same sign. By the orientation of C , and edge-convexity, $\bar{n} \times n = \mu e$ for some scalar $\mu > 0$. Recall that for any three vectors $(a \times b) \times c = (a \cdot c)b - (b \cdot c)a$. Therefore

$$(\bar{y} \cdot \bar{n}) e = (e \cdot \bar{n}) \bar{y} + (\bar{y} \times e) \times \bar{n} = 0 + (\bar{y} \times e) \times \bar{n} = -\alpha \bar{n} \times n = -\alpha \mu e.$$

Therefore $-\alpha \mu = \bar{y} \cdot \bar{n} > 0$. We conclude that $\alpha < 0$ and $\beta \leq 0$, as required.

Figure 8 shows why the argument fails if the polyhedron is not edge convex: $y \cdot \bar{n} > 0$ and $y \cdot n \leq 0$ imply $y \times \bar{e} \cdot c \geq 0$, for a concave edge.

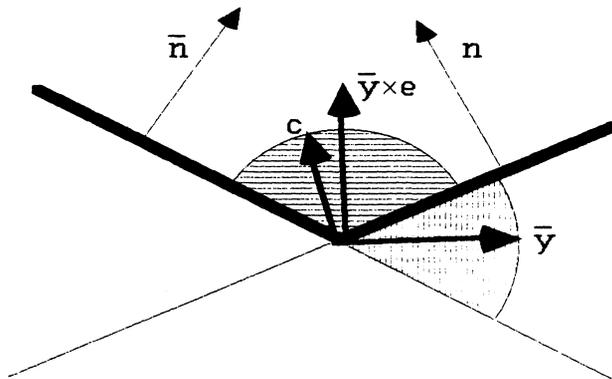


FIGURE 8

REMARK 4. The methods used in the proof of the Spin Lemma, which is of central importance in this paper, have a common theme with methods used by the author to prove the infinitesimal rigidity of

differentiable surfaces with Gauss curvature $\mathcal{K} > 0$, (see [K₁], [K₂], [K₃]). We'll briefly illustrate the analogy.

Let \mathbf{x} be the position vector of a surface which deforms with time t such that the surface metric $d\mathbf{x} \cdot d\mathbf{x}$ is constant to first order in t at time $t = 0$. The velocity $\mathbf{z} = \mathbf{x}_t$ (assumed to be a C'' field) of each point on the surface then satisfies $d\mathbf{z} \cdot d\mathbf{x} = 0$ on the surface at $t = 0$. It is well known, in the theory of infinitesimal deformations of a surface, that a unique differentiable vector field \mathbf{y} exists such that $d\mathbf{z} = \mathbf{y} \times d\mathbf{x}$. The field \mathbf{y} , which can be assumed never to vanish, has the property that the locus where $\mathbf{y} \cdot \mathbf{n} = 0$ (\mathbf{n} , as usual is the unit surface normal) is a differentiable curve. If $\mathbf{y} \cdot \mathbf{n}$ doesn't vanish on the boundary then the locus is either empty or is a disjoint union of simple closed curves. Let σ be such a curve. Differentiating $\mathbf{y} \cdot \mathbf{n}$ along σ we obtain $\mathbf{y} \cdot d\mathbf{n} = 0$ ($\mathbf{n} \cdot d\mathbf{y}$ is known to be zero). Since $\mathcal{K} > 0$, $d\mathbf{x} \cdot d\mathbf{n} \neq 0$ and so \mathbf{y} can never be parallel to $d\mathbf{x}$. Therefore we can write $d\mathbf{z} = \mathbf{y} \times d\mathbf{x} = \lambda \mathbf{n}$, where λ , say, is positive. If the surface satisfies the hemisphere condition $\mathbf{c} \cdot \mathbf{n} > 0$ then $\mathbf{c} \cdot d\mathbf{z} > 0$. However, this contradicts $\int_{\sigma} \mathbf{c} \cdot d\mathbf{z} = 0$. Therefore

If $\mathbf{y} \cdot \mathbf{n}$ doesn't vanish on the boundary of a surface satisfying the hemisphere condition then $\mathbf{y} \cdot \mathbf{n}$ doesn't vanish on the surface.

This lemma is the main tool used in the research mentioned above. It's clear that σ and the cycle \mathcal{C} in the proof of the Spin Lemma play analogous roles.

4. Infinitesimal rigidity with boundary conditions. In the next lemma, we do not need edge-convexity or hemisphere conditions. However we need the following notation:

\mathcal{N} = the union of the nonboundary faces of C and

\mathcal{S} = the 1-skeleton of C , minus the boundary edges of C .

Figure 9 shows an example of a polyhedron (A) with disconnected set \mathcal{N} of non-boundary faces (B) and a disconnected set \mathcal{S} of non-boundary edges.

THE TRANSLATION LEMMA 5. *If contiguous faces of a polyhedron C do not lie in the same plane then a motion with zero spin defined on an Alexandrov refinement C' of C is*

- (a) *a translation on each component of \mathcal{S} .*
- (b) *a translation on each component of \mathcal{N} .*

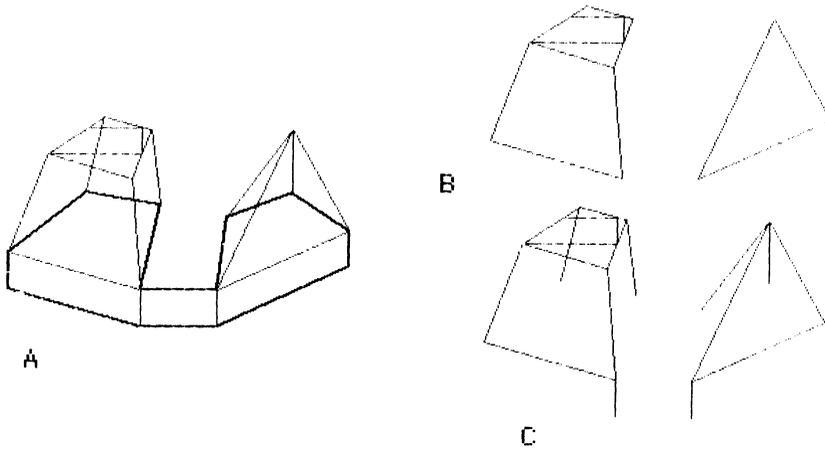


FIGURE 9

Proof. $\mathbf{y} \cdot \mathbf{n} \equiv 0$ and (4) imply that the rotation vector \mathbf{y} on any face f of C' is orthogonal to the normal to any face (original or conditional) sharing an edge with f . If such adjoining face is not parallel to f , which is the case when the shared edge is original, then \mathbf{y} is parallel to the shared edge. Thus by (2) $\Delta z = 0$ on each original nonboundary edge. Therefore z is constant on each component of the union of all original nonboundary edges of C , i.e., on each component of \mathcal{S} .

