PACIFIC JOURNAL OF MATHEMATICS
Vol. 152, No. 2, 1992

A NOTE ON GENERATORS FOR ARITHMETIC SUBGROUPS OF ALGEBRAIC GROUPS

M. S. RAGHUNATHAN

In this paper we construct systems of generators for arithmetic subgroups of algebraic groups.

1.1. Let k be a global field and G an absolutely almost simple simply connected (connected) k-algebraic group. We fix once and for all a *faithful* k-representation of G in some $\mathrm{GL}(n)$ and identify G with its image *under this representation.* In the sequel we will freely use results from Borel-Tits **[1]** without citing that reference repeatedly. Practically all facts about reductive algebraic groups used are to be found there. Let S be a finite set of valuations of k containing all the archimedean valuations and Λ be the ring of S-integers in $k : \Lambda = \{x \in k | x$ an integer in the completion k_v of k at v for all valuations $v \notin S\}$. For a subgroup $H \subset G$, we set $H(\Lambda) = H \cap \mathrm{GL}(n, \Lambda)$. More generally for an ideal $\mathfrak{a} \neq 0$ in Λ, we set

$$H(\mathfrak{a}) = \{x \in H(\Lambda) | x \equiv 1 \pmod{\mathfrak{a}}\}.$$

We fix a maximal k-split torus T in G. We assume that $\dim T \geq 2$ i.e. that k-rank $G \geq 2$. Let Φ denote the root system of G with respect to T. We fix a lexicographic ordering on $X(T)$, the character group of T and denote by Φ^+ (resp. Φ^-) the positive (resp. negative) roots with respect to this ordering. We also denote by Δ the corresponding simple system of roots. For $\phi \in \Phi$, let $U(\phi)$ denote the root group corresponding to ϕ: $U(\phi)$ is the unique T-stable k-split subgroup of G whose Lie algebra is the span of the root spaces $\{\mathfrak{g}^{r\phi} | r \text{ integer} > 0\}$ (here for $\psi \in \Phi$, $\mathfrak{g}^{\psi} = \{v \in \mathfrak{g} | \operatorname{Ad} t(v) = \psi(t)v\}$, $\mathfrak{g}$ being the Lie algebra of G). With this notation our main result is

1.2. THEOREM. *The group $\Gamma(\mathfrak{a})$ generated by $\{U(\phi)(\mathfrak{a}) | \phi \in \Phi\}$ for any non-zero ideal $(\mathfrak{a}) \subset \Lambda$ has finite index in $G(\mathfrak{a})$.*

Note. Tits **[8]** has obtained this result for Chevalley groups. However the methods of this paper are very different and make no use of Tits' results.

1.3. We denote by U^+ (resp. U^-) the group generated by $U(\phi)$, $\phi \in \Phi^+$ (resp. Φ^-). For $\phi \in \Phi$, let $G(\phi)$ denote the (k-rank 1) subgroup generated by $U(\phi)$ and $U(-\phi)$. We denote by T_ϕ the connected component of the identity in kernel ϕ and by $Z(T_\phi)$ the centraliser of T_ϕ in G. Then $Z(T_\phi)$ is reductive and $G(\phi)$ is its maximal normal semisimple subgroup all of whose k-simple factors are isotropic. For $\alpha \in \Delta$ let $V^+(\alpha)$ (resp. $V^-(\alpha)$) denote the subgroup of U^+ (resp. U^-) generated by the $U(\phi)$, $\phi \in \Phi_\alpha^+$ (resp. Φ_α^+) $= \{\phi \in \Phi^+$ (resp. $\Phi_-)|\phi$ not a multiple of $\alpha\}$. Then $V^+(\alpha)$ and $V^-(\alpha)$ are normalised by $Z(T_\alpha)$. The centraliser $Z(T)$ of T normalises all the $U(\phi)$, $\phi \in \Phi$, and hence in particular U^+, U^-, $V^+(\alpha)$ and $V^-(\alpha)$ for all $\alpha \in \Delta$. We will establish the following

1.4. *Claim.* Let $\mathfrak{a}$ be a nonzero ideal in Λ and (as in Theorem 1.2) let $\Gamma(\mathfrak{a})$ denote the subgroup of $G(\Lambda)$ generated by $\{U(\phi)(\mathfrak{a})|\phi \in \Phi\}$. Then for any $g \in G(k)$ there is a non-zero ideal $\mathfrak{a}'$ (depending on g) in Λ such that $g\Gamma(\mathfrak{a}')g^{-1} \subset \Gamma(\mathfrak{a})$.

