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ON THE TENSOR PRODUCT OF THETA REPRESENTATIONS OF GL_3

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Let V be the theta representation of $\widetilde{GL}_3$—the two fold central extension of GL_3. Let W be a spherical representation of GL_3. We show that there is a nonzero GL_3 invariant trilinear form on $V \otimes V^* \otimes W$ if and only if W is a lift from SL_2. In this case the form is unique up to a scalar.

Introduction. Let k be a global field and A its ring of adeles. Let σ be an irreducible 3 dimensional representation of the Galois group Γ of k. Assume, for simplicity, that $\sigma(\Gamma) \subset SL_3(\mathbf{C})$. Then, according to Langlands there exists an automorphic representation $\pi \subset L^2(PGL_3(k)\backslash PGL_3(\mathbf{A}))$ such that the corresponding L-functions are equal. Consider the symmetric square of the representation σ. Then, conjecturally, the corresponding L function will have a pole only if the symmetric square representation contains a copy of trivial representation. But this means that there is a quadratic form invariant under σ and therefore $\sigma(\Gamma) \subset SO_3(\mathbf{C})$. Since $SO_3(\mathbf{C}) = {}^LSL_2$, the automorphic representation π should be a lift of an automorphic representation of SL_2. Let π_v be a local component of π. If it is spherical, π_v is the local lift of a representation of SL_2 if $\pi_v = \mathrm{ind}_B^{PGL_3}\chi$ where χ is a character of the diagonal subgroup of PGL_3 given by

$$\chi\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = \mu\left(\frac{a}{c}\right)$$

for some unramified character $\mu\colon k_v^* \to \mathbf{C}^*$.

On the other hand, Patterson and Piatetski-Shapiro **[PP]** have constructed the symmetric square L-function corresponding to a cuspidal automorphic representation π of PGL_3. Moreover, they showed that the residue at $s = 1$ of this L-function is

$$\int_{PGL_3(k)\backslash PGL_3(\mathbf{A})} \varphi(g)\theta(g)\theta'(g)\,dg$$

where $\varphi \in \pi$ and θ, θ' are "theta functions" of Kazhdan and Patterson **[KP]**. They are certain automorphic forms on $\widetilde{GL}_3$—the two fold central extension of GL_3. Let F be a local field. In **[FKS]** we have constructed a smooth model (θ_3, V) of the local component of "theta functions". Let (π, W) be an irreducible representation of $\mathrm{PGL}_3(F)$. From what was explained above, it is natural to ask whether there is a GL_3 invariant trilinear form on $V \otimes V^* \otimes W$. We have the following result:

THEOREM. *Let F be a local field of the characteristic $\neq 2$. Let (π, W) be a spherical representation of* PGL_3. *Then there exists a* GL_3 *invariant trilinear form on $V \otimes V^* \otimes W$ if and only if π is the lift of a representation of* SL_2. *Moreover, the form is unique up to a scalar.*

We remark that the article of Prasad **[P]** was perhaps the first result indicating relationship between special values of L-functions and invariant functionals.

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Preliminaries and notation. Let P_1 (resp. P_2) be the standard $(2, 1)$ (resp. $(1, 2)$ parabolic subgroup of GL_3. Let $P_1 = M_1 U_1$ and $P_2 = M_2 U_2$ be standard Levi decompositions. We shall use the letter N to denote the unipotent group of uppertriangular matrices of GL_2 and GL_3 and the letter T to denote the group of diagonal matrices of GL_2 and GL_3. It will be clear from the context which is meant. Finally put $N_1 = N \cap M_1$, $N_2 = N \cap M_2$ and $B = TN$.

Let $P = MU$ be a parabolic subgroup and (π, V) a smooth module. Define $V(U) = \mathrm{span}\{v - \pi(u)v | v \in V, u \in U\}$. Then $V_U = V/V(U)$ is the module of coinvariants.

Let X be an algebraic variety over the field F. Then $S(X)$ will denote the space of locally constant, compactly supported functions on X. Obviously, $S(X_1 \times X_2) = S(X_1) \otimes S(X_2)$. Let q be an algebraic function on X. Then one can define a representation π of N on $S(X)$ by

$$\pi\left(\begin{pmatrix} 1 & n \\ & n \end{pmatrix}\right) f(x) = \phi(nq(x)) f(x)$$

where ϕ is an additive character of F. It is easy to check that $S(X)_N = S(Y)$ where Y is the subvariety of X defined by $q = 0$.

