PACIFIC JOURNAL OF MATHEMATICS
Vol. 150, No. 2, 1991

THE REMAINDER TERMS ASPECT OF THE THEORY OF THE RIEMANN ZETA-FUNCTION

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Assuming the Riemann hypothesis for Riemann zeta-function $\zeta(s)$, let $R(u)$ and $S(t)$ denote the remainder terms for the prime number theorem (suitably normalized) and the zero counting formula for $\zeta(s)$ respectively. We analyze the relation between $R(u)$ and $S(t)$, which generalizes A. Guinand's work.

0. Introduction. In this paper, the general form of transform T on $L^2(0, \infty)$ defined by a kernel φ is given by

$$Tf(t) = \lim_{U \to \infty} \int_0^U f(u)\varphi(tu)\, du \quad \text{in } L^2(0, \infty)$$

for $f \in L^2(0, \infty)$. Throughout this paper, we assume the Riemann hypothesis for the Riemann zeta-function $\zeta(s)$. It is concerned with the remainder terms $R(u)$ in the prime number theorem (suitably normalized) and $S(t)$ in the zero counting formula for the Riemann zeta-function; see (2.1) and (2.4). Gallagher's version [**2**, Theorem 1] of the Guinand's summation formula, (0.0) below, gives a symmetric relation between $R(u)$ and $S(t)$:

$$\int_0^\infty f(u)\, dR(u) = \int_0^\infty \hat{f}(t)\, dS(t) \tag{0.0}$$

where $\hat{f}(t) = \int_0^\infty f(u) \cos tu\, du$, for a suitable class of functions $f(u)$ with "good" growth condition on $f(u)$. In particular, it is possible to take as $f(\cdot)$ in (0.0) the characteristic function of the interval $[0, u]$, and on taking integration by parts, we get

$$\sqrt{\frac{2}{\pi}} \frac{R(u)}{u} = -\int_0^\infty \frac{S^*(t)}{t} k(tu)\, dt \tag{0.1}$$

where

$$S^*(t) = S(t) - S(0^+) \quad \text{and} \quad k(\theta) = \sqrt{\frac{2}{\pi}}\, \theta \frac{d}{d\theta} \frac{\sin\theta}{\theta}.$$

A. Guinand [3] observed that $k(\theta)$ is the kernel of a Hankel transform of order 3/2, which is an involution on $L^2(0, \infty)$. So we have

$$-\frac{S^*(t)}{t} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{R(u)}{u} k(tu)\, du, \tag{0.2}$$

since $S(t) = O(\log t / \log\log t)$ due to Littlewood [6], and hence $S^*(t)/t \in L^2(0, \infty)$.

Equations (0.1) and (0.2) suggest a symmetric relation between $R(u)$ and $S(t)$. For a function $f(x)$, the Riemann-Liouville fractional integral is defined by

$$I_\alpha f(x) = f_\alpha(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-y)^{\alpha-1} f(y)\, dy & (\alpha > 0), \\ f(x) & (\alpha = 0); \end{cases} \tag{0.3}$$

and we have

$$I_{\alpha+\beta} f(x) = I_\alpha I_\beta f(x)$$

for $\alpha \geq 0$, $\beta \geq 0$. For α an integer $f_\alpha(x)$ turns out to be an iterated integral. As a generalization of Guinand's work, we consider the Riemann-Liouville fractional integral on $R(u)$ and $S(t)$, and construct the kernel

$$K(\alpha, m, \theta) = \sqrt{\frac{2}{\pi}} \theta^{\alpha+1} \left(\frac{d}{d\theta}\right)^{m+1} \frac{\sin_\alpha(\theta)}{\theta^{\alpha+1}} \qquad (\theta > 0)$$

where m is an integer and α is real, and the equation similar to (0.1)

$$\sqrt{\frac{2}{\pi}} \frac{R_\alpha(u)}{u^{m+1}} = (-1)^{m+1} \int_0^\infty \frac{S_m^*(t)}{t^{\alpha+1}} K(\alpha, m, tu)\, dt. \tag{0.4}$$

Note that $S_1(t) = O(\log t / (\log\log t)^2)$ due to Littlewood [6], and hence

$$S_m^*(t) = S_m(t) - \frac{1}{m!} S(0^+) t^m = O(t^m) \qquad (t \to +\infty) \tag{0.5}$$

for $m \geq 1$, and

$$S_m^*(t) = O(t^{m+1}) \qquad (t \to 0^+) \tag{0.6}$$

for $m \geq 0$.

We give first in Theorem 1 the conditions on α, m for which equation (0.4) holds good, and then in Theorem 2 the conditions on α, m for which $K(\alpha, m, \theta)$ defines a bounded and invertible transform on $L^2(0, \infty)$. In particular, $K(m, m, \theta)$ for $m \geq -1$ is an involution on $L^2(0, \infty)$. Thus corresponding to (0.2) we will be able to show in Theorem 3

$$\frac{S_\alpha^*(t)}{t^{m+1}} = (-1)^{m+1}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{R_m(u)}{u^{\alpha+1}}K(\alpha, m, tu)\,du \tag{0.7}$$

for suitable α and m, and furthermore

$$\int_0^\infty \left|\frac{S_m^*(t)}{t^{m+1}}\right|^2 dt = \frac{2}{\pi}\int_0^\infty \left|\frac{R_m(u)}{u^{m+1}}\right|^2 du \tag{0.8}$$

for $m = 0, 1, 2, 3, \ldots$.