1.5. Let $\widetilde{\Gamma} = \{g \in G(k)|$ for any nonzero ideal $\mathfrak{a} \subset \Lambda$, there is a nonzero ideal $\mathfrak{a}' \subset \Lambda$ such that $g\Gamma(\mathfrak{a}')g^{-1}$ and $g^{-1}\Gamma(\mathfrak{a}')g \subset \Gamma(\mathfrak{a})\}$. It is then evident that $\widetilde{\Gamma}$ is a subgroup of $G(k)$. Since $Z(T)$ normalises $U(\phi)$ for all $\phi \in \Phi$, it is easily seen that $Z(T)(k) \subset \widetilde{\Gamma}$. We will presently show that $U(\pm\alpha)(k) \subset \widetilde{\Gamma}$ for *all* $\alpha \in \Delta$. This will prove the claim since the $\{U(\pm\alpha)(k)|\alpha \in \Delta\}$ and $Z(T)(k)$ generate all of $G(k)$. Suppose then that $\alpha \in \Delta$ and $u \in U(\pm\alpha)(k)$. Then u normalises $U(\phi)$, $\phi \in \Phi_\alpha^+$ (resp. Φ_α^-). It follows that we can, for any non-zero ideal $\mathfrak{a} \subset \Lambda$, find a non-zero ideal $\mathfrak{b} \subset \Lambda$ such that $uU(\phi)(\mathfrak{b})u^{-1} \subset U(\phi)(\mathfrak{a})$ for all $\phi \in \Phi_\alpha^\pm$. If we denote by $\Gamma_\alpha(\mathfrak{b})$ the group generated by $U(\phi)(\mathfrak{b})$, $\phi \in \Phi_\alpha^+$ or Φ_α^-, this means that $u\Gamma_\alpha(\mathfrak{b})u^{-1}$ $(\subset \Gamma_\alpha(\mathfrak{a})) \subset \Gamma(\mathfrak{a})$. Thus to establish the claim we need only show that for any nonzero ideal $\mathfrak{b}$ in Λ, there is a non-zero ideal $\mathfrak{c}$ in Λ with $\Gamma(\mathfrak{c}) \subset \Gamma_\alpha(\mathfrak{b})$ for all $\alpha \in \Delta$. This follows from the following stronger result.

1.6. LEMMA. *Let $\alpha, \beta \in \Delta$ be such that $\alpha + \beta \in \Phi$. Then there is an element $t = t(\alpha, \beta)$ in Λ, $t \neq 0$ such that for any ideal $\mathfrak{a} \neq 0$ in Λ, the group generated by $\{U(r\alpha + s\beta)(\mathfrak{a})|r \cdot s \neq 0,\ r\alpha + s\beta \in \Phi\}$ and $U_\beta(\mathfrak{a})$ (resp. $U_\beta(\mathfrak{a})$) contains $U(\alpha)(t\mathfrak{a}^3)$ (resp. $U(-\alpha)(t\mathfrak{a}^3)$).*

Proof. We treat the case of $U(\alpha)$; the other case, viz. of $U(-\alpha)$, is entirely analogous. Consider first the case when Φ is reduced i.e. $2\phi \notin \Phi$ for any $\phi \in \Phi$. Let α, β be as above then the commutator map $(x, y) \to xyx^{-1}y^{-1}$ of $G \times G$ in G defines a k-morphism