The diagonals of a nonboundary face of C join vertices which lie in the edges of a component of \mathcal{S} (as in Figure 9). Therefore $\Delta z = 0$ on all of the diagonals of nonboundary faces. Thus, for each conditional face of a nonboundary face of C , the rotation vector \mathbf{y} is parallel to each of its edges. By the definition of a face and of an Alexandrov refinement, every such conditional face has at least two non-parallel edges and therefore $\mathbf{y} = \mathbf{0}$ on each conditional face of a nonboundary face. By (1) z is constant on each face and therefore, by continuity, on each component of \mathcal{N} .

We are now prepared for the infinitesimal rigidity theorem.

THEOREM A. *Let C be a cap with Alexandrov refinement C' satisfying the hemisphere condition $\mathbf{c} \cdot \mathbf{n} > 0$ and let \mathbf{z} be any motion on C' satisfying the boundary condition $\mathbf{c} \cdot \mathbf{z} = \text{constant}$ on each boundary component of C separately. Then \mathbf{z} is a rigid motion.*

In particular (since the boundary condition $\mathbf{c} \cdot \mathbf{z} = 0$ is called a glidebending) a polyhedral cap with spherical image in the interior of a hemisphere is infinitesimally rigid under glidebending.

Proof. By the Spin Lemma, \mathbf{z} is equivalent to a motion $\mathbf{z}' = \mathbf{z} - \alpha \mathbf{c} \times \mathbf{x}$, which has zero spin. By the Translation Lemma \mathbf{z}' is a translation on each component of the union \mathcal{N} of the nonboundary faces of C . The proof that \mathbf{z} is a rigid motion on all of C' requires a little more detail. We need to show that the translated components of \mathcal{N} , if there is more than one, are connected by boundary components of C which are also translated thus forcing C to be translated as a unit.

By the boundary condition $\mathbf{c} \cdot \Delta \mathbf{z}' = \mathbf{c} \cdot \Delta \mathbf{z} = 0$ on the boundary of C' . Let f be an original boundary face and f' any conditional face of f . Since the spin of \mathbf{z}' is zero its rotation vector \mathbf{y}' is parallel to f' . This means that for any vector \mathbf{e} in f' there is a scalar λ such that $\mathbf{y}' \times \mathbf{e} = \lambda \mathbf{n}$. If f' is a boundary face of C' and \mathbf{e} is an edge vector of f' in the boundary of C then $0 = \mathbf{c} \cdot \Delta \mathbf{z}' = \mathbf{c} \cdot \mathbf{y}' \times \mathbf{e} = \mathbf{c} \cdot \lambda \mathbf{n}$. By the hemisphere condition $\mathbf{c} \cdot \mathbf{n} > 0$, $\lambda = 0$. Therefore \mathbf{y}' is parallel to all of the edge vectors \mathbf{e} of f' which belong to the boundary of C' (which of course equals the boundary of C). It follows that $\Delta \mathbf{z}' = 0$ on each edge of f' which is a boundary edge of C (the heavy lines in Figure 10). Therefore \mathbf{z}' is constant on these edges.

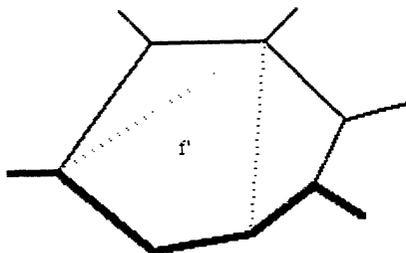


FIGURE 10

By the Translation Lemma (a) \mathbf{z}' is also constant on those edges of f which are nonboundary edges of C since they are in \mathcal{S} (the thinner solid lines in Figure 10). We now have shown that \mathbf{z}' is constant on each original edge of f . We assumed (Section 2) that the boundary of any face is connected and thus \mathbf{z}' is constant on the boundary of f . Since diagonals of f join points on the boundary of f (dotted lines in Figure 10), $\Delta \mathbf{z}' = 0$ on all diagonals as well as edges of f . Hence \mathbf{z}' is constant on each f' and so on each boundary face f . Therefore \mathbf{z}' is constant on each boundary component of C as well as on each component of \mathcal{N} . Since the components of \mathcal{N} are connected by strips of boundary faces, \mathbf{z}' is constant on C . Therefore \mathbf{z} is a

rigid motion. (Note that this conclusion holds even if \mathcal{N} is empty because in this case the set of boundary faces is connected.)

5. Infinite collars and the rigidity of closed polyhedral surfaces. Given a cap C with Alexandrov refinement C' satisfying the hemisphere condition $\mathbf{c} \cdot \mathbf{n} > 0$ we attach infinite cylindrical polyhedral collars with infinite edges and faces parallel to \mathbf{c} to each boundary component. This may be done as follows. Let $v_1 v_2 \cdots v_n$ be a maximal plane polygonal arc having the two properties (Figure 11A):

1. its vertices are successive original boundary vertices of a boundary component of C and
2. it lies in a plane parallel to \mathbf{c} .

Each boundary component can be expressed as the union of a finite set of such arcs where the endpoint of each arc is the initial point of the next. Attach to each such arc in a boundary component the infinite plane set bounded by $v_1 v_2 \cdots v_n$ and the rays from its endpoints, v_1 and v_n , parallel to \mathbf{c} and both directed the same way. We call this set a *natural collar face*. A natural collar face is the union of *infinite triangular regions* whose finite sides are the segments $v_{i-1} v_i$ and whose other sides are two rays parallel to \mathbf{c} (Figure 11B).

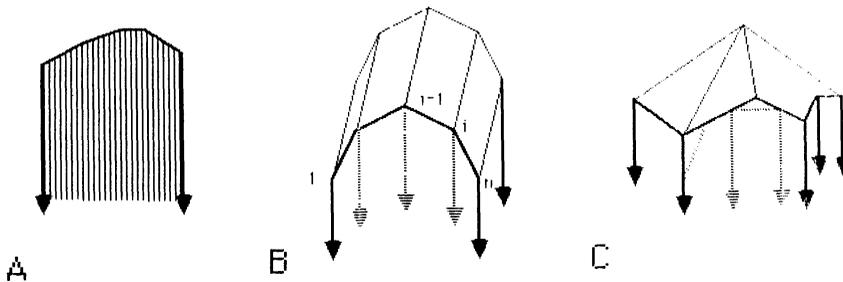


FIGURE 11

The rays on a given boundary component will all be taken to point in the same direction as \mathbf{c} or all in the opposite direction so that the regions go to infinity in the same direction. If the rays point opposite to \mathbf{c} , the natural collar face is a convex plane set. This follows easily from edge-convexity, our assumption that $\mathbf{c} \cdot \mathbf{n} > 0$ on C and the fact that none of the $v_{i-1} v_i$ is parallel to \mathbf{c} .