1. Averaging operators. To pursue the object of this paper, we have to introduce the averaging operator.

For $f \in L^2(0, \infty)$, we define the transforms A_α and B_β by

$$A_\alpha f(x) = x^{-\alpha}\int_0^x y^{\alpha-1}f(y)\,dy \qquad (\alpha > \tfrac{1}{2}),$$

$$B_\beta f(x) = x^{-\beta}\int_x^\infty y^{\beta-1}f(y)\,dy \qquad (\beta < \tfrac{1}{2}).$$

LEMMA 1.1. *The A_α and B_β are bounded operators on $L^2(0, \infty)$ and*

$$\|A_\alpha\|_2 \le (\alpha - \tfrac{1}{2})^{-1}, \quad \|B_\beta\|_2 \le (\tfrac{1}{2} - \beta)^{-1}.$$

Proof. By (9.9.8) and (9.9.9) of [4], the results follow immediately.

LEMMA 1.2. *For all values of α and β for which A_α and B_β are defined, we have*

$$A_\alpha B_\beta = \frac{B_\beta + A_\alpha}{\alpha - \beta} = B_\beta A_\alpha,$$

$$A_\alpha A_\beta = \frac{A_\beta - A_\alpha}{\alpha - \beta} = A_\beta A_\alpha \qquad (\alpha \ne \beta),$$

$$B_\alpha B_\beta = \frac{B_\beta - B_\alpha}{\beta - \alpha} = B_\beta B_\alpha \qquad (\alpha \ne \beta).$$

Proof. Note first that the unitary involution J, defined by

$$Jf(x) = \frac{1}{x}f\left(\frac{1}{x}\right),$$

intertwines the operators A_α and $B_{1-\alpha}$: $JA_\alpha = B_{1-\alpha}J$.

It suffices to prove the first two equations on the left. The first equation on the right then follows by conjugation by J, the second equation on the right by symmetry, and the third pair of equations follows from the second pair by conjugation by J.

For continuous $f(x)$ with compact support in $(0, \infty)$, we have

$$\begin{aligned}
A_\alpha B_\beta f(x) &= x^{-\alpha}\int_0^x y^{\alpha-1}y^{-\beta}\int_y^\infty z^{\beta-1}f(z)\,dz\,dy \\
&= x^{-\alpha}\left[\frac{y^{\alpha-\beta}}{\alpha-\beta}\int_y^\infty z^{\beta-1}f(z)\,dz\right]_{y=0}^{y=x} \\
&\quad + x^{-\alpha}\int_0^x \frac{y^{\alpha-\beta}}{\alpha-\beta}y^{\beta-1}f(y)\,dy, \\
A_\alpha A_\beta f(x) &= x^{-\alpha}\int_0^x y^{\alpha-1}y^{-\beta}\int_0^y z^{\beta-1}f(z)\,dz\,dy \\
&= x^{-\alpha}\left[\frac{y^{\alpha-\beta}}{\alpha-\beta}\int_0^y z^{\beta-1}f(z)\,dz\right]_{y=0}^{y=x} \\
&\quad - x^{-\alpha}\int_0^x \frac{y^{\alpha-\beta}}{\alpha-\beta}y^{\beta-1}f(y)\,dy
\end{aligned}$$

from which the first two equations on the left follows. The general cases then follow by continuity of bounded operators.

LEMMA 1.3. *The operator* $\mathrm{Id}-(2\alpha-1)A_\alpha$ *is unitary on* $L^2(0, \infty)$ *for each* $\alpha > \frac{1}{2}$.

Proof. We show first

$$A_\alpha^* = B_{1-\alpha} \quad \text{for } \alpha > 1/2. \tag{1.1}$$

It suffices by continuity to show that

$$(A_\alpha f, g) = (f, B_{1-\alpha}g)$$

for continuous functions f, g with compact support. The left is

$$\begin{aligned}
&\int_0^\infty x^{-\alpha}\int_0^x y^{\alpha-1}f(y)\,dy\,\overline{g}(x)\,dx \\
&\qquad = -\int_0^\infty\int_0^x y^{\alpha-1}f(y)\,d\int_x^\infty z^{-\alpha}\overline{g}(z)\,dz.
\end{aligned}$$

On integrating by parts, the integrated terms drop out, leaving

$$\begin{aligned}
&\int_0^\infty\int_x^\infty z^{-\alpha}\overline{g}(z)\,dz\,d\int_0^x y^{\alpha-1}f(y)\,dy \\
&\qquad = \int_0^\infty f(x)x^{\alpha-1}\int_x^\infty z^{-\alpha}\overline{g}(z)\,dz\,dx,
\end{aligned}$$

which is the right side.

Now for each real $\alpha > 1/2$, we have by (1.1)

$$(\mathrm{Id} - (2\alpha - 1)A_\alpha)^* = \mathrm{Id} - (2\alpha - 1)B_{1-\alpha}$$

and by Lemma 1.2

$$\begin{aligned}
&(\mathrm{Id} - (2\alpha - 1)B_{1-\alpha})(\mathrm{Id} - (2\alpha - 1)A_\alpha) \\
&\quad = \mathrm{Id} - (2\alpha - 1)B_{1-\alpha} - (2\alpha - 1)A_\alpha + (2\alpha - 1)^2 \frac{A_\alpha + B_{1-\alpha}}{2\alpha - 1} \\
&\quad = \mathrm{Id},
\end{aligned}$$

which proves Lemma 1.3.