$$c\colon U(-\beta) \times U(\alpha + \beta) \to U(\alpha).$$

As Φ is *reduced*, $U(\phi)$ is abelian and hence k-isomorphic to a k-vector space and c is easily seen to be a k-bilinear map. Let $U_c(\alpha)$ denote the group generated by Image c. Then $U_c(\alpha)$ is a k-algebraic subgroup—in fact a k-vector subspace of $U(\alpha)$. Since c is compatible with the action of $Z(T)$ on both sides, $U_c(\alpha)$ is $Z(T)$-stable as well. It is easy to see that our lemma follows if the following holds: $U_c(\alpha) = U(\alpha)$. In fact one concludes that there is a $t \in \Lambda\backslash\{0\}$ such that $U(\alpha)(t\mathfrak{a}^2)$ (resp. $U_{-\alpha}(t\mathfrak{a}^2)$) is contained in the group generated by $\{U(r\alpha + s\beta)(\mathfrak{a}) | r \cdot s \neq 0\}$ and $U_\beta(\mathfrak{a})$ (resp. $U_\beta(\mathfrak{a})$). Evidently this equality holds if the following two conditions are satisfied:

C1: $U(\alpha)$ as a $Z(T)$-module is irreducible over k.

C2: The map c is non-trivial.

By using split semisimple subgroups of G containing T (Borel-Tits [**1**, Theorem 7.2]) one sees easily that C2 fails only if $\operatorname{Char} k = \langle\alpha, \alpha\rangle / \langle\beta, \beta\rangle = 2$ or 3. When C2 fails and $\operatorname{char} k = 2$ we consider the k-morphism

$$c'\colon U(-\beta) \times U(\alpha + 2\beta) \to U(\alpha) \cdot U(\alpha + \beta) = U^*$$

obtained by restricting the commutator map in G. Now U^* is a direct product of $U(\alpha)$ and $U(\alpha+\beta)$ and this direct product decomposition is compatible with the action of $Z(T)$. Thus c' may be regarded as a pair (c'_1, c'_2) where

$$c'_1\colon U(-\beta) \times U(\alpha + 2\beta) \to U(\alpha)$$

is a k-morphism which for fixed $u \in U(\alpha + 2\beta)$ is a homogeneous quadratic polynomial on $U(-\beta)$ and for fixed x in $U(-\beta)$ is linear on $U(\alpha + 2\beta)$ while

$$c'_2\colon U(-\beta) \times U(\alpha + 2\beta) \to U(\alpha + \beta)$$

is bilinear. To prove the lemma once again it suffices to show that the group $U_{c'}(\alpha)$ generated by the image of c' contains all of $U(\alpha)$. Now if C1 holds, this is indeed the case. To see this observe that $U(\alpha)$ and $U(\alpha + \beta)$ are distinct isotypical T-submodules of U^*—as a T-module U^* is semisimple. Thus if c'_1 is non-trivial $U_{c'}(\alpha) \cap U(\alpha)$ is a nontrivial $Z(T)$-stable k-vector space hence is all of $U(\alpha)$. That c'_1 is non-trivial is checked using the Chevalley commutation relations in a Chevalley group containing T and contained in H. Finally if characteristic of $k = 3$, C2 fails and C1 holds, we consider the commutator map restricted to $U(-(\alpha+3\beta)) \times U(2\alpha+3\beta)$ as a k-morphism of this variety into $U(\alpha)$. One sees easily that it is bilinear and non-trivial. This leaves us to deal with the situation when C1 fails. From the classification of Tits **[6]** of groups over global fields, it is easy to conclude that if C1 fails one has necessarily $\operatorname{char} k = 2$ and G is a group of Type C_n with Tits index as below

$$|\!-\!\circ\!-\!|\!-\!\circ\!-\!|\ldots|\!-\!\overset{\beta}{\circ}\!-\!|\overset{\alpha}{=\!=}\circ$$

(C2 also fails in this case). But in this case one has a description of G as the special unitary group of a non-degenerate hermitian form h over a quaternion algebra (over k) with respect to an involution whose fixed point set is of dimension 3 (over k) (such that h has Witt index $n/2$— n is necessarily even). Explicit matrix computation leads us in this case to the conclusion that $U_{c'}(\alpha) = U^*$ (in the notation introduced above).