The collection of the natural collar faces attached to a boundary component will be called a *cylindrical collar*. The edges $v_{i-1} v_i$ and the rays from v_1 and v_n of the natural collar faces will be called *natural edges of the collar*. The ray edges are, of course, the intersections of

contiguous natural collar faces. We note that contiguous natural collar faces lie in different planes.

The union of C and the collars is an oriented polyhedron with infinite faces, which we'll denote by D . In general D will not be edge-convex (recall Figure 1). In fact, the collars themselves need not be edge-convex and may self-intersect (recall Figure 1A and D).

We also have a *refined cylindrical collar* in the manner of Alexandrov. We add a finite number of new vertices in the interiors of the edges (including rays) of a natural collar face and joining vertices, new or old, by diagonals which lie, except for their endpoints, in the interior of the natural collar face but do not intersect in the interior of the face. Moreover, we allow as diagonals rays in a natural collar face which are parallel to \mathbf{c} starting at points of a boundary edge of C (including points in the interior of the boundary edge) (Figure 11C). Again, no diagonals may intersect in the interior of a natural collar face. A refinement of D will be denoted by D' .

Let an infinitesimal motion \mathbf{z} of the form (1) be given on D' which is a rigid motion on each face of D' . We show that \mathbf{z} is equivalent to a motion with zero spin on the collar, using arguments related to those in [B, Sec. 16, p. 126]. Since D' contains only a finite number of vertices there exists a plane π perpendicular to \mathbf{c} which cuts any given collar into two parts: a finite cylinder and a half-infinite cylinder Q which contains no vertices of D' . Number the faces of D' which intersect Q (these are infinite collar faces with edges parallel to \mathbf{c}) consecutively $f_1, f_2, \dots, f_n, f_{n+1} = f_1, \dots$ and let \mathbf{y}_i be the rotation vector of \mathbf{z} on f_i . By (3) $\mathbf{y}_i - \mathbf{y}_{i-1}$ is parallel to \mathbf{c} and hence so is $\mathbf{y}_i - \mathbf{y}_1$ since it may be written

$$\mathbf{y}_i - \mathbf{y}_1 = (\mathbf{y}_i - \mathbf{y}_{i-1}) + (\mathbf{y}_{i-1} - \mathbf{y}_{i-2}) + \dots + (\mathbf{y}_2 - \mathbf{y}_1).$$

Define a velocity field \mathbf{z}' equivalent to \mathbf{z} by $\mathbf{z}' = \mathbf{z} - \mathbf{y}_1 \times \mathbf{x}$. The rotation vector of \mathbf{z}' on f_i is $\mathbf{y}'_i = \mathbf{y}_i - \mathbf{y}_1$ and, since it is parallel to \mathbf{c} , the spin of \mathbf{z}' on f_i is zero. Therefore, by (4), the spin of \mathbf{z}' is zero on the natural collar face containing f_i . Hence

(7) the spin of \mathbf{z}' is zero on the collar.

It follows that $\mathbf{c} \cdot \mathbf{z}'$ is constant on each original collar face and hence on the whole collar including its common boundary with C . In other words we have proved:

THE COLLAR LEMMA 6. *Let a refined cylindrical collar with faces parallel to \mathbf{c} , be attached, as described above, to a boundary component*

of a refined cap satisfying the hemisphere condition $\mathbf{c} \cdot \mathbf{n} > 0$. Then any motion \mathbf{z} on the resulting refined polyhedron is equivalent, modulo an infinitesimal rotation, to a motion \mathbf{z}' satisfying $\mathbf{c} \cdot \mathbf{z}' = \text{constant}$ on the collar.

REMARK 7. The unboundedness of the cylindrical collar was used in the arguments leading to this statement only to insure that no vertices appeared in Q . It would still be true if, for example, a half-infinite cylinder containing no vertices were cut off from the collar by a plane perpendicular to \mathbf{c} leaving a bounded collar with no diagonals intersecting the free boundary (for example, a rectangular box with one face removed). The fact that the collar goes to infinity will be used to produce rigid closed surfaces by the Darboux transformation.

REMARK 8. We would like to have $\mathbf{c} \cdot \mathbf{z}' = \text{constant}$ on each boundary component of C' so we can apply Theorem A. However, the operation of subtracting a rotation from \mathbf{z} in order to achieve this condition has to be done to one boundary component at a time. Performing the operation on a second boundary component will, in general, destroy the condition on the first one. Therefore additional boundary conditions will be needed if there is more than one boundary component on the cap. This will be discussed in Section 6.

Suppose, for now, that there is only one boundary component. It follows from the Collar Lemma and Theorem A that the restriction $\mathbf{z}'|_{C'}$ is a rigid motion and hence so is $\mathbf{z}|_{C'}$. We will show that \mathbf{z} is a rigid motion on all of D' :

By the Collar Lemma and the Spin Lemma $\mathbf{z}'|_{C'}$ is equivalent to a motion $\mathbf{z}'' = \mathbf{z}' - \alpha \mathbf{c} \times \mathbf{x}$, $\alpha = \text{constant}$, which has zero spin on the cap C' . Since the spin of \mathbf{z}' is zero on the collar by (7) and the rigid motion $\alpha \mathbf{c} \times \mathbf{x}$ has zero spin on the collar, \mathbf{z}'' has zero spin on the entire collar. Thus \mathbf{z}'' has spin identically equal to zero on D' .