Let h be a bounded measurable function on $(0, \infty)$. For each function $f \in L^1(0, \infty)$, we define

$$(Hf)(x) = \int_0^\infty f(y)h(xy)\,dy.$$

Provided in addition H is a bounded operator on $L^1(0, \infty) \cap L^2(0, \infty)$, i.e.

$$\|Hf\|_2 \ll \|f\|_2 \quad \text{for } f \in L^1(0, \infty) \cap L^2(0, \infty) \tag{1.2}$$

we may extend the operator H by continuity to a bounded operator on all of $L^2(0, \infty)$. In particular, in this case

$$(Hf)(x) = \lim_{Y \to \infty} \int_0^Y f(y)h(xy)\,dy \quad \text{in } L^2(0, \infty)$$

for each $f \in L^2(0, \infty)$.

LEMMA 1.4. *Let H be the bounded operator on $L^2(0, \infty)$ defined as above by a bounded measurable kernel h satisfying* (1.2). *Then for each $\alpha > 1/2$, we have*

$$A_\alpha H = HB_{1-\alpha}. \tag{1.3}$$

Moreover, the operator (1.3) *is defined as above by the kernel $A_\alpha h$.*

Proof. We observe first that $A_\alpha h$ is a bounded measurable function. Next, for $f \in L^1 \cap L^2$, we have

$$\begin{aligned}
A_\alpha Hf(x) &= x^{-\alpha} \int_0^x y^{\alpha-1} \int_0^\infty h(yz)f(z)\,dz\,dy \\
&= \int_0^\infty \left\{ x^{-\alpha} \int_0^x y^{\alpha-1} h(yz)\,dy \right\} f(z)\,dz
\end{aligned}$$

and

$$HB_{1-\alpha}f(x) = \int_0^\infty h(xy)y^{\alpha-1}\int_y^\infty z^{-\alpha}f(z)\,dz\,dy$$
$$= \int_0^\infty \left\{z^{-\alpha}\int_0^z y^{\alpha-1}h(xy)\,dy\right\} f(z)\,dz\,,$$

with both interchanges justified by Fubini's theorem. Each of the factors $\{\cdots\}$ reduces to $A_\alpha h(xz) = A_\alpha g(x)$ where $g(x) = h(xz)$.

Denoting by H_α the operator defined by $A_\alpha h$ on $L^1 \cap L^2$, we thus have

$$H_\alpha f = A_\alpha H f = HB_{1-\alpha} f \tag{1.4}$$

for $f \in L^1 \cap L^2$. For such f,

$$\|H_\alpha f\|_2 = \|A_\alpha H f\|_2 \le \frac{1}{\alpha - 1/2}\|Hf\|_2 \ll \|f\|_2\,,$$

with an implicit constant depending on α. It follows that H_α is a bounded operator on $L^1 \cap L^2$ and thus H_α extends by continuity to a bounded operator on L^2. By continuity, the equation (1.4) now holds for $f \in L^2$, giving the assertions of Lemma 1.4.

2. An explicit formula. It is known that $(s-1)\zeta(s)$ is an entire function and

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} \qquad (\operatorname{Re} s > 1) \tag{2.0}$$

where $\Lambda(n)$ is the von Mangoldt function.

Put

$$P(u) = -\frac{1}{2}{\sum_{n\le e^u}}' \Lambda(n) n^{-1/2}\,, \quad Q(u) = -2\sinh\frac{1}{2}u\,,$$

the prime on the summation here and in the following means if e^u is an integer, then the last term is weighted with $1/2$. A version of the prime number theorem states that $\sum_{n\le x}\Lambda(n) = x + O(x^{1/2}\log^2 x)$ see Davenport [**1**].

Define

$$R(u) = P(u) - Q(u)\,. \tag{2.1}$$

We have that $R(u) = O(u^2)$ which implies the prime number theorem with remainder term, and for $\alpha \geq 0$

$$(2.2) \qquad R_\alpha(u) = -\frac{1}{2}\frac{1}{\Gamma(\alpha+1)}\sum_{n\leq e^u}{}' \Lambda(n)n^{-1/2}(u-\log n)^\alpha + \sinh_\alpha\left(\frac{1}{2}u\right)\Big/\left(\frac{1}{2}\right)^{\alpha+1};$$

note that $I_\alpha(f)|_{cy} = c^\alpha I_\alpha(g)|_y$ where $g(y) = f(cy)$.

Now define for $t > 0$ not an ordinate of a zero of $\zeta(s)$

$$N(t) = \#\left\{\frac{1}{2}+i\gamma \,\middle|\, \zeta\left(\frac{1}{2}+i\gamma\right) = 0,\ 0 < \gamma \leq t\right\},$$

and $N(t) = \frac{1}{2}(N(t^+) + N(t^-))$ for t the imaginary part of a zero of $\zeta(s)$. By the argument principle, we see that for $T > 0$ not an ordinate of a zero of $\zeta(s)$

$$N(T) = \frac{1}{2\pi i}\int_{\Gamma_T}\frac{\zeta'}{\zeta}(s)\,ds$$

where Γ_T is the line running from $\infty + iT$ to $\frac{1}{2}+iT$ to $\frac{1}{2}-iT$ to $\infty - iT$, and a Cauchy principal value is taken at each zero of $\zeta(s)$ on Γ_T.