Consider now the case when Φ is not reduced. Let Φ_0 be the reduced system associated to Φ and Δ_0 the corresponding simple system. If $\alpha, \beta \in \Delta_0$ we are reduced to the preceding case. If $2\beta \in \Delta_0$ since $U(\beta) \supset U(-2\beta)$ and $U(\beta) \supset U(2\beta)$ we are again reduced to the preceding case. Then we are left with the case $\beta \in \Delta_0$, $2\alpha \in \Delta_0$. In this case one notes that the preceding considerations show that $U(2\alpha)(t\mathfrak{a}_2)$ is contained in the group generated by $\{U(r\alpha+s\beta)(\mathfrak{a}) | r, s \neq 0, r\alpha+s\beta \in \Phi\}$ and $U(-\beta)$. This reduces the lemma to proving that the map $c\colon U(-\beta)\times U(\alpha+\beta) \to U(\alpha)/U(2\alpha)$ obtained from the commutator map is such that Image c generates all of $U(\alpha)/U(2\alpha)$. This is easily checked. Hence the lemma.

1.7. Let $\mathfrak{a} \subset \Lambda$ be a non-zero ideal. Then $G(\alpha)(\Lambda)$ normalises $V(\alpha)(\mathfrak{a})$. Consequently $G(\alpha)(\Lambda)$ normalises $\Gamma_\alpha(\mathfrak{a})$ and hence also $\Psi_\alpha \overset{\text{def}}{=} \Gamma_\alpha(\mathfrak{a}) \cap G(\alpha)(k)$. We also set $\Psi_\alpha = \Psi_\alpha(\Lambda)$. Observe that for any $g \in G(\alpha)(k)$, and a non-zero ideal $\mathfrak{a} \subset \Lambda$, there is an ideal $\mathfrak{b}$ (depending on $\mathfrak{a}$ and g) such that $g\Psi_\alpha(\mathfrak{b})g^{-1}$ is contained in $\Psi_\alpha(\mathfrak{a})$: this follows from Claim 1.4 combined with Lemma 1.6, which shows that $\Gamma(t\mathfrak{a}^3)$ is contained in $\Gamma_\alpha(\mathfrak{a})$. It is easy to see from this that the following collection T of subsets of $G(\alpha)(k)$ is the family of open sets for a topology on $G(\alpha)(k)$: $\mathrm{T} = \{\Omega \subset G(\alpha)(k) |$ for every $x \in \Omega$, there is a non-zero ideal $\mathfrak{a}(x)$ in Λ such that $x\Psi_\alpha(\mathfrak{a}(x))$ is contained in $\Omega\}$. (That T constitutes a topology is seen easily from the fact that $\Psi_\alpha(\mathfrak{a}) \cap \Psi_\alpha(\mathfrak{b})$ contains $\Psi_\alpha(\mathfrak{a}\mathfrak{b})$ and that if $\mathfrak{a} \neq 0$, $\mathfrak{b} \neq 0$, then

$\mathfrak{a}\mathfrak{b} \neq 0$.) Let L and R denote respectively the left and right uniform structures on $G(\alpha)(k)$ for the topology T. Then we assert that a sequence $x_n \in G(\alpha)(k)$ is Cauchy for L if and only if it is Cauchy for R. Assume that x_n is Cauchy for L. Let $l \geq 0$ be an integer such that $x_n^{-1}x_m \in \Psi_\alpha(\mathfrak{a})$ for all $m, n \geq l$. Let $t \in \Lambda\backslash\{0\}$ be as in Lemma 1.6. For an ideal $\mathfrak{a} \neq 0$ let $\mathfrak{a}' \neq 0$ be an ideal such that $x_l\Psi_\alpha(\mathfrak{a}')x_l^{-1}$ is contained in $\Psi_\alpha(\mathfrak{a})$. Since x_n is Cauchy for L there is an integer $l(\mathfrak{a}') > 0$ such that $x_n^{-1}x_m \in \Psi_\alpha(\mathfrak{a}')$ for $m, n \geq l(\mathfrak{a}')$. Then for $m, n \neq \max(l, l(\mathfrak{a}'))$ we have $x_m x_n^{-1} = x_n x_n^{-1} x_m x_n^{-1} = x_l \cdot x_l^{-1}x_n \cdot x_n^{-1}x_m(x_l^{-1}x_n)^{-1} \cdot x_l^{-1} \in \Psi_\alpha(\mathfrak{a})$. Thus x_n is Cauchy for R as well. The converse is proved analogously. It follows that there is a canonical identification of the completions of $G(\alpha)(k)$ with respect to R and L and we denote this common completion by $\widehat{G}(\alpha)(k)$. Then $\widehat{G}(\alpha)(k)$ is a topological group in a natural fashion. The closure of $U(\alpha)(k)$ (resp. $U(-\alpha)(k)$) in $\widehat{G}(\alpha)(k)$ is obviously the same as the completion $\overline{U}(\alpha)(k)$ (resp. $\overline{U}(\alpha)(k)$) of $U(\alpha)(k)$ (resp. $U(-\alpha)(k)$) in the congruence subgroup topology. If $\overline{G}(\alpha)(k)$ denotes the completion $G(\alpha)(k)$ with respect to the congruence subgroup topology we have natural commutative diagrams as follows:

$$\begin{tikzcd} \widehat{G}(\alpha)(k) \arrow[rr, "\pi"] & & G(\alpha)(k) \\ & G(\alpha)(k) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} \widehat{G}(\alpha)(k) \arrow[rr, "\pi"] & & \overline{G}(\alpha)(k) \\ & \overline{U}(\pm\alpha)(k) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Since $\overline{U}(\pm\alpha)(k)$ generate $\overline{G}(\alpha)(k)$ (as an abstract group) (Raghunathan [5]) one sees that π is surjective. We will now prove the following result.

1.8. Proposition. *Let $G(\alpha)(k)^+$ denote the normal subgroup of $G(k)$ generated by $U^+(\alpha)(k)$. Then $G(\alpha)(k)^+$ centralises the kernel of π $(= C)$.*

Proof. One knows from the work of Tits [7] that any noncentral normal subgroup of $G(\alpha)(k)$ contains $G(\alpha)(k)^+$. Thus it suffices to show that C $(=$ kernel $\pi)$ is centralised by an element x in $G(k)^+$ which is not central in $G(\alpha)$—the centraliser of C in $G(\alpha)$ is a normal subgroup of $G(\alpha)$. We know that Ψ_α contains a non-trivial element of $U(\alpha)(\Lambda)$ (Lemma 1.6). Let u be such an element; then u can be written as a product:

$$u = x_r x_{r-1} \cdots x_1 ,$$

where for $1 \le i \le r$, $x_i \in U(\phi_i)(\Lambda)$ with $\phi_\varepsilon \Psi_\alpha^\pm$. Let

$$u_i = x_i x_{i-1} \cdots x_2 x_1 .$$

Let A_i be the following assertion: for any ideal $\mathfrak{a} \subset \Lambda$, $\mathfrak{a} \ne 0$, there is a nonzero ideal $f_i(\mathfrak{a}) \subset \Lambda$ such that $\rho u_i \rho^{-1} u_i^{-1} \in \Gamma_\alpha(\mathfrak{a})$ for all $\rho \in G(\alpha)(f_i(\mathfrak{a}))$. Then A_0 holds if we set $f_0(\mathfrak{a}) = \mathfrak{a}$. Assume that A_l holds for some l with $1 \le l < r$ and we will show then that A_{l+1} holds as well. Let $\mathfrak{a}' \subset \mathfrak{a}$ be a non-zero ideal such that $x_{l+1}\Gamma_\alpha(\mathfrak{a}')x_{l+1}^{-1} \subset \Gamma_\alpha(\mathfrak{a})$ (Claim 1.4 and Lemma 1.6). Let $f_{l+1}(\mathfrak{a}) = f_l(\mathfrak{a}') \cap \mathfrak{a}$. Then for $\rho \in G_\alpha(\mathfrak{b})\mathfrak{b} = f_{l+1}(a)$, we have $\rho x_{l+1}\rho^{-1}x_{l+1}^{-1} \in \Gamma_\alpha$ while $x_{l+1}\rho u_l \rho^{-1} u_l^{-1} x_{l+1}^{-1} \in x_{l+1}\Gamma_\alpha(\mathfrak{a}')x_{l+1}^{-1} \subset \Gamma_\alpha(\mathfrak{a})$. But one has