By familiar arguments, similar to those in the proof of the Translation Lemma, \mathbf{z}'' is a translation: \mathbf{z}'' is constant on all edges which separate nonparallel faces by (4) (including the natural collar edges which are rays separating the nonparallel natural collar faces). The set \mathcal{S} of such edges is connected (since the skeleton of D is connected and D has no boundary) so that \mathbf{z}'' is constant on \mathcal{S} . Since any conditional ray edge is parallel to \mathbf{c} , and the rotation vector $\mathbf{y}'_i - \alpha \mathbf{c}$ of \mathbf{z}'' is parallel to \mathbf{c} on the terminal collar Q , \mathbf{z}'' is also constant on all conditional ray edges of the collar. These rays start at points

of \mathcal{F} (their initial points are in the boundary of C'). Let \mathcal{F}' equal \mathcal{F} enlarged to include all conditional rays. Then \mathbf{z}'' is constant on \mathcal{F}' . Diagonal segments connect points of \mathcal{F}' (Figure 11C) so \mathbf{z}'' is constant on these. This implies that \mathbf{z}'' is a translation on D' . \mathbf{z} , being equivalent to \mathbf{z}'' on D' , is thus a rigid motion on D' .

We have proved the second infinitesimal rigidity theorem.

THEOREM B. *Let C be a cap with Alexandrov refinement C' , satisfying the hemisphere condition $\mathbf{c} \cdot \mathbf{n} > 0$ and having exactly one boundary component. Attach an infinite cylindrical polyhedral collar with Alexandrov refinement to the boundary component of C whose faces are parallel to \mathbf{c} . The resulting surface is infinitesimally rigid.*

REMARK 9. If the collar is truncated by removing a semi-infinite cylinder containing no vertices the resulting surface with boundary is still infinitesimally rigid. This is an example of a general slicing principle described in [W₅].

EXAMPLE 1. Figure 12 shows a (schematically drawn) polyhedral spherical cap with truncated collar (which could go either up or down). Since it is infinitesimally rigid, the surface on the right, obtained by closing it with a second cap, is an infinitesimally rigid (nonconvex) polyhedron.

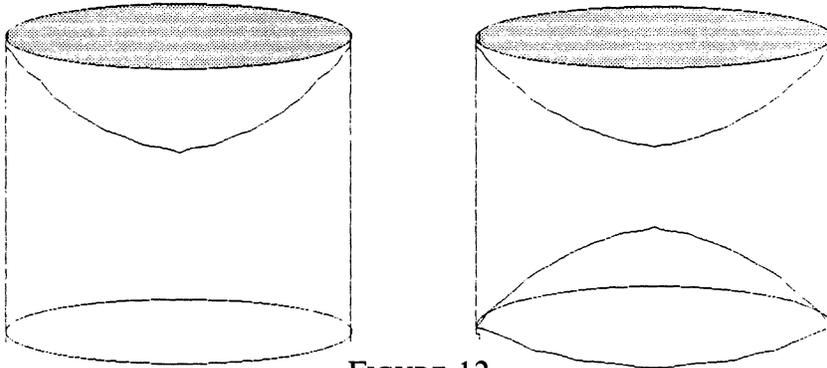


FIGURE 12

We now apply the Darboux transformation to caps whose collars all go to infinity in the same direction, so that the union of cap and collars lies in a half-space, in order to obtain closed boundaryless polyhedra which are rigid. Several examples of these were mentioned in the Introduction. The *Darboux transformation* consists of two parts. Let x_1, x_2, x_3 be rectangular coordinates on E^3 . The first part is given by

$$D_1: \quad x'_1 = x_1/x_3, \quad x'_2 = x_2/x_3, \quad x'_3 = 1/x_3.$$

This may be interpreted as a projective transformation of the extended Euclidean space interchanging the x_1, x_2 plane and the “plane at infinity”. It is well known how to give this a precise meaning by embedding E^3 in the projective space P^3 in a canonical way and distinguishing a plane in P^3 . We may always choose the coordinate system so that the cap with its collars lies in the half-space $x_3 > 0$ with the collar rays in the direction of the positive x_3 axis (Figure 13A). D_1 takes the cap with its cylindrical collars to a boundaryless polyhedron (Figure 13B) consisting of an edge-convex polyhedron with boundary with polyhedral cones (the images of the cylindrical collars) attached to the boundary which have a vertex p at the origin 0 of the coordinate system. Strictly speaking p is not part of the image of the cap-with-collars; however we will add it to the image and refer to the new set as *the completed image polyhedron* of the cap-with-collars under the Darboux transformation. We note that if there is more than one collar, copies of p appear for each cone. If these are identified (see Section 6), p becomes a pinchpoint of the completed image polyhedron.

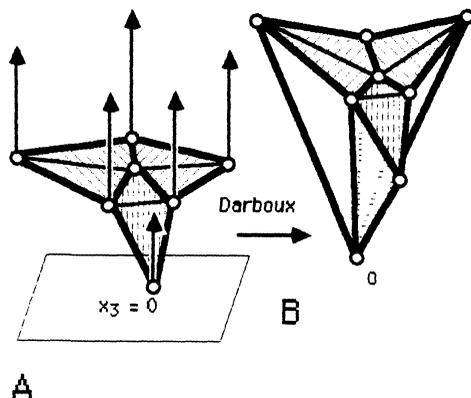


FIGURE 13

The second part of the Darboux transformation transforms infinitesimal motions on a surface. It is given by

$$D_2: \quad z'_1 = z_1/x_3, \quad z'_2 = z_2/x_3, \quad z'_3 = -(\mathbf{x} \cdot \mathbf{z})/x_3$$

where $\mathbf{z} = (z_1, z_2, z_3)$ (the components of all vectors are with respect to the standard rectangular coordinate system) is the velocity field of a motion on the original surface and \mathbf{x} is the position vector from the origin of the coordinate system to the surface. It is well known and, easy to check, that $\mathbf{z} = (z_1, z_2, z_3)$ is a rigid motion on any

set of points if and only if $\mathbf{z}' = (z'_1, z'_2, z'_3)$ is a rigid motion on the image of the set under D_1 (see, for example, [E, p. 58–61] or [B, p. 126] (which gives the reference [Da], Livre VIII, Chapitre III); and [CW, Section 3.6] which discusses the projective invariance of the infinitesimal rigidity of panel structures).

Since a cap with one collar can sustain only rigid motions, by Theorem B, it follows that the same is true for its image under D_1 .

How must a polyhedron look in order that it be such an image? Calculation using D_1 shows that an oriented polyhedron C in the half-space $x_3 > 0$ satisfies the hemisphere condition $\mathbf{c} \cdot \mathbf{n} > 0$ at a point, where \mathbf{c} is a unit vector in the direction of the positive x_3 axis, if and only if the function $\mathbf{x}' \cdot \mathbf{n}'$ satisfies $\mathbf{x}' \cdot \mathbf{n}' > 0$ at the corresponding point on its D_1 transform.