In view of $\zeta(\bar{s}) = \overline{\zeta(s)}$, we have

$$(2.3) \qquad N(T) = M(T) + S(T)$$

where

$$(2.4) \qquad M(T) = \frac{-1}{\pi}\int_0^T \operatorname{Re}\frac{\zeta'}{\zeta}(\tfrac{1}{2}+it)\,dt,$$

$$S(T) = \frac{1}{\pi}\int_\infty^{1/2}\operatorname{Im}\frac{\zeta'}{\zeta}(\sigma+iT)\,d\sigma = \frac{1}{\pi}\arg\zeta(\tfrac{1}{2}+iT).$$

The argument is defined by continuous horizontal movement from $\infty+iT$ to $\frac{1}{2}+iT$ starting with the value zero. Meanwhile, comparing (2.3) with the zero counting formula shown in Davenport [**1**], we see that

$$(2.5) \qquad M(t) = \frac{1}{\pi}\left[\arg\left(-\frac{1}{2}+it\right) + \arg(\pi^{-(1/2)(1/2+it)}) + \arg\Gamma\left(\frac{5}{4}+i\frac{t}{2}\right)\right]$$

$$= \frac{t}{2\pi}\left(\log\frac{t}{2\pi} - 1\right) + O(1) \quad \text{for large } t,$$

and

$$M'(t) \sim \frac{1}{2\pi} \log t. \tag{2.6}$$

Our first object is to construct the following "explicit formula."

PROPOSITION 1. *For $\alpha \geq 0$, we have*

$$R_\alpha(u) = \int_0^\infty \frac{\sin_\alpha(tu)}{t^{\alpha+1}}\, dS(t). \tag{Δ}$$

Before proceeding with the proof of Proposition 1, we need several lemmas.

LEMMA 2.1. *If $\alpha \geq 0$, $u > 0$, and $f(v) = e^{sv}$, then*

$$(I_\alpha f)(u) = \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} e^{sv}\, dv = \frac{e^{su}}{s^\alpha} + L(\alpha, s, u)$$

for s on the slit plane cut along the line from the origin to $-i\infty$; and s^α is defined by analytic continuation starting with $1^\alpha = 1$. Moreover, $(I_\alpha f)(u)$ is a holomorphic function on the entire plane of s with $L(\alpha, s, u) \ll_\alpha u^{\alpha-1}/|s| + (|\sigma u|^{\alpha-1} + 1)/|s|^\alpha$, $\sigma = \operatorname{Re} s$, and we define $L(0, s, u) \equiv 0$.

Proof. We start by considering, for $\alpha > 0$,

$$\begin{aligned}
e^{-su} \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} e^{sv}\, dv &= \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} e^{s(v-u)}\, dv \\
&= \frac{1}{s^\alpha} \frac{1}{\Gamma(\alpha)} \int_0^{su} v^{\alpha-1} e^{-v}\, dv \\
&= \frac{1}{s^\alpha} \frac{1}{\Gamma(\alpha)} \left\{ \Gamma(\alpha) - \int_{su}^{\sigma u} v^{\alpha-1} e^{-v}\, dv - \int_{\sigma u}^\infty v^{\alpha-1} e^{-v}\, dv \right\} \\
&= \frac{1}{s^\alpha} + O_\alpha\left(\frac{|su|^{\alpha-1}}{|s|^\alpha} e^{-\sigma u} + \frac{|\sigma u|^{\alpha-1} + 1}{|s|^\alpha} e^{-\sigma u} \right);
\end{aligned}$$

the estimate of this remainder term will be given at the end of this section. This proves Lemma 2.1 by taking, for $\alpha > 0$,

$$L(\alpha, s, u) = -\frac{1}{s^\alpha} \frac{e^{su}}{\Gamma(\alpha)} \left\{ \int_{su}^{\sigma u} v^{\alpha-1} e^{-v}\, dv + \int_{\sigma u}^\infty v^{\alpha-1} e^{-v}\, dv \right\}.$$

REMARK. We see by Lemma 2.1 that

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} \sinh sv\, dv &= (\sinh_\alpha s)(u) \\
&= \frac{1}{2} \left[\frac{e^{su}}{s^\alpha} + \frac{e^{-su}}{(-s)^\alpha} \right] + \widetilde{L}(\alpha, s, u)
\end{aligned} \tag{2.7}$$

where $\widetilde{L}(\alpha, s, u)$ is bounded by

$$O\left(\frac{u^{\alpha-1}}{|s|} + \frac{|\sigma u|^{\alpha-1}+1}{|s|^{\alpha}}\right),$$

and $\widetilde{L}(0, s, u) \equiv 0$.

LEMMA 2.2. *Consider, for $\alpha \geq 0$ and $u > 0$, $(\sinh_\alpha s)(u)$ as defined by (2.7) on the slit plane in Lemma 2.1. Then for $c > 0$, $\alpha \geq 0$, $\eta \geq 0$*

$$\lim_{T\to\infty} \frac{1}{\pi i}\int_{c-iT}^{c+iT} \frac{(\sinh_\alpha s)(u)}{s} e^{-s\eta}\, ds$$
$$= \begin{cases} \frac{1}{\Gamma(\alpha+1)}(u-\eta)^{\alpha} & (u > \eta), \\ 0 & (u < \eta) \text{ or } (u = \eta,\ \alpha > 0), \\ \frac{1}{2} & (u = \eta,\ \alpha = 0). \end{cases}$$

Proof. The result follows by (1.5.3) of Titchmarsh [7] and (2.7).

Proof of Proposition 1. Consider, for $c > 1$ and $T > 0$ not an ordinate of a zero of $\zeta(s)$,

$$J(T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{(\sinh_\alpha(s-\frac{1}{2}))(u)}{s-\frac{1}{2}}\frac{\zeta'}{\zeta}(s)\, ds;$$

note that $(\sinh_\alpha s)(u)/s$ is an even function of s.