We need to recall some facts about the principal series representations of $\mathrm{GL}_3(F)$. Let St denote the Steinberg representation of GL_2.

Let λ be a multiplicative character of F^* such that $\lambda^2 = 1$. Put $\mathrm{St}_\lambda = \mathrm{St} \otimes \lambda(\det)$.

Let μ be a character of F^*. It defines a character $\chi_\mu \colon T \to \mathbf{C}^*$ by the following formula:

$$\chi_\mu \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = \mu\left(\frac{a}{c}\right).$$

Let $\pi(\mu) = \mathrm{ind}_B^{\mathrm{GL}_3} \chi_\mu$ (normalized induction). To describe the composition series of $\pi(\mu)$ we need to introduce σ_1, σ_2 representations of GL_3 defined as follows:

$$0 \to 1 \to \mathrm{ind}_{P_1}^{\mathrm{GL}_3} 1 \to \sigma_1 \to 0,$$
$$0 \to 1 \to \mathrm{ind}_{P_2}^{\mathrm{GL}_3} 1 \to \sigma_2 \to 0.$$

Here the induction is not normalized! We need the following result about the principal series representations. A reader can find details in Cartier's article [**C**, §III].

Proposition 1. *The representations $\pi(\mu)$ are irreducible and $\pi(\mu) \cong \pi(\mu^{-1})$ unless μ is of the two following types:*

(a) $\mu = \lambda|\cdot|^{\pm 1/2}$, $\lambda^2 = 1$. *The composition series consists of $\mathrm{ind}_P^{\mathrm{GL}_3} \lambda(\det)$ and $\mathrm{ind}_P^{\mathrm{GL}_3} \mathrm{St}_\lambda$.*

(b) $\mu = |\cdot|^{\pm 1}$. *The composition series consists of the trivial representation, the Steinberg representation, σ_1 and σ_2.*

The central extension and theta representation. Let $(\cdot, \cdot) \colon F^* \times F^* \to \{\pm 1\}$ be the Hilbert symbol. Let $\widetilde{\mathrm{GL}}_n(F)$ be the 2-fold central extension of $\mathrm{GL}_n(F)$ and $\mathbf{s} \colon \mathrm{GL}_n \to \widetilde{\mathrm{GL}}_n$ the section as in [**FKS**]. The extension can be characterized in the following way:

$$\mathbf{s}(\mathrm{diag}(a_i))\mathbf{s}(\mathrm{diag}(b_i)) = \mathbf{s}(\mathrm{diag}(a_i b_i)) \prod_{i<j} (a_i, b_j),$$

where $\mathrm{diag}(a_i)$ denotes the diagonal matrix with entries a_i. Moreover the section $\mathbf{s}$ is an isomorphism on N and we will identify N and $\mathbf{s}(N)$. Fix a nontrivial additive character $\phi \colon F \to \mathbf{C}^*$. Define a function $\gamma = \gamma_\phi \colon F^* \to \mathbf{C}^*$ by

$$\gamma(a) = \frac{|a|^{1/2} \int \phi(ax^2)\, dx}{\int \phi(x^2)\, dx}.$$

LEMMA 1 (Weil [W1]). *The function* γ *has the following properties*:

(a) $\gamma(ab) = \gamma(a)\gamma(b)(a, b)$,

(b) $\overline{\gamma}_\phi = \gamma_{\overline{\phi}}$. □

DEFINITION 1. Let $C_2(F)$ be the space of locally constant functions on F^* such that

(a) $f(x) = 0$ if $|x| > c$,

(b) $f(y^2x) = f(x)$ if $|x|, |y^2x| < 1/c$,

where c is a constant depending on f.