Since the function $\mathbf{x}' \cdot \mathbf{n}'$ is the directed distance from the origin O to the plane of a face with normal \mathbf{n}' , O lies in the intersections of the negative half-spaces (defined in the usual way as the half-spaces opposite to the ones into which the vectors \mathbf{n}' point) of the oriented planes containing the faces of the transform (oriented to agree with the orientation of the faces) if and only if $\mathbf{c} \cdot \mathbf{n} > 0$. Therefore, from Theorem B we conclude

THEOREM B'. *Let C be a polyhedron with Alexandrov refinement C' which is oriented, edge-convex and has a connected boundary to which a polyhedral cone with Alexandrov refinement with vertex p has been attached. Suppose*

1. *that p lies in the intersection of all negative (positive) half-spaces of the coherently oriented facial planes of C and*

2. *there exists a plane through p such that C and the cone lie (except for p) in an open half-space determined by the plane.*

Then the polyhedron formed by C and the attached cone is infinitesimally rigid.

In this way we can obtain many examples of infinitesimally rigid closed polyhedra as mentioned in the Introduction. These surfaces need not be edge-convex on the star of p .

EXAMPLE 2. Figure 13 shows a cap and its Darboux transform. The cap has three faces and an infinite polyhedral collar attached to its boundary. In this case the collar has a nonconvex cross-section and has six faces. The Darboux transform is a closed nonconvex

infinitesimally rigid polyhedral sphere. The collar becomes a cone with vertex at the origin.

COROLLARY 10 (*Alexandrov [A]*). *A closed convex polyhedron D with Alexandrov refinement D' is infinitesimally rigid.*

Proof. Let p be any vertex of D and π a supporting plane of D at p such that D lies, except for p , in an open half-space of π . Take the set of all faces of D which have vertex p as the cone and the closure of its complement in D as the polyhedron C of Theorem B'.

EXAMPLE 3. The present method allows a generalization of Alexandrov's Theorem to polyhedra which are immersions of manifolds of higher genus. An example offered by Connelly is the following: Let T be a torus of genus n with an open disk removed embedded in E^3 . T may be deformed isotopically through embeddings to a polyhedron embedded in E^3 which is edge-convex and satisfies the hemisphere condition. Figure 14A, B, C, illustrates a deformation to the plane of a torus of genus two with a connected boundary embedded in a standard way. The third stage, Figure 14C, can clearly be further deformed in the desired way (see 2-cell with (untwisted) strips, [M₂, p. 157]). Since the boundary is connected we may attach a single cylindrical collar to it (which will be self-intersecting and not edge-convex) with generators parallel to c . The resulting cap-with-collar is infinitesimally rigid by Theorem B. We place it in the half-space $x_3 > 0$ with the collar rays parallel to the x_3 axis. Applying the transformation D_1 we obtain an immersion of a closed polyhedron of genus n which is infinitesimally rigid. The image of the cap is edge-convex but the cone which is the image of the collar is not edge-convex nor embedded (Figure 14D).

EXAMPLE 4. Figure 15 shows an infinitesimally rigid pentagramal bipyramid with one vertex at infinity. (Its Euler characteristic is 2 and therefore it is a combinatorial sphere.) The pentagram lies in a plane and is the common boundary of the cap and the cylindrical collar. A Darboux transformation takes the unbounded bipyramid in Figure 15A to an infinitesimally rigid bounded bipyramid like the one in Figure 15B.

EXAMPLE 5. Figure 1B shows an edge-convex, infinitesimally rigid polyhedron with two branch points like that on the Riemann surface of $w = z^{1/2}$. Similar k -sheeted infinitesimally rigid polyhedra corresponding to $z^{1/k}$ can be constructed.

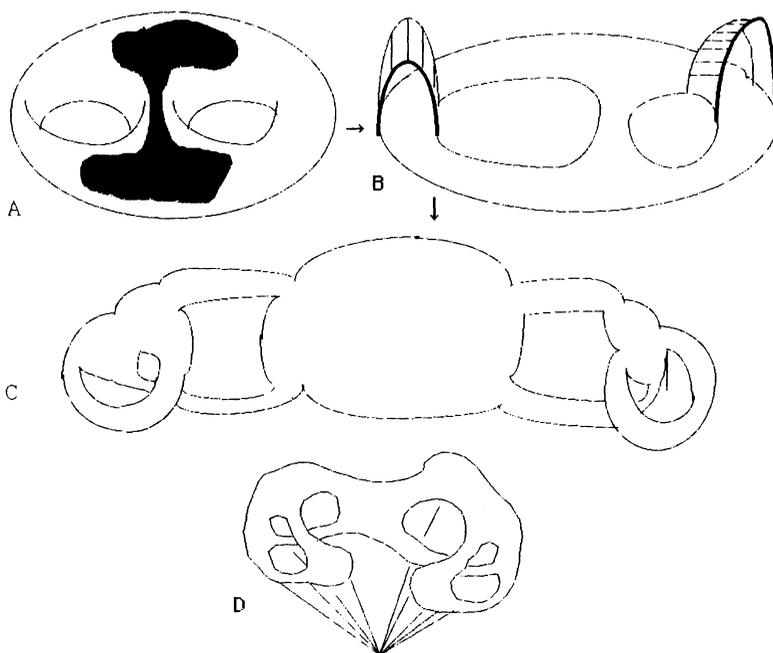


FIGURE 14

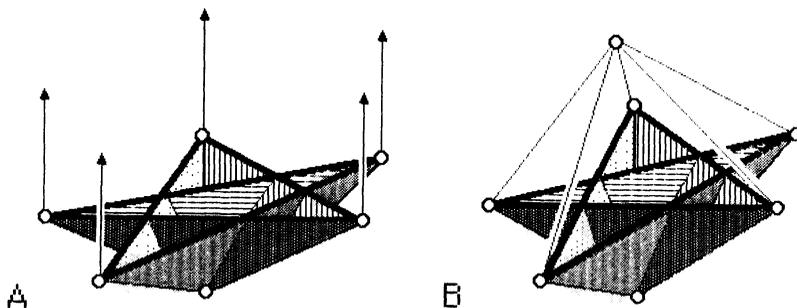


FIGURE 15

6. Extensions. If the cap C has more than one boundary component then adding collars may not by itself guarantee that a motion on the resulting cap-with-collars will be equivalent to one for which $\mathbf{c} \cdot \mathbf{z} = \text{const.}$ on each boundary component of C (Remark 8). Moreover, if no further conditions are imposed then a value of the velocity $\mathbf{z}' = D_2(\mathbf{z})$ at the copy of p on a cone is not determined by the values of \mathbf{z} on the collar antecedent to the cone under D_1 since p has no antecedent. We will assume the minimal condition that a continuous velocity field \mathbf{z}' on the completed image of a cap-with-collars exists and then explore additional conditions that will enable us to utilize Theorem A to prove its infinitesimal rigidity. Therefore we require

for each copy of p :

(8) $\mathbf{z}'(p)$ is the limit of $\mathbf{z}'(q)$ as q approaches p on a completed cone.