$$\begin{aligned} \rho u_{l+1}\rho^{-1}u_{l+1}^{-1} &= \rho x_{l+1} u_l \rho^{-1} u_l^{-1} x_{l+1}^{-1} \\ &= (\rho x_{l+1}\rho^{-1}x_{l+1}^{-1}) \cdot x_{l+1}(u_l\rho^{-1}u_l^{-1})x_{l+1}^{-1} \end{aligned}$$

so that $\rho u_{l+1}\rho^{-1}u_{l+1}^{-1}$ belongs to $\Gamma_\alpha(\mathfrak{a})$. We conclude that for each ideal $\mathfrak{a} \subset \Lambda$, $\mathfrak{a} \ne 0$, there is an ideal $\mathfrak{a}' \ne 0$ such that $[u, G(\alpha)(a')] \subset \Psi_\alpha(a)$. Passing to the completions it is now clear that this means that u centralises C in $\widehat{G}(\alpha)(k)$ proving Proposition 1.8.

1.9. Let $\widehat{G}(\alpha)(k)^+$ denote the closure of $G(\alpha)(k)^+$ in $\widehat{G}(\alpha)(k)$. Then $\widehat{G}(\alpha)(k)^+ \xrightarrow{\pi_0} \overline{G}(\alpha)(k)$ is a central extension where π_0 is the restriction of π to $\widehat{G}(\alpha)(k)^+$. Let C_0 denote the kernel of π_0. Then C_0 is a closed subgroup of C; and since C is the projective limit of the family $\{G(\alpha)(\mathfrak{a})/\Psi_\alpha(\mathfrak{a}) | \mathfrak{a}$ a nonzero ideal in $\Lambda\}$ of *discrete* groups, it follows that C_0 is the projective limit of a family of *discrete* abelian groups

$$C_0 \simeq \varprojlim C_i .$$

We have for $i > j$ a map $f_{ij} : C_i \to C_j$ which may be assumed to be surjective as also the natural map $f_i \colon C_0 \to C_i$. Now for every i the central extension $\widehat{G}(\alpha)(k)^+/(\text{kernel } f_i)$ of $\overline{G}(\alpha)(k)$ is a *locally compact* central extension *split over* $G(k)^+$. But from Prasad-Raghunathan [3] one knows that the universal locally compact central extension $\widetilde{G}(k)(k)^+ \to \overline{G}(\alpha)(k)$ *split over* $G(k)^+$ has $\ker\phi$ a subgroup of the group μ_k of roots unity in k. It is now easy to deduce from this that C_0 is a finite cyclic group of order at most $|\mu_k|$. Since $G(k)/G(k)^+$ is finite (Margulis [2]) one concludes that C is finite. The following result is immediate from the finiteness of C.

1.10. PROPOSITION. *For any non-zero ideal* $\mathfrak{a}$, Ψ_α *is an S-arithmetic subgroup of* $G(\alpha)$.

Proof. If $U \subset \widehat{G}(\alpha)(k)^+$ is any open subgroup, then $U \cap G(k)^+$ is an S arithmetic subgroup, since C is finite and (hence) π maps $\widehat{G}(\alpha)(k)$ onto $\overline{G}(k)$. Since for any $\mathfrak{a} \neq 0$, $\Psi_\alpha(\mathfrak{a})$ contains a subgroup of the form $U \cap G(k)$ with U open in $\widehat{G}(\alpha)(k)$ our contention follows.

1.11. COROLLARY. *If* $P(\alpha) = Z(T) \cdot U(\alpha)$ *then for any ideal* $\mathfrak{a} \neq 0$ *in* Λ, *there is a finite subset* $\Sigma_\alpha(\mathfrak{a})$ *in* $G(\alpha)(k)$ *such that*

$$Z(T_\alpha)(k) = \Psi_\alpha(\mathfrak{a}) \cdot \Sigma_\alpha(\mathfrak{a}) \cdot P(\alpha)(k)$$

(*this is a theorem due to Borel; for a proof see Raghunathan* [4, *Chapter* XIII]).