The theta representation θ_2 of $\widetilde{\mathrm{GL}}_2$ can be realized on the space of functions f on F^* such that $|x|^{1/4}f(x) \in C_2(F)$. The action of $\widetilde{\mathrm{GL}}_2$ is given by the following formulae [F]:

$$\theta_2\left(\mathbf{s}\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) f(x) = |a|^{1/2} f(ax),$$

$$\theta_2\left(\mathbf{s}\begin{pmatrix} z & \\ & z \end{pmatrix}\right) f(x) = (x, z)\gamma(z) f(x),$$

$$\theta_2\left(\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\right) f(x) = \phi(nx) f(x),$$

$$\theta_2\left(\mathbf{s}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) f(x) = c\gamma(x)|x|^{1/2} \int_F |y|^{1/2} f(xy^2)\phi(xy)\, dy$$

for some constant c.

PROPOSITION 2. *Let* λ *be a multiplicative character of* F^* *such that* $\lambda^2 = 1$. *Then* $\theta_2 \cong \theta_2 \otimes \lambda(\det)$.

Proof. Let θ be the even Weil representation of $\widetilde{\mathrm{SL}}_2$. Let G be the subgroup of $\widetilde{\mathrm{GL}}_2$ consisting of the elements whose determinant is a square in F^*. It is easy to see that θ extends to G and that $\theta_2 = \mathrm{ind}_G^{\widetilde{\mathrm{GL}}_2} \theta$. Since $\widetilde{\mathrm{GL}}_2/G \cong F^*/(F^*)^2$ the proposition follows.

DEFINITION 2. Let H be a group and C its center. We say that H is a Heisenberg group if H/C is abelian.

To give a characterization of θ_3 we need a simple result about Heisenberg groups (see [KP, §0.3]):

LEMMA 2. *Let H be a Heisenberg group and C its center. Let δ be a character of C. Assume that δ is faithful on $[H, H] \subset C$. Then there is unique irreducible representation π_δ of H such that C acts by multiplication by δ. Moreover, $\pi_\delta \otimes \pi_{\bar{\delta}}$ is just the regular representation of H/C.* □

Let $\widetilde{T}$ be the inverse image of T in $\widetilde{\mathrm{GL}_3}$. Let Z be the center of GL_3 and $\widetilde{Z}$ the inverse image in $\widetilde{\mathrm{GL}}$. It is easy to check that $\widetilde{Z}$ is the center of $\widetilde{\mathrm{GL}_3}$. The group $\widetilde{T}$ is a Heisenberg group with center $C = \widetilde{Z} \cdot \mathbf{s}(T^2)$ where T^2 is the group of diagonal matrices whose entries are squares. Define a character δ of C by

$$\delta(\mathbf{s}(z)\mathbf{s}(t^2)\zeta) = \gamma(z)\zeta, \qquad \zeta \in \{\pm 1\}.$$

Let π_δ be the corresponding representation of $\widetilde{T}$. Define ρ to be, as usual,

$$\rho\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = \left|\frac{a}{c}\right|.$$

In **[FKS]** we have the following theorem.

THEOREM 1. *There is a unique representation (θ_3, V) of $\widetilde{\mathrm{GL}_3}$ such that $V_N \cong \rho^{1/2} \otimes \pi_\delta$. The properties of θ_3 are*:

(1) $\theta_3(\mathbf{s}(z)) = \gamma(z)\,\mathrm{Id}$, $z \in Z$.

(2) $V_{U_1} \cong \theta_2 \otimes |\det|^{1/4}$, $V_{U_2} \cong \theta_2 \otimes |\det|^{-1/4}$.

(3) *Let $V_0 = V(U_1) \cap V(U_2)$; then $V_0 \cong S(F^* \times F)$ with the action of $\widetilde{B}$ given by*

$$\theta_3\left(\mathbf{s}\begin{pmatrix} a & b & \\ & d & \\ & & 1 \end{pmatrix}\right) f(x, y) = (x, d)|ad|^{1/2} f(ax, bx + dy),$$

$$\theta_3\left(\begin{pmatrix} 1 & & u \\ & 1 & v \\ & & 1 \end{pmatrix}\right) f(x, y) = \phi(ux + vy) f(x, y).$$

(4) $V(U_1) + V(U_2) = V(N)$.

In particular, it follows that we have a filtration of V as a $\widetilde{B}$ module such that the quotients are V_0, $V_{U_1}(N_1)$, $V_{U_2}(N_2)$ and V_N.