We calculate the limit of \mathbf{z}' as q approaches p on a face f' of a cone using the components of $\mathbf{z} = D_2^{-1}(\mathbf{z}')$. Place the cap-with-collars, as before, in the half-space $x_3 > 0$ with the collar faces parallel to the x_3 axis and take $\mathbf{c} = (0, 0, 1)$. Let f be the antecedent of f' under D_1 and (x_1, x_2, x_3) an arbitrary point of f . Substituting the expressions for the components of \mathbf{z} on f from (1), with $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$, into D_2 we get

$$\begin{aligned} D_2(\mathbf{z}) = (1/x_3)(a_1 + y_2x_3 - y_3x_2, a_2 + y_3x_1 - y_1x_3, \\ - (a_1 + y_2x_3 - y_3x_2)x_1 - (a_2 + y_3x_1 - y_1x_3)x_2 \\ - (a_3 + y_1x_2 - y_2x_1)x_3). \end{aligned}$$

Taking the limit as $x_3 \rightarrow \infty$, keeping x_1, x_2 fixed, we obtain $(y_2, -y_1, -a_3)$. Similarly, for a contiguous face \bar{f} we obtain $(\bar{y}_2, -\bar{y}_1, -\bar{a}_3)$.

REMARK 11. This equality is logically independent of the assumption of continuity at p . These two expressions for the limit do not depend on the choice of x_1, x_2 but only on quantities determined by the values of the field \mathbf{z} on the respective collar faces. Since we can choose x_1, x_2 so that the point (x_1, x_2, x_3) goes to infinity on the shared edge of the two contiguous collar faces to obtain the limits simultaneously the two limits are equal and, thus, the limit in (8) exists independently of the assumption of continuity at p . An equivalent argument can be given without using the Darboux transformation.

Therefore $y_2, -y_1, -a_3$ are constant on a given collar and we may write $\mathbf{z}'(p) = (y_2, -y_1, -a_3)$.

We now seek conditions at p which will insure that $\mathbf{c} \cdot \mathbf{z} = \text{const.}$ on a collar. Taking the dot product of (1) with \mathbf{c} we obtain $\mathbf{c} \cdot \mathbf{z} = y_1x_2 - y_2x_1 + a_3$. Since y_1, y_2, a_3 are the same for all faces of a collar $\mathbf{c} \cdot \mathbf{z}$ is a linear function of x_1 and x_2 on the entire collar. Since the faces of a collar are not all parallel it follows that $\mathbf{c} \cdot \mathbf{z} = k = \text{const.}$ on a collar if and only if $y_1 = y_2 = 0$ and $a_3 = k$. Comparing this with the expression for the velocity $\mathbf{z}'(p)$ from the preceding paragraph we find that a necessary and sufficient condition that $\mathbf{c} \cdot \mathbf{z} = k = \text{const.}$ on a collar (and, hence, on its common boundary with the cap) is that $\mathbf{z}'(p) = (0, 0, -k)$.

Thus, the condition that $\mathbf{c} \cdot \mathbf{z}$ be constant on a collar (and hence on its boundary with the cap) is equivalent to the mechanical

Condition 1. The vertex at p is constrained to slide on a rod through p parallel to \mathbf{c} at time zero.

Suppose C has more than one boundary component with collars attached. Let p_i denote the vertices of the cones corresponding to the collars under D_1 . Although the p_i coincide at the origin they are separate points on the surface allowed to have different velocities $\mathbf{z}'(p_i) = (0, 0, a_i)$.

We summarize the above discussion so far as

THE SLIDING LEMMA 12. *Let C be a cap with Alexandrov refinement C' , satisfying the hemisphere condition $\mathbf{c} \cdot \mathbf{n} > 0$, with infinite cylindrical polyhedral collars whose faces are parallel to \mathbf{c} attached to each boundary component. Let \mathbf{z}' be a motion on the completed Darboux transform $D_1(C' + \text{collars})$. If the vertices p_i are constrained to slide independently on a rod through O parallel to \mathbf{c} then \mathbf{z}' is a rigid motion on $D_1(C' + \text{collars})$.*

A natural constraint, perhaps simpler than the sliding constraint of this lemma, is

Condition 2. The p_i are identified at p , with a single velocity, i.e., are mechanically fastened together, but are not otherwise fixed. This means that the instantaneous velocities $\mathbf{z}'(p_i)$ at the vertices p_i are all equal, and $\mathbf{c} \cdot \mathbf{z}'(p_i) = \mathbf{c} \cdot \mathbf{z}'(p) = y_1 x_2 - y_2 x_1 + a = k$. This implies that the rotation vectors at all points on all collars are of the form $\mathbf{y} = (y_1, y_2, u)$ with y_1, y_2 constant. Subtracting a rigid motion whose rotation vector is $(-y_1, -y_2, 0)$ from \mathbf{z} we obtain a motion \mathbf{z}'' on the cap-with-collars whose rotation vector is parallel to $\mathbf{c} = (0, 0, 1)$ on all collars. That is, $\mathbf{c} \cdot \mathbf{z}''$ is constant on the collars. Thus we have

THEOREM C. *Let C be a cap with Alexandrov refinement C' , satisfying the hemisphere condition $\mathbf{c} \cdot \mathbf{n} > 0$ with infinite cylindrical polyhedral collars, parallel to \mathbf{c} , attached to each boundary component. Let \mathbf{z}' be a motion on the completed Darboux transform $D_1(C' + \text{collars})$, with all the vertices p_i identified. Then \mathbf{z}' is a rigid motion on $D_1(C' + \text{collars})$.*

EXAMPLE 6. Figure 3 illustrates a cap with two collars, which is infinitesimally rigid under this restriction.

The surfaces described in the Sliding Lemma and Theorem C may be described in a way independent of the transformation D_1 , analogous to the description in Theorem B'. We leave this to the reader.

Problem 13. The restrictions on the motions of the vertices p_i in the Sliding Lemma, or Theorem C, are necessitated by our methods. We do not have an example to show that these are essential to infinitesimal rigidity. It is possible, as far as we know, that Theorem C would be true without identification of the vertices p_i for distinct collars.