Since $\frac{\zeta'}{\zeta}(s) \ll (\log|t|)^2$ for $s = \sigma + it (-\frac{1}{2} \leq \sigma)$ and a suitable sequence of t with infinity the limit of $|t|$, we have that by computation of residues and for large T

$$\begin{aligned} J(T) &= -\frac{\sinh_\alpha\left(\frac{1}{2}u\right)}{\left(\frac{1}{2}\right)^{\alpha+1}} - \int_0^T \frac{(\sin_\alpha t)(u)}{t}\, dM(t) \\ &\quad + \sum_{\substack{0<\gamma<T \\ \zeta(1/2+i\gamma)=0}} \frac{\sin_\alpha(\gamma u)}{\gamma^{\alpha+1}} + o(1) \\ &= -\frac{\sinh_\alpha\left(\frac{1}{2}u\right)}{\left(\frac{1}{2}\right)^{\alpha+1}} + \int_0^T \frac{\sin_\alpha(tu)}{t^{\alpha+1}}\, dS(t) + o(1), \end{aligned}$$

since $(\sin_\alpha t)(u)/t = \sin_\alpha(tu)/t^{\alpha+1}$. On the other hand, by using

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \Lambda(n) n^{-1/2} e^{-(s-1/2)\log n},$$

we have

$$J(T)=-\sum_{n=1}^{\infty}\Lambda(n)n^{-1/2}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\left(\sinh_\alpha\left(s-\frac{1}{2}\right)\right)(u)}{s-\frac{1}{2}}e^{-(s-1/2)\log n}\,ds.$$

By virtue of Lemma 2.2, (2.2), and making $T\to\infty$, we get

$$R_\alpha(u)=\int_0^\infty \frac{\sin_\alpha(tu)}{t^{\alpha+1}}\,dS(t).$$

This completes the proof of Proposition 1.

We now give the estimate of the remainder term in the proof of Lemma 2.1. It is based on the following

PROPOSITION 2. *If $s=\sigma+it$ is on the slit plane cut along the line from the origin to $-i\infty$, then, for $\alpha>0$,*

$$\int_{\sigma+it}^{\infty+it} v^{\alpha-1}e^{-v}\,dv=O_\alpha(e^{-\sigma}(|s|^{\alpha-1}+1))$$

with $t\geq 0$, or $t\leq 0$ and $\sigma>0$, and

$$\int_{\sigma+it}^{-\infty+it} v^{\alpha-1}e^{v}\,dv=O_\alpha(e^{-\sigma}(|s|^{\alpha-1}+1))$$

with $t\geq 0$, or $t\leq 0$ and $\sigma<0$.

Proof. We only prove the first assertion, and the second assertion will follow by a similar argument.

If $-1\leq\sigma\leq 1$, then, for $|t|<1$,

$$\int_{\sigma+it}^{\infty+it} v^{\alpha-1}e^{-v}\,dv=O(1)=O(e^{-\sigma}),$$

and for $|t|>1$, by integrating by parts $[\alpha]$ times,

$$\begin{aligned}\int_{\sigma+it}^{\infty+it} v^{\alpha-1}e^{-v}\,dv &= -e^{-v}v^{\alpha-1}\Big|_{v=\sigma+it}^{v=\infty+it}+(\alpha-1)\int_{\sigma+it}^{\infty+it}v^{\alpha-2}e^{-v}\,dv\\ &=O(e^{-\sigma}|s|^{\alpha-1})+O\left(\int_{\sigma+it}^{\infty+it}|v|^{\alpha-[\alpha]-2}e^{-v}\,dv\right)\\ &=O(e^{-\sigma}|s|^{\alpha-1})+O\left(e^{-\sigma}\int_{\sigma+it}^{\infty+it}|v|^{\alpha-[\alpha]-2}\,dv\right)\\ &=O(e^{-\sigma}|s|^{\alpha-1}).\end{aligned}$$

So, for $-1 \le \sigma \le 1$, we have

$$\int_{\sigma+it}^{\infty+it} v^{\alpha-1}e^{-v}\,dv = O(e^{-\sigma}(|s|^{\alpha-1}+1)). \tag{2.8}$$

If $1 < \sigma$, then $|s| > 1$ and

$$\begin{aligned}
\int_{\sigma+it}^{\infty+it} v^{\alpha-1}e^{-v}\,dv &= -e^{-v}v^{\alpha-1}\Big|_{v=\sigma+it}^{v=\infty+it} + (\alpha-1)\int_{\sigma+it}^{\infty+it} v^{\alpha-2}e^{-v}\,dv \\
&= O(e^{-\sigma}|s|^{\alpha-1}) + O\left(\int_{\sigma+it}^{\infty+it} |v|^{\alpha-[\alpha]-2}e^{-v}\,dv\right) \\
&= O(e^{-\sigma}|s|^{\alpha-1}) + O\left(e^{-\sigma}\int_{\sigma+it}^{\infty+it} |v|^{\alpha-[\alpha]-2}\,dv\right) \\
&= O(e^{-\sigma}|s|^{\alpha-1}).
\end{aligned}$$

Finally if $\sigma < -1$, then $|s| > 1$ and

$$\begin{aligned}
\int_{\sigma+it}^{\infty+it} v^{\alpha-1}e^{-v}\,dv &= \int_{\sigma+it}^{-1+it} v^{\alpha-1}e^{-v}\,dv + \int_{-1+it}^{\infty+it} v^{\alpha-1}e^{-v}\,dv \\
&= -e^{-v}v^{\alpha-1}\Big|_{v=\sigma+it}^{v=-1+it} + (\alpha-1)\int_{\sigma+it}^{-1+it} v^{\alpha-2}e^{-v}\,dv \\
&\quad + O(e^{-\sigma}(|1+it|^{\alpha-1}+1)), \qquad \text{by (2.8)} \\
&= O(e^{-\sigma}(|s|^{\alpha-1}+|1+it|^{\alpha-1})) \\
&\quad + O\left(e^{-\sigma}\int_{\sigma+it}^{-1+it} |v|^{\alpha-2}\,dv\right) \\
&\quad + O(e^{-\sigma}(|1+it|^{\alpha-1}+1)) \\
&= O(e^{-\sigma}(|s|^{\alpha-1}+|1+it|^{\alpha-1}+1)) \\
&= O(e^{-\sigma}(|s|^{\alpha-1}+1)).
\end{aligned}$$

This completes the proof of the first assertion of Proposition 2.