REMARK. Note that the dual representation θ_3^* is obtained by replacing ϕ by $\bar{\phi}$.

Proof of the Theorem. Let (π, W) be a representation of GL_3. Then the existence of a nontrivial trilinear GL_3 invariant form is equivalent to the existence of a nontrivial GL_3 intertwining map from $V \otimes V^*$ to W^*. Hence we have to compute $\dim \mathrm{Hom}_G(V \otimes V^*, W)$. Assume that $\pi = \mathrm{ind}_B^G \chi$. Then by the Frobenius reciprocity we have $\dim \mathrm{Hom}_G(V \otimes V^*, W) = \dim \mathrm{Hom}_T((V \otimes V^*)_N, \rho\chi)$. In other words we have restricted the problem to computing the T-equivariant functionals on $(V \otimes V^*)_N$. From $\widetilde{B}$ filtration of V it follows that $(V \otimes V^*)_N$ has a filtration whose quotients are $(V_0 \otimes V_0^*)_N$, $(V_{U_1}(N_1) \otimes V_{U_1}^*(N_1))_{N_1}$, $(V_{U_2}(N_2) \otimes V_{U_2}^*(N_2))_{N_2}$ and $V_N \otimes V_N^*$.

Let Γ_μ be the functional on $S(F^*)$ given by

$$\Gamma_\mu(f) = \int_{F^*} f(x)\mu(x^{-1})\frac{dx}{|x|}.$$

Obviously, we have the following simple proposition:

PROPOSITION 3. *The functional Γ_μ is unique up to a nonzero constant μ-equivariant functional on $S(F^*)$ with respect to the standard action of F^*.* □

Next we need to describe F^* equivariant functionals on $C_2(F)$.

PROPOSITION 4 (see [W2]). *Let μ be a character of F^*. The functional Γ_μ extends to $C_2(F)$ if $\mu^2 \neq 1$.*

Proof. Let $\mathcal{O}$ be the ring of integers of F and ϖ a uniformizing element. Put $q = |\varpi|^{-1}$. Assume that $\mu^2(x) = |x|^{-s}$. Let $f \in C_2(F)$. Consider the integral

$$\Lambda_s(f) = \int_{F^*} (f(x) - f(\varpi^2 x))|x|^s \frac{dx}{|x|}.$$

Obviously $\Lambda_s(f)$ is defined for every s and if $\mathrm{Re}(s) > 0$ then

$$\Lambda_s(f) = (1 - q^s)\Gamma_\mu(f).$$

This formula extends the functional Γ_μ to $C_2(F)$ if $\mu^2(x) = |x|^{-s}$ and $s \neq 0$. If μ^2 is ramified then Γ_μ extends by taking the Principal Value integral. The proposition is proved.

PROPOSITION 5. *Let χ be a character of T. The space of χ-equivariant linear functionals on $V_N \otimes V_N^*$ is at most 1-dimensional. It has the* *dimension one if and only if* $\chi = \rho\lambda$

$$\lambda\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = \mu_1(a)\mu_2(b)\mu_3(c),$$

$\mu_i^2 = 1$ *and* $\mu_1 \cdot \mu_2 \cdot \mu_3 = 1$.

Proof. It follows from Lemma 2.

Let μ be a multiplicative character of F^*. Let $\delta_{1,2}(\mu)$, $\delta_{1,3}(\mu)$ and $\delta_{1,3}(\mu)$ be the characters of T defined by

$$\begin{aligned} \delta_{12}(\mu)(t) &= \mu(a)\mu(b)^{-1}, \\ \delta_{23}(\mu)(t) &= \mu(b)\mu(c)^{-1}, \\ \delta_{13}(\mu)(t) &= \mu(a)\mu(c)^{-1} \end{aligned}$$

where $t = \operatorname{diag}(a, b, c)$. Let $W^{T,\chi}$ denote the space of χ-equivariant functionals on a smooth T module W.

PROPOSITION 6.