REMARK 14. Faces of a polyhedron have been isometric embeddings in E^3 of polygonal regions. The concept of polyhedron can be extended by taking as a *generalized face* of a polyhedron an oriented polyhedron (as defined in Section 2) immersed (but not necessarily embedded) in a plane (Figure 16A). If a generalized face has a single boundary component then the methods of the previous sections continue to apply. For instance

EXAMPLE 7. A polyhedral cap consisting of a single generalized face which is a 2-cell with strips (for example, the surface shown in Figure 16B) has a connected boundary containing all the vertices. Adding a cone or a cylinder as described above yields an infinitesimally rigid polyhedron. Since a two-cell with strips immersed in a plane is not a plane convex set it must occur as a boundary face. Several 2-cells with strips and other faces can be attached together along boundary segments in various ways to form a cap.

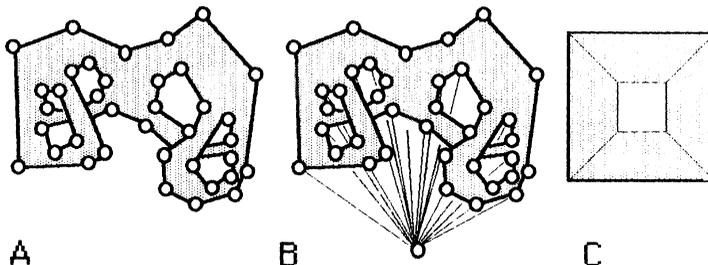


FIGURE 16

However, connected facial boundaries are essential to our method. If the boundary of a generalized face has more than one component then

the above theory fails. Figure 16C shows a refined cap with one ring-shaped face which is a square with a central square removed. Even if one of the boundary components is held fixed the other one can move infinitesimally perpendicular to the plane of the face. The reader can easily find explicit formulas for such a nonrigid motion.

7. Informal remarks on related topics. The 1-skeleton or *framework* of a polyhedron in E^3 may be thought of as a set of rigid bars with joints at their endpoints. Two bars with endpoints at the same joint are allowed to rotate freely relative to each other. The framework of a polyhedron is *infinitesimally rigid* if at any instant any velocity field on the framework which is a rigid motion (i.e., of form (1)) on each bar coincides with the velocity field of a rigid motion of E^3 .

It is easy to see that a framework consisting of a triangle with joints at its vertices and bars for its sides is infinitesimally rigid and that therefore any velocity field on it must coincide with a rigid motion of the ambient space. The restriction of this motion to the triangular region of the triangle is thus of form (1). Suppose now that a velocity field is given on the framework of a polyhedron (or refined polyhedron) all of whose faces are triangles which is a rigid motion on each bar. The velocity field extends to a motion on the polyhedron which is a rigid motion on each triangular face. Thus any polyhedron with triangular faces which is infinitesimally rigid (or infinitesimally rigid with respect to certain boundary conditions, such as glidebending) has a framework which is infinitesimally rigid (resp. infinitesimally rigid with respect to the corresponding boundary conditions on the framework). In particular, each of our examples of bounded infinitesimally rigid polyhedra above, if triangulated, gives rise to an infinitesimally rigid framework.

Conversely, if the framework of a polyhedron with arbitrary polyhedral faces is infinitesimally rigid then obviously the polyhedron itself is infinitesimally rigid. Thus, for triangulated polyhedra infinitesimal rigidity and infinitesimal rigidity of frameworks are equivalent. Roth [**R**, **R1**] has shown, using methods related to those in Gluck [**G**], that the framework of a polyhedron which is convex (in the large) is infinitesimally rigid if and only if the faces of the polyhedron are triangles.

We have shown that allowing new vertices on the edges of any face and allowing the face to crease along diagonals joining new or old vertices does not destroy the infinitesimal rigidity of our polyhedra, provided diagonals do not intersect in the interior of a face (see [**W**₂]

for a general theorem on such refinements). If we place new vertices in the interior of a face and diagonalize then the result may (Figure 17A) or may not (Figure 17B) destroy the infinitesimal rigidity of the polyhedron.

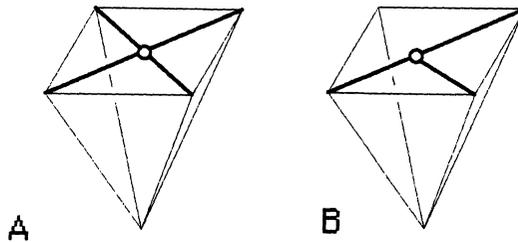


FIGURE 17

W. Whiteley has made extensive studies of the motions of frameworks and polyhedra using methods of statics $[W_2]$, $[W_3]$, $[W_5]$. In $[W_5]$ he discusses, in these terms, the effect on the infinitesimal rigidity of infinitesimally rigid polyhedra, and their frameworks, of adding new vertices and diagonals and gives necessary and sufficient conditions (one of which uses a refinement of a theorem of Clerk Maxwell—see $[W_1]$ and $[CW_2]$) under which vertices may appear in the faces of such polyhedra while maintaining infinitesimal rigidity.

Whiteley has shown that if a polyhedron with a triangulated refinement is infinitesimally rigid, then the polar polyhedron is also infinitesimally rigid with a triangulated refinement $[W_3]$. This polarity, applied to our examples from Sections 5 and 6, creates other examples for our growing collection of infinitesimally rigid polyhedra (Figure 18).

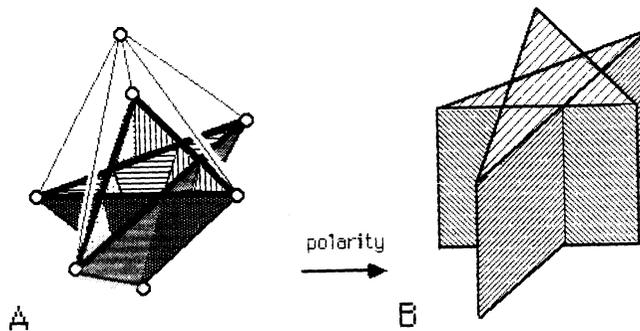


FIGURE 18

A polyhedron is *rigid* (as contrasted with infinitesimally rigid) if any continuous deformation which moves its faces to congruent faces

moves the entire polyhedron to a congruent polyhedron. Thus rigidity is concerned with motion over an interval of the deformation parameter, say time, while infinitesimal rigidity is concerned with instantaneous velocity, at time zero, and no actual motion need occur. It is known that infinitesimal rigidity implies rigidity [AR]. However, vertices of a triangulation of a polyhedron which lie in the interior of a natural face destroy infinitesimal rigidity so that the possibility remains open that such a triangulated polyhedron is not rigid.