Now we obtain from Proposition 2

$$\int_{\sigma u}^{\infty} v^{\alpha-1}e^{-v}\,dv = O(e^{-\sigma u}(|\sigma u|^{\alpha-1}+1))$$

needed in the proof of Lemma 2.1.

As for the following quantity

$$\int_{su}^{\sigma u} v^{\alpha-1}e^{-v}\,dv,$$

we apply the Cauchy integral theorem and get for $\sigma > 0$, or $\sigma < 0$ and $t > 0$,

$$\begin{aligned}\int_{su}^{\sigma u} v^{\alpha-1}e^{-v}\,dv &= \int_{su}^{\infty+itu} v^{\alpha-1}e^{-v}\,dv + \int_{\infty}^{\sigma u} v^{\alpha-1}e^{-v}\,dv\\ &= O(e^{-\sigma u}(|su|^{\alpha-1}+|\sigma u|^{\alpha-1}+1)),\end{aligned}$$

by Proposition 2; and if $t < 0$ and $\sigma < 0$, then by the Cauchy integral theorem

$$\begin{aligned}\int_{su}^{\sigma u} v^{\alpha-1}e^{-v}\,dv &= (-1)^{\alpha-1}\int_{-\sigma u}^{-su} v^{\alpha-1}e^{v}\,dv\\ &= (-1)^{\alpha-1}\left\{\int_{-\infty}^{-\sigma u} v^{\alpha-1}e^{v}\,dv + \int_{-su}^{-\infty+itu} v^{\alpha-1}e^{v}\,dv\right\}\\ &= O(e^{-\sigma u}(|su|^{\alpha-1}+|\sigma u|^{\alpha-1}+1));\end{aligned}$$

by the second assertion of Proposition 2.

Hence we have given the estimates in the proof of Lemma 2.1.

3. Theorems. Consider equation (Δ) in Proposition 1. On taking integration by parts $m+1$ times, we get formally

$$\sqrt{\frac{2}{\pi}}\frac{R_\alpha(u)}{u^{m+1}} = (-1)^{m+1}\int_0^\infty \frac{S_m^*(t)}{t^{\alpha+1}}K(\alpha, m, tu)\,dt \tag{3.1}$$

where

$$K(\alpha, m, \theta) = \sqrt{\frac{2}{\pi}}\theta^{\alpha+1}\left(\frac{d}{d\theta}\right)^{m+1}\frac{\sin_\alpha(\theta)}{\theta^{\alpha+1}} \qquad (\theta > 0). \tag{3.2}$$

Since for $\alpha \geq 0$, by considering the Taylor series expansion of $\sin\theta$,

$$\sin_\alpha\theta = \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(2n+2+\alpha)}\theta^{2n+1+\alpha} \qquad (\theta > 0), \tag{3.3}$$

and the series on the right-hand side is defined for any real $\alpha \in \mathbb{R}$, we define (3.2) for any real $\alpha \in \mathbb{R}$.

In this section, we give suitable conditions on real α and integer m for which $K(\alpha, m, \theta)$ defines a bounded and invertible transform on $L^2(0, \infty)$. In view of $S(t) = O(\log t/\log\log t)$, (0.5), and (0.6), we see that

$$\frac{S_m^*(t)}{t^{\alpha+1}} \in L^2(0, \infty) \tag{3.4}$$

for $\frac{1}{2} < \alpha - m + 1 < \frac{3}{2}$, $m \geq 0$, $\alpha \geq 0$.

We show first

THEOREM 1. *Equation* (3.1) *holds for* $\alpha > m-1$, $\alpha \geq 0$, $m = 0, 1, 2, 3, \ldots$.

The following lemma is helpful in the proof of Theorem 1.

LEMMA 3.1. *For a function* $f(x)$ *on* $[0, \infty)$, *put*

$$M(\alpha, x) = \sup_{0 \leq t \leq x} |f_\alpha(t)|.$$

Then the following estimate holds uniformly in α:

$$|f_\alpha(t)| \ll M^{1-\alpha}(0, t) \cdot M^\alpha(1, t) \qquad (0 \leq \alpha \leq 1).$$

Proof. Note first that if either $g(x)$ or $h(x)$ is monotonic on $[a, b]$, then

$$\int_a^b g(x)\, dh(x) \ll \sup_{a \leq x \leq b} |g(x)| \cdot \sup_{a \leq x \leq b} |h(x)|.$$

We may suppose $f(x)$ is non-constant.

Now for $0 < \lambda < x$

$$\begin{aligned} f_\alpha(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y)\, dy \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{x-\lambda} (x-y)^{\alpha-1} f(y)\, dy \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{x-\lambda}^x (x-y)^{\alpha-1} f(y)\, dy \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{x-\lambda} (x-y)^{\alpha-1}\, df_1(y) \\ &\quad - \frac{1}{\Gamma(\alpha+1)} \int_{x-\lambda}^x f(y)\, d(x-y)^\alpha \\ &\ll \alpha\lambda^{\alpha-1} M(1, x) + \lambda^\alpha M(0, x) \\ &\ll M^{1-\alpha}(0, x) M^\alpha(1, x), \end{aligned}$$

by taking $\lambda = \alpha M(1, x)/M(0, x) < x$.