(a) $$\dim(V_{U_1}(N_1) \otimes V_{U_1}^*(N_1))_{N_1}^{T,\chi} = \begin{cases} 1 & \text{if } \chi = \rho\delta_{12}(\mu), \\ 0 & \text{otherwise}. \end{cases}$$

(b) $$\dim(V_{U_2}(N_2) \otimes V_{U_2}^*(N_2))_{N_2}^{T,\chi} = \begin{cases} 1 & \text{if } \chi = \rho\delta_{23}(\mu), \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Using property (2) of θ_3 and the description of θ_2 it is easy to check that $V_{U_1}(N_1) \cong S(F^*)$ and therefore $(V_{U_1}(N_1) \otimes V_{U_1}^*(N_1))_{N_1} \cong S(F^*)$ with the action of T given by

$$\theta_3 \otimes \theta_3^* \left(\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \right) f(x) = \left|\frac{a}{c}\right| \left|\frac{a}{b}\right|^{1/2} f\left(\frac{a}{b}x\right).$$

Part (a) now follows from Proposition 3. Part (b) is proved analogously.

PROPOSITION 7.

$$(V_0 \otimes V_0^*)_N^{T,\chi} = \begin{cases} 1 & \text{if } \chi = \rho\delta_{13}(\mu), \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Using property (3) of θ_3 it follows that

$$(V_0 \otimes V_0^*)_{U_1} \cong S(F^* \times F)$$

with the action of TN_1 given by

$$\theta_3 \otimes \theta_3^* \left(\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \right) f(x, y) = \left| \frac{ab}{c^2} \right| f\left(\frac{a}{c} x, \frac{b}{c} y \right) \quad \text{and}$$

$$\theta_3 \otimes \theta_3^* \left(\begin{pmatrix} 1 & n & \\ & 1 & \\ & & 1 \end{pmatrix} \right) f(x, y) = f(x, nx + y).$$

After taking the Fourier transform in the second variable the action becomes

$$\theta_3 \otimes \theta_3^* \left(\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \right) f(x, y) = \left| \frac{a}{c} \right| f\left(\frac{a}{c} x, \frac{c}{b} y \right) \quad \text{and}$$

$$\theta_3 \otimes \theta_3^* \left(\begin{pmatrix} 1 & n & \\ & 1 & \\ & & 1 \end{pmatrix} \right) f(x, y) = f(x, y)\phi(nxy).$$

Therefore $(V_0 \otimes V_0^*)_N \cong S(F^*)$ with the action of T given by

$$\theta_3 \otimes \theta_3^* \left(\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \right) f(x) = \left| \frac{a}{c} \right| f\left(\frac{a}{c} x \right).$$

The proposition follows from Proposition 3.

Let us call T equivariant functionals appearing in Proposition 5 (resp. Propositions 6 and 7) of type I (resp. II and III). Since $V_N \otimes V_N^*$ is a quotient of $(V \otimes V^*)_N$, functionals of type I extend to $(V \otimes V^*)_N$. In the next several propositions we are studying extension of the functionals of type II and III to $(V \otimes V^*)_N$.

Proposition 8. *The functionals of type* II *extend to* $(V \otimes V^*)_N$ *if and only if* $\mu^2 \neq 1$.

Proof. Since V_{U_1} is a quotient of V it follows that $(V_{U_1} \otimes V_{U_1}^*)_{N_1}$ is a quotient of $(V \otimes V^*)_N$. Recall that $V_{U_1} \cong \theta_2 \otimes |\det|^{1/4}$. The value of a $\rho\delta_{12}(\mu)$ equivariant functional on $(V_{U_1}(N_1) \otimes V_{U_1}^*(N_1))_{N_1}$ is given by the following integral.

$$I_\mu(f \otimes f^*) = \int_{F^*} |x|^{1/2} f(x) f^*(x) \mu(x^{-1}) \frac{dx}{|x|}.$$

If $f \in \theta_2 \otimes |\det|^{1/4}$ and $f^* \in \theta_2^* \otimes |\det|^{1/4}$ it follows from the description of θ_2 that $|x|^{1/4} f(x)$ and $|x|^{1/4} f^*(x) \in C_2(F)$. Therefore

I_μ defines a $\rho\delta_{12}(\mu)$ equivariant functional on $(V \otimes V^*)_N$ if $\mu^2 \neq 1$ by Proposition 4. It remains to deal with μ, $\mu^2 = 1$. Let $\varphi \in V_{U_1}$ be a function given by

$$\varphi(x) = \begin{cases} |x|^{-1/4} & \text{if } |x| \leq 1 \text{ and } x \text{ is a square}, \\ 0 & \text{otherwise}. \end{cases}$$