Connelly [C₂] has eliminated this possibility by showing that a convex polyhedron triangulated in any way (including with vertices in the interior of natural faces) is second-order rigid and that this implies that it is rigid (as is its framework). His methods apply to retriangulations, with new interior vertices, of convex faces of any infinitesimally rigid polyhedral framework (e.g., Figure 17A). This shows the second-order rigidity of our polyhedral frameworks, with facial vertices added inside any convex (e.g., any nonboundary) faces, and the face retriangulated with bars disjoint in their interior.

Finally we recall Connelly's striking example of a non-rigid, i.e., flexible, closed polyhedron embedded in E^3 [C₁]. It is homeomorphic to a sphere and is, of course, nonconvex. Thus, there are essential limits to the class of closed polyhedra which are rigid.

REFERENCES

- [A] A. D. Alexandrov, *Konvexe Polyeder*, Akademie-Verlag, Berlin, 1958.
- [AZ] A. D. Alexandrov and V. A. Zalgaller, *Intrinsic Geometry of Surfaces*, Translations of Mathematical Monographs, Vol. 15, American Mathematical Society, Providence, 1967.
- [AR] L. Asimov and B. Roth, *The rigidity of graphs*, II, J. Math. Anal. and Appl., **68** (1979), 171–190.
- [B] H. Busemann, *Convex Surfaces*, Interscience, New York, 1958.
- [C₁] R. Connelly, *A counterexample to the rigidity conjecture for polyhedra*, Inst. Hautes Etudes Sci. Publ. Math., **47** (1978), 333–338.
- [C₂] ———, *The rigidity of certain cabled frameworks and the second order rigidity of arbitrarily triangulated convex surfaces*, Advances in Math., **37** (1980), 272–299.
- [CW₁] H. Crapo and W. Whiteley, *Statics of frameworks and motions of panel structures, a projective geometric introduction*, Structural Topology, **6** (1982), 43–82.
- [CW₂] ———, *Plane stresses and projected polyhedra*, preprint, Champlain Regional College, 900 Riverside Drive, St. Lambert, Quebec, J4P-3B8, 1985.
- [Da] G. Darboux, *Theorie General des Surfaces*, vol. III, Paris 1894, vol. IV, Paris, 1896.

- [D] M. Dehn, *Über die Starrheit konvexer Polyeder*, Math. Ann., **77** (1916), 466–473.
- [E] N. W. Efimov, *Flächenverbiegung im Grossen mit einem Nachtrag von E. Rembs und K. P. Grottemeyer*, Akademie-Verlag, Berlin, 1957.
- [E₁] ———, *Qualitative problems of the theory of deformations of surfaces*, Transl. Amer. Math. Soc., Series 1, Vol. 6: Differential Geometry and the Calculus of Variations, (1962), 274–423, (English translation of the first part of the above).
- [F] A. Fogelsanger, *The generic rigidity of minimal cycles*, Ph.D. Thesis, Cornell University, May 1988.
- [G] H. Gluck, *Almost all simply connected closed surfaces are rigid*, Geometric Topology, Lecture Notes in Math., Vol. 438, Springer-Verlag, Berlin, 1975, 225–239.
- [K₁] E. Kann, *A new method for infinitesimal rigidity of surfaces with $K > 0$* , J. Differential Geom., **4** (1970), 5–12.
- [K₂] ———, *An elementary proof of a finite rigidity problem by infinitesimal rigidity methods*, Proc. Amer. Math. Soc., **60** (1976), 252–258.
- [K₃] ———, *Glidebending of general caps: An infinitesimal treatment*, Proc. Amer. Math. Soc., **84** (1982), 247–255.
- [Ko] D. Koutroufiotis, *Monotopy of convex caps*, Archiv der Mathematik, **XXVII** (1976), 657–662.
- [L] S. Lefschetz, *Introduction to Topology*, Princeton University Press, Princeton, 1949.
- [Ly] L. A. Lusternik, *Convex Figures and Polyhedra*, D. C. Heath and Co., Boston, 1966.
- [M₁] E. Moise, *Elementary Geometry from an Advanced Standpoint*, Second Ed., Addison-Wesley, Reading, 1974.
- [M₂] ———, *Geometric Topology in Dimensions 2 and 3*, Springer-Verlag, 1977.
- [MR] T. Minagawa and T. Rado, *On the infinitesimal rigidity of surfaces*, Osaka Math. J., **4** (1952), 241–285.
- [N] L. Nirenberg, *Rigidity of a class of closed surfaces*, Nonlinear Problems, The University of Wisconsin Press, 1963, 177–193.
- [P] A. V. Pogorelov, *Extrinsic Geometry of Convex Surfaces*, Transl. of Math. Monographs, Vol. 35, Amer. Math. Soc., Providence, 1973.
- [Ra] W. J. M. Rankine, *On the Application of Barycentric Perspective to the Transformation of Structures*, Phil. Mag. Series, **4** **26** (1863), 387–388.
- [R] B. Roth, *Rigid and flexible frameworks*, Amer. Math. Monthly, **88** (1981), 6–21.
- [R1] ———, *Rigidity of Convex Surfaces*, Chapter 4 in “The Geometry of Rigid Structures”, H. Crapo and W. Whiteley eds. to appear.
- [RW] B. Roth and W. Whiteley, *Tensegrity frameworks*, Trans. Amer. Math. Soc., **265** (1981), 419–446.
- [W₁] W. Whiteley, *Motions, stresses and projected polyhedra*, Structural Topology, **7** (1982), 13–38.
- [W₂] ———, *Infinitesimally rigid polyhedra I: Statics of frameworks*, Trans. Amer. Math. Soc., **285** (1984), 431–465.
- [W₃] ———, *Rigidity and polarity I: Statics of sheet structures*, Geom. Dedicata, **22** (1987), 329–362.
- [W₄] ———, *Infinitesimally rigid polyhedra II: Modified spherical frameworks*, Trans. Amer. Math. Soc., **306** (1988), 115–137.

[W₅] W. Whiteley *Infinitesimally Rigid Polyhedra IV: Motions of Hinged-Panel Manifolds*, preprint Champlain Regional College, 900 Riverside Drive, St. Lambert, Quebec, J4P-3P2 1986.

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