This proves Lemma 3.1.

Proof of Theorem 1. It suffices to show that the integral constant

$$S_m^*(t) \left(\frac{d}{dt}\right)^m \frac{\sin_\alpha(tu)}{t^{\alpha+1}}\Bigg|_{t=0}^{t=\infty} = 0 \tag{3.5}$$

for $\alpha > m-1$, $\alpha \geq 0$, $m \geq 0$. Lemma 3.1 and the power series (3.3) give

$$\sin_\beta(\theta) \ll \begin{cases} \max(\theta^{\beta-1}, 1), & \theta \to +\infty, \\ \theta^{\beta+1}, & \theta \to 0^+ \end{cases} \tag{3.6}$$

for all $\beta \in \mathbb{R}$. Thus, the Leibniz formula yields

$$\begin{aligned} \left(\frac{d}{dt}\right)^m \frac{\sin_\alpha(tu)}{t^{\alpha+1}} &\ll \sum_{j=0}^{m} \binom{m}{j} |\sin_{\alpha-j}(tu)| \cdot t^{-\alpha-1-m+j} \\ &\ll \max\{t^{-m-2}, t^{-\alpha-1}\} \qquad (t \to +\infty), \end{aligned} \tag{3.7}$$

and the power series (3.3) gives

$$\left(\frac{d}{dt}\right)^m \frac{\sin_\alpha(tu)}{t^{\alpha+1}} \ll 1 \qquad (t \to 0^+). \tag{3.8}$$

Now (3.5) follows immediately from (0.5), (0.6), and (3.7), (3.8). This proves Lewmma 3.2 Theorem 1.

We next show the following.

THEOREM 2. *The kernel* (3.2) *defines a bounded and invertible transform on* $L^2(0, \infty)$ *for* $m - \frac{1}{2} < \alpha < m + \frac{3}{2}$, $m = -1, 0, 1, 2, \ldots$. *In particular, for* $\alpha = m = -1, 0, 1, 2, \ldots$, *it defines an involution on* $L^2(0, \infty)$.

The proof of Theorem 2 is based on the following lemma.

LEMMA 3.2. *We have*

$$K(\alpha+1, m+1, \theta) = (\mathrm{Id} - (m+\alpha+3)A_{m+2})K(\alpha, m, \theta)$$

for $m + \alpha + 3 > 0$, $-\infty < \alpha < +\infty$, $m = -1, 0, 1, 2, 3, \ldots$.

Proof. Recall (3.2) and (3.3). We see that

$$K(\alpha, m, \theta) = \sqrt{\frac{2}{\pi}}\theta^{\alpha+1} \left(\frac{d}{d\theta}\right)^{m+1} \sum_{l=0}^{\infty} \frac{(-1)^l}{\Gamma(2l+\alpha+2)}\theta^{2l}.$$

Thus

$$
\begin{aligned}
&(m+\alpha+3)A_{m+2}K(\alpha\,, m\,, \theta)\\
&\quad = \sqrt{\frac{2}{\pi}}(m+\alpha+3)\theta^{-m-2}\int_0^\theta \eta^{m+1}\eta^{\alpha+1}\left(\frac{d}{d\eta}\right)^{m+1}\\
&\qquad \cdot \sum_{l=0}^{\infty}\frac{(-1)^l}{\Gamma(2l+\alpha+2)}\eta^{2l}\,d\eta\\
&\quad = \sqrt{\frac{2}{\pi}}\theta^{-m-2}\left\{\eta^{m+\alpha+3}\left(\frac{d}{d\eta}\right)^{m+1}\sum_{l=0}^{\infty}\frac{(-1)^l}{\Gamma(2l+\alpha+2)}\eta^{2l}\Bigg|_{\eta=0}^{\eta=\theta}\right.\\
&\qquad\qquad \left. - \int_0^\theta \eta^{m+\alpha+3}\left(\frac{d}{d\eta}\right)^{m+2}\sum_{l=0}^{\infty}\frac{(-1)^l}{\Gamma(2l+\alpha+2)}\eta^{2l}\,d\eta\right\}\\
&\quad = K(\alpha\,, m\,, \theta)\\
&\qquad - \sqrt{\frac{2}{\pi}}\theta^{-m-2}\int_0^\theta \eta^{m+\alpha+3}\left(\frac{d}{d\eta}\right)^{m+2}\sum_{l=0}^{\infty}\frac{(-1)^l}{\Gamma(2l+\alpha+2)}\eta^{2l}\,d\eta\\
&\quad = K(\alpha\,, m\,, \theta) - K(\alpha+1\,, m+1\,, \theta)\,.
\end{aligned}
$$

This proves Lemma 3.2.

Proof of Theorem 2. Note that $K(\alpha\,, -1\,, \theta) = \sqrt{\frac{2}{\pi}}\sin_\alpha(\theta)$ and $K(-1\,, -1\,, \theta) = \sqrt{\frac{2}{\pi}}\cos\theta$. Lemma 3 and Theorem 1 of Kueh [5] show that $K(\alpha\,, -1\,, \theta)$ $(-\frac{3}{2} < \alpha < \frac{1}{2})$ defines a bounded and invertible transform on $L^2(0\,, \infty)$. Thus Theorem 2 follows immediately from Lemma 3.2 and Lemmas 1.1, 1.2, 1.3, 1.4. We see also that the kernel $K^{-1}(\alpha\,, m\,, \theta)$ of the inverse of the transform defined by $K(\alpha\,, m\,, \theta)$ satisfies, for the same condition as in Lemma 3.2,

$$
K^{-1}(\alpha+1\,, m+1\,, \theta) = (\mathrm{Id} - (m+\alpha+3)A_{\alpha+2})K^{-1}(\alpha\,, m\,, \theta) \tag{3.9}
$$

and

$$
K^{-1}(\alpha\,, -1\,, \theta) = \sqrt{\frac{2}{\pi}}\sin\left(\theta - \frac{\pi}{2}\right) \qquad \left(-\frac{3}{2} < \alpha < \frac{1}{2}\right)\,.
$$

By Theorems 1, 2, and (3.4), equation (0.8) holds.