1.12. THEOREM. *Let* $\mathfrak{a}$ *be a nonzero ideal in* Λ. *Then there is a finite set* $\Sigma(\mathfrak{a}) \subset G(k)$ *such that* $G(k) = \Gamma(\mathfrak{a}) \cdot \Sigma(\mathfrak{a}) \cdot P(k)$ *where* $P = Z(T) \cdot U$.

Proof. Let $N(T)$ be the normaliser of T in G and $W = N(T)/Z(T)$ the *k-Weyl* group of G. Then W is generated by reflection σ_α corresponding to the simple roots α in Δ and each σ_α has a representative s_α in $(N(T) \cap G(\alpha))(k)$. One has $G(k) = U(k)WP(k)$, where W is identified with a set of representatives of its elements in $N(T)(k)$. Let l be an integer ≥ 0 and $W(l)$ the set of elements of W of length l with respect to the set $\{s_\alpha | \alpha \in \Delta\}$ of generators. We will prove the following statement by induction on l. For any ideal $\mathfrak{a} \neq 0$ in Λ, there is a *finite* set $\Sigma_l(\mathfrak{a})$ such that $U(k)W(l)P(k)$ is contained in $\Gamma(\mathfrak{a}) \cdot \Sigma_l(\mathfrak{a})(k)$. When $l = 1$, this is simply Corollary 1.12. Assume that the assertion holds for $l < r$. Let $g = uwp$ in $G(k)$ be such that length $w = r$, $u \in U^+(k)$ and $p \in P(k)$. Then $w = s_\alpha w'$ for some w' of length $r - 1$ and $\alpha \in \Delta$. Also one can write $u = u' \cdot u''$ with $u' \in U(\alpha)(k)$ and $u'' \in V(\alpha)(k)$. Since $G(\alpha)$ normalises $V(\alpha)(k)$ we see that $g = xyw'p$ where $x \in G(\alpha)(k)$ and $y \in V(\alpha)(k)$. Let $\Sigma_\alpha(a)$ be as in Corollary 1.1. Clearly then $g \in \Psi_\alpha(\mathfrak{a}) \cdot \Sigma_\alpha(\mathfrak{a})U(k)W(r-1)P(k)$. Now let $\mathfrak{b}(\alpha) = \mathfrak{b} \neq 0$ an ideal such that $x\Gamma(\mathfrak{b})x^{-1} \subset \Gamma(\mathfrak{a})$ for all x in the *finite* set $\Sigma_\alpha(\mathfrak{a})$. By the induction hypothesis we can find a finite set $\Sigma_{r-1}(\mathfrak{b})\cdot G(k)$ such that $\Gamma(\mathfrak{b})\cdot\Sigma_{r-1}(\mathfrak{b})P(k)$ contains $U(k)W(r-1)\cdot P(k)$. Thus $g\in\Psi_l(\mathfrak{a})\cdot\Sigma_\alpha(\mathfrak{a})\cdot\Gamma(\mathfrak{b})\cdot\Sigma_{r-1}(\mathfrak{b})\cdot P(k)$; and this last set is contained in $\Psi_\alpha(\mathfrak{a})\Gamma(\mathfrak{a})\Sigma_\alpha(\mathfrak{a})\Sigma_{r-1}(b)\cdot P(k)$. Since $\Psi_\alpha(\mathfrak{a})\subset\Gamma(\mathfrak{a})$ and $\Sigma_\alpha(\mathfrak{a})\Sigma_{r-1}(\mathfrak{b})$ is finite, our claim for r follows if we set $\Sigma_r(\mathfrak{a})$ to be $\bigcup_{\alpha\in\Delta}\Sigma_\alpha(\mathfrak{a})\cdot\Sigma_{r-1}(\mathfrak{b}(\alpha))$ $(\mathfrak{b}(\alpha))$ also depends on $\mathfrak{a}$). This proves the theorem.