Let

$$v = |\varpi|^2 \varphi \otimes \varphi^* - |\varpi| \theta_2 \otimes \theta_2^* \left(\begin{pmatrix} \varpi^2 & \\ & 1 \end{pmatrix} \right) \varphi \otimes \varphi^* .$$

The projection of v on $(V_{U_1} \otimes V_{U_1}^*)_{N_1}$ lies in $(V_{U_1}(N_1) \otimes V_{U_1}^*(N_1))_{N_1} \cong S(F^*)$ and is given by

$$\begin{aligned} &|\varpi|^2 \varphi(x) \varphi^*(x) - |\varpi|^3 \varphi(\varpi^2 x) \varphi^*(\varpi^2 x) \\ &= \begin{cases} -|\varpi| & \text{if } |x| = |\varpi|^{-2} \text{ and } x \text{ is a square}, \\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

It follows that $I_\mu(v) < 0$. On the other hand, if the functional I_μ extends to $(V \otimes V^*)_N$ then the equivariance implies $I_\mu(v) = 0$. Contradiction. Similar conclusions can be obtained for the characters $\rho\delta_{23}(\mu)$. The proposition is proved.

Let $m_{ij}(\mu)$ be the multiplicity of $\rho\delta_{ij}(\mu)$ equivariant functionals on $(V \otimes V^*)_N$. If the principal series representation $\pi(\mu)$ is irreducible then $\pi(\mu) = \operatorname{ind}_B^G \delta_{ij}(\mu)$ for all $1 \leq i < j \leq 3$ [C]. In particular, $m_{ij}(\mu)$ is independent of i, j. Therefore, we have obtained the following corollary.

COROLLARY 1. *The functional* $\rho\delta_{13}(\mu)$ *of type* III *extends to* $(V \otimes V^*)_N$ *if* $\mu \neq |\cdot|^{\pm 1}$, $\mu^2 \neq |\cdot|^{\pm 1}$ *and* $\mu^2 \neq 1$. *If* $\mu^2 = 1$ *and* $\mu \neq 1$ *then it does not extend.* □

It remains to deal with $\rho\delta_{13}(1)$.

PROPOSITION 9. *The functional* $\rho\delta_{13}(1)$ *of type* III *does not extend to* $(V \otimes V^*)_N$.

Proof. The value of a $\rho\delta_{13}(\mu)$ equivariant functional on $(V_0 \otimes V_0^*)_N$ is given by the following integral:

$$I_\mu(f \otimes f^*) = \int_{F^*} \int_F f(x, y) f^*(x, y) \mu(x^{-1}) \frac{dx}{|x|} \, dy .$$

Let φ be a function on $F^* \times F$ given by

$$\varphi(x, y) = \begin{cases} 1 & \text{if } |x| \leq 1 \text{ and } |y| = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Then $\varphi \in V(U_1)$ and let

$$v = |\varpi|\varphi \otimes \varphi^* - \theta_3 \otimes \theta_3^* \left(\begin{pmatrix} \varpi & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \varphi \otimes \varphi^* .$$

The projection of v on $(V(U_1) \otimes V^*(U_1))_{U_1}$ lies in $(V_0 \otimes V_0^*)_{U_1} \cong S(F^* \times F)$ and is given by

$$\begin{aligned} &|\varpi|\varphi(x, y)\varphi^*(x, y) - |\varpi|\varphi(\varpi x, y)\varphi^*(\varpi x, y) \\ &\quad = \begin{cases} -|\varpi| & \text{if } |x| = |\varpi|^{-1} \text{ and } |y| = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that $I_1(v) < 0$. On the other hand, if the functional I_1 extends to $(V \otimes V^*)_N$ then the equivariance implies $I_1(v) = 0$. Contradiction. The proposition is proved.