Finally, we prove the following

THEOREM 3. *Equation* (0.7) *holds for* $\alpha > m-1\,,\ \alpha \geq 0\,,\ m = 0\,, 1\,, 2\,, \ldots$.

We need the estimate

$$R_1(u) = -\frac{\pi}{2}M'(0)u + O(1) \tag{3.10}$$

in the proof of Theorem 3. Taking $\alpha = 1$ in Proposition 1, we obtain

$$R_1(u) = \int_0^\infty \frac{1-\cos tu}{t^2}\,dS(t)\,.$$

Note that $S(t) = N(t) - M(t)$. So

$$\begin{aligned} R_1(u) &= \int_0^\infty \frac{1-\cos tu}{t^2}\,dN(t) - \int_0^\infty \frac{1-\cos tu}{t^2}M'(t)\,dt \\ &= -\,M'(0)\int_0^\infty \frac{1-\cos tu}{t^2}\,dt \\ &\quad -\int_0^\infty \frac{1-\cos tu}{t^2}(M'(t)-M'(0))\,dt + O(1) \\ &= -\,\frac{\pi}{2}M'(0)u + O(1)\,, \end{aligned}$$

since

$$\int_0^\infty \frac{1-\cos tu}{t^2}\,dt = \frac{\pi}{2}u\,,$$

and $M'(t)$ is an even function making $M'(t) - M'(0) = O(t^2)$ as $t \to 0^+$, and $M'(t) \sim \frac{1}{2\pi}\log t$ for large t, by (2.6).

Proof of Theorem 3. By Theorem 2,

$$\frac{S_1^*(t)}{t^2} = \sqrt{\frac{2}{\pi}}\lim_{U\to\infty}\int_0^U \frac{R_1(u)}{u^2}K(1\,,\,1\,,\,tu)\,du\,.$$

Now estimate (3.10) makes the above integral converge in the ordinary sense. Thus

$$S_1^*(t) = \frac{2}{\pi}\int_0^\infty R_1(u)\left(\frac{d}{du}\right)^2\frac{\sin_1(tu)}{u^2}\,du\,. \tag{3.11}$$

In addition,

$$\int_0^U R_1(u)\left(\frac{d}{du}\right)^2\frac{\sin_1(tu)}{u^2}\,du$$

is bounded uniformly with respect to t in any compact set as $U \to \infty$. So, after applying fractional integral operator on both sides of (3.11), we get for $\alpha \geq 0$

$$S_{\alpha+1}^*(t) = \frac{2}{\pi}\int_0^\infty R_1(u)\left(\frac{d}{du}\right)^2\frac{\sin_{\alpha+1}(tu)}{u^{\alpha+2}}\,du\,. \tag{3.12}$$

Now set

$$f(t) = \frac{2}{\pi}\int_0^\infty R_1(u)\left(\frac{d}{du}\right)^2 \frac{\sin_\alpha(tu)}{u^{\alpha+1}}\,du \qquad (\alpha > 0). \tag{3.13}$$

By using Lemma 2 of Kueh [5] and (3.6), we see that similarly for $\alpha > 0$

$$\int_0^U R_1(u)\left(\frac{d}{du}\right)^2 \frac{\sin_\alpha(tu)}{u^{\alpha+1}}\,du$$

is bounded uniformly with respect to t in any compact set as $U \to \infty$. Hence, on integrating both sides of (3.13), we get

$$f_1(t) = \frac{2}{\pi}\int_0^\infty R_1(u)\left(\frac{d}{du}\right)^2 \frac{\sin_{\alpha+1}(tu)}{u^{\alpha+2}}\,du \qquad (\alpha > 0)$$

and, in view of (3.12),

$$S^*_{\alpha+1}(t) = f_1(t) \qquad (\alpha > 0). \tag{3.14}$$

Thus, on differentiating both sides of (3.14), we get

$$\begin{aligned} S^*_\alpha(t) = f(t) &= \frac{2}{\pi}\int_0^\infty R_1(u)\left(\frac{d}{du}\right)^2 \frac{\sin_\alpha(tu)}{u^{\alpha+1}}\,du \\ &= -\frac{2}{\pi}\int_0^\infty R(u)\frac{d}{du}\frac{\sin_\alpha(tu)}{u^{\alpha+1}}\,du \qquad (\alpha > 0). \end{aligned}$$

We now repeat the same argument as in Theorem 1 and get

$$\frac{S^*_\alpha(t)}{t^{m+1}} = (-1)^{m+1}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{R_m(u)}{u^{\alpha+1}}K(\alpha, m, tu)\,du$$

with $\alpha > m-1$, $\alpha > 0$, $m = 0, 1, 2, \ldots$. Also by (0.2), the above equation holds for $\alpha = m = 0$.

This completes the proof of Theorem 3.

The author takes pleasure in thanking Professor P. X. Gallagher for his suggestions and many helpful discussions on the subject.

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Received January 23, 1989 and in revised form October 25, 1990.

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