1.14. COROLLARY. *For a non-zero ideal $\mathfrak{a}$ in Λ, $\Gamma(\mathfrak{a})$ is an arithmetic subgroup of G.*

Proof. Let $\Sigma\subset G(k)$ be a finite set such that $\Gamma(\mathfrak{a})\cdot\Sigma P(k)=G(k)$. Then if $g\in G(\Lambda)$ we have $g=x\zeta p$ with $p\in P(k)x\in\Gamma(\mathfrak{a})$ and $\zeta\in\Sigma$. Since Σ is a *finite* set we conclude that there is a $\lambda\in\Lambda\backslash\{0\}$ such that the following holds: if $p=z\cdot u$, $z\in Z(T)(k)$, $u\in U(k)$, and ξ is *any* matrix entry of z, u, z^{-1} or u^{-1}, then $\lambda\xi\in\Lambda$. It is also easy to see that if B is *any* k-simple component of $Z(T)$, $B\subset G(\alpha)$ for *some* $\alpha\in\Delta$. Thus $B\cap\Gamma(\mathfrak{a})$ is an S-arithmetic subgroup of B so that $Z(T)\cap\Gamma(\mathfrak{a})$ is an arithmetic subgroup of $Z(T)$. Hence $P\cap\Gamma(\mathfrak{a})$ is an S-arithmetic subgroup of P. In particular $\prod_{V\in S}{}^\circ P(k_v)/{}^\circ P\cap\Gamma(\mathfrak{a})$ is compact where ${}^\circ P=\{\ker\chi|\chi$ a character on P defined over $k\}$. From the fact that z and z^{-1} have both entries of the form ξ/λ with $\xi\in\Lambda$, one easily deduces that z belongs to a finite set modulo ${}^\circ P$. From the compactness of ${}^\circ P/{}^\circ P\cap\Gamma(\mathfrak{b})$ for any $\mathfrak{b}\neq 0$ and the discreteness of the set $\{p\in{}^\circ P|$ the entries of p and p^{-1} belong to $\lambda^{-1}\}$, one sees easily now that there is a finite set Σ' such that $p\in P(k)\cap\Gamma(\mathfrak{b})\cdot\Sigma'$ for *all* $g\in G(k)$. Now choose $\mathfrak{b}$ such that $x\Gamma(\mathfrak{b})x^{-1}\subset\Gamma(\mathfrak{a})$ for *all* $x\in\Sigma$. Then one has clearly

$$g\in\Gamma(\mathfrak{a})\cdot\Sigma\cdot\Sigma.$$

Since $\Sigma\cdot\Sigma'$ is finite we have shown that $\Gamma(\mathfrak{a})$ has finite index in $G(\Lambda)$. Hence the corollary.

Added in proof. T. N. Venkataramana recently drew my attention to two papers of G. A. Margulis (*Arithmetic Properties of Discrete Groups*, Russian Mathematical Surveys, **29:1** (1974), 107–156 and *Arithmeticity of non-uniform lattices in weakly non compact groups*, Functional Analysis and its Applications, Vol. 9 (1975), 31–38), which contain results that imply our main theorem. The methods of the present paper are however very different, and I believe, more transparent.

REFERENCES

[1] A. Borel, and J. Tits, *Groupes reductifs*, Publ. Math. Inst. Haut. Etud. Sci., **27** (1965), 55–150.
[2] G. A. Margulis, *Finiteness of quotient groups of discrete groups* (Russian), Funktsional. Anal. i Prilozhen., **13** (1979), 28–39.
[3] G. Prasad and M. S. Raghunathan, *On the congruence subgroup problem: determination of the mletaplectic kernel*, Invent. Math., **71** (1983), 21–42.
[4] M. S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer Verlag, 1972.
[5] ——, *On the congruence subgroup problem* II, Invent. Math., **85** (1986), 73–117.
[6] J. Tits, *Classification of algebraic semisimple groups*, Proc. Symp. Pure Math., Amer. Math. Soc., **9** (1969), 33–62.
[7] ——, *Algebraic and abstract simple groups*, Ann. of Math., **80** (1964), 313–329.
[8] ——, *Systems générateurs de groupes de congruence*, C. R. Acad. Sci. Paris, **283** (1976), Serie A, 693–695.

Received January 5, 1989 and in revised form June 23, 1989.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH
HOMI BHABHA ROAD
BOMBAY 400 005, INDIA