COROLLARY 2. *Let χ be a character of T. Then* $\dim(V \otimes V^*)_N^{T,\chi} \leq 1$. □

Let μ_1, μ_2, μ_3 be three characters of F^* such that $\mu_i^2 = 1$ and $\mu_1 \cdot \mu_2 \cdot \mu_3 = 1$. Let χ be a character of T defined by $\chi(\operatorname{diag}(a, b, c)) = \mu_1(a)\mu_2(b)\mu_3(c)$. Define $\pi(\mu_1, \mu_2, \mu_3) = \operatorname{ind}_B^G \chi$ (normalized induction). It is unitary irreducible representation. We are now ready to formulate our main result:

THEOREM. *Let (π, W) be a quotient of a principal series representation of* GL_3. *Then the space of* GL_3 *invariant trilinear forms on* $V \otimes V^* \otimes W$ *is* 0 *or* 1 *dimensional. The dimension is* 0 *unless π is one of the following*:

(a) $\pi(\mu_1, \mu_2, \mu_3)$, $\mu_i^2 = 1$ *and* $\mu_1\mu_2\mu_3 = 1$,
(b) $\pi(\mu)$, $\mu^2 \neq |\cdot|^{\pm 1}$ *and* $\mu \neq |\cdot|^{\pm 1}$,
(c) *trivial representation,*
(d) $\operatorname{ind}_P^{\mathrm{GL}_3} \mu$, $\mu^2 = 1$,
(e) $\operatorname{ind}_P^{\mathrm{GL}_3} \mathrm{St}_\mu$, $\mu^2 = 1$,
(f) σ_1, σ_2 *and* St.

In cases (a)–(d) *the dimension is* 1. *In cases* (e) *and* (f) *the dimension is* ≤ 1.

Proof. Clearly the dimension is 0 unless π is one of the representations in (a)–(f). Since representations in (a) and (b) are irreducible these two cases follow from Corollary 2. The trace $\mathrm{tr}\colon V \times V^* \to \mathbf{C}$

is a GL_3 invariant trilinear form for $\pi = 1$. We can similarly deal with the representations in (d). Indeed, $V_U \otimes V_U^*$ is a quotient of $(V \otimes V^*)_U$. Since $V_U \cong \theta_2 \otimes |\det|^{1/4}$ and $\theta_2 \cong \theta_2 \otimes \mu(\det)$ by Proposition 2 we can define an appropriate P-equivariant functional on $(V \otimes V^*)_U$ defining a map from $V \otimes V^*$ into $\mathrm{ind}_P^G \mu$. The theorem is proved.

COROLLARY. *Let (π, W) be a spherical representation of GL_3. Then there exists a GL_3 invariant trilinear form on $V \otimes V^* \otimes W$ if and only if π is the lift of a representation of SL_2. In this case the form is unique up to a scalar.*

Proof. Note that there is only one nontrivial unramified character μ of F^* such that $\mu^2 = 1$. Therefore if $\pi(\mu_1, \mu_2, \mu_3)$ is spherical then $\pi(\mu_1, \mu_2, \mu_3) \cong \pi(\mu, 1, \mu^{-1})$ for some unramified character μ, $\mu^2 = 1$.

A final remark. Recently Bump and Ginsburg [**BG**] have generalized the work of Patterson and Piatetski-Shapiro to construct an integral representation of the symmetric square L-function corresponding to a cuspidal automorphic representation π of PGL_n. As in the case $n = 3$, the residue at $s = 1$ of the L-function is

$$\int_{\mathrm{PGL}_n(k)\backslash \mathrm{PGL}_n(\mathbf{A})} \varphi(g)\theta(g)\theta'(g)\, dg$$

where $\varphi \in \pi$ and θ, θ' are "theta functions" of $\widetilde{\mathrm{GL}}_n$—the two fold central extension of GL_n. The result of Bump and Ginzburg suggests the following generalization of our result:

CONJECTURE. *Let F be a local field of the characteristic $\neq 2$ and let (θ, V) be the theta representation of $\widetilde{\mathrm{GL}}_n$. Let (π, W) be a spherical representation of PGL_n. Then there exists a GL_n invariant trilinear form on $V \otimes V^* \otimes W$ if and only if π is the lift of a representation of $\mathrm{Sp}(2m)$ if $n = 2m+1$ or π is the lift of a representation of $\mathrm{SO}(2m)$ if $n = 2m$.*

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