PACIFIC JOURNAL OF MATHEMATICS
Vol. 168, No. 1, 1995

MAPS ON INFRA-NILMANIFOLDS

–Rigidity and applications to Fixed-point Theory

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We show that Bieberbach's rigidity theorem for flat manifolds still holds true for any continuous maps on infra-nilmanifolds. Namely, every endomorphism of an almost crystallographic group is semi-conjugate to an affine endomorphism. Applying this result to Fixed-point theory, we obtain a criterion for the Lefschetz number and Nielsen number for a map on infra-nilmanifolds to be equal.

0. Infra-nilmanifolds. Let G be a connected Lie group. Consider the semi-group $\operatorname{Endo}(G)$, the set of all endomorphisms of G, under the composition as operation. We form the semi-direct product $G \rtimes \operatorname{Endo}(G)$ and call it $\operatorname{aff}(G)$. With the binary operation

$$(a, A)(b, B) = (a \cdot Ab, AB),$$

the set $\operatorname{aff}(G)$ forms a semi-group with identity $(e, 1)$, where $e \in G$ and $1 \in \operatorname{Endo}(G)$ are the identity elements. The semi-group $\operatorname{aff}(G)$ "acts" on G by

$$(a, A) \cdot x = a \cdot Ax.$$

Note that (a, A) is not a homeomorphism unless $A \in \operatorname{Aut}(G)$. Clearly, $\operatorname{aff}(G)$ is a subsemi-group of the semi-group of all continuous maps of G into itself, for $((a, A)(b, B))x = (a, A)((b, B)x)$ for all $x \in G$. We call elements of $\operatorname{aff}(G)$ *affine endomorphisms.*

Suppose G is a connected, simply connected, nilpotent Lie group; $\operatorname{Aff}(G) = G \rtimes \operatorname{Aut}(G)$ is called the group of affine automorphisms of G. Let $\pi \subset \operatorname{Aff}(G)$ be a discrete subgroup such that $\Gamma = \pi \cap G$ has finite index in π. Then $\pi\backslash G$ is compact if and only if Γ is a lattice of G. In this case, π is called an *almost crystallographic group.* If moreover, π is torsion-free, π is an *almost Bieberbach group.* Such a group is the fundamental group of an infra-nilmanifold. According to Gromov and Farrell-Hsiang, the class of infra-nilmanifolds coincides with the class of almost flat manifolds.

1. Generalization of Bieberbach's Theorem. In 1911, Bieberbach proved that any automorphism of a crystallographic group is conjugation by an element of $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$. Recently this was generalized to almost crystallographic groups, see [**1**], [**3**] and [**4**].

We shall generalize this result to all homomorphisms (not necessarily isomorphisms). Topologically, this implies that every continuous map on an infra-nilmanifold is homotopic to a map induced by an affine endomorphism on the Lie group level. It can be stated as: every endomorphism of an almost crystallographic group is semi-conjugate to an affine endomorphism.

THEOREM 1.1. *Let $\pi, \pi' \subset \mathrm{Aff}(G)$ be two almost crystallographic groups. Then for any homomorphism $\theta : \pi \to \pi'$, there exists $g = (d, D) \in \mathrm{aff}(G)$ such that $\theta(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in \pi$.*

COROLLARY 1.2. *Let $M = \pi \backslash G$ be an infra-nilmanifold, and $h : M \to M$ be any map. Then h is homotopic to a map induced from an affine endomorphism $G \to G$.*

COROLLARY 1.3 [**3**, **4**]. *Homotopy equivalent infra-nilmanifolds are affinely diffeomorphic.*

Now we consider the uniqueness problem: How many g's are there? Let $\Phi = \pi/(G \cap \pi) \subset \mathrm{Aut}(G)$ and $\Phi' = \pi'/(G \cap \pi') \subset \mathrm{Aut}(G)$ be the holonomy groups of π and π'. Let Ψ' be the image of $\theta(\pi)$ in Φ'. So $\Phi' \subset \mathrm{Aut}(G)$. Let $G^{\Psi'}$ denote the fixed point set of the action. For $c \in G$, $\mu(c)$ denotes conjugation by c. Therefore, $\mu(c)(x) = cxc^{-1}$ for all $x \in G$. The group of all inner automorphisms is denoted by $\mathrm{Inn}(G)$.

PROPOSITION 1.4 (Uniqueness). *With the same notation as above, suppose $\theta(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in \pi$. Then $\theta(\alpha) \cdot \gamma = \gamma \cdot \alpha$ for all $\alpha \in \pi$ if and only if $\gamma = \xi \cdot g$, where $\xi = (c, \mu(c^{-1}))$, for $c \in G^{\Psi'}$. Therefore, D is unique up to* $\mathrm{Inn}(G)$. *If θ is an isomorphism, then $c \in G^{\Phi'}$. In particular, if π is a Bieberbach group with $H^1(\pi; \mathbb{R}) = 0$ and θ is an isomorphism, then such a g is unique.*

EXAMPLE 1.5. The subgroup $\Gamma = \pi \cap G$ of of an almost crystallographic group π is characteristic, but not fully invariant. The homomorphism θ in the Theorem 1.1 may not map the maximal normal nilpotent subgroup Γ of π into that of π'. This causes a lot of trouble. Let π be an orientable 4-dimensional Bieberbach group with holonomy group $\mathbb{Z}_2$. More precisely, $\pi \subset \mathbb{R}^4 \rtimes O(4) = E(4) \subset \mathrm{Aff}(\mathbb{R}^4)$ is generated by $(e_1, I), (e_2, I)$, (e_3, I), (e_4, I) and (a, A), where $a = (1/2, 0, 0, 0)^t$, and A is diagonal matrix with diagonal entries $1, -1, -1$ and 1. Note that $(a, A)^2 = (e_1, I)$. The subgroup generated by (e_1, I), (e_2, I), (e_3, I), and (a, A) forms a 3-dimensional Bieberbach group $\mathcal{G}_2$, and $\pi = \mathcal{G}_2 \times \mathbb{Z}$. Consider the endomorphism $\theta : \pi \to \pi$ which is the composite $\pi \to \mathbb{Z} \to \pi$, where the first map is the projection onto $\mathbb{Z} = \langle (e_4, I) \rangle$ and the second map sends (e_4, I) to (a, A). Thus the homomorphism θ does not map the maximal normal abelian subgroup $\mathbb{Z}^4$ (generated by the 4 translations) into itself. Such a $\mathbb{Z}^4$ is characteristic but not fully invariant in π. Let

$$d = \begin{bmatrix} x \\ 0 \\ 0 \\ y \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and let $g = (d, D)$. Then it is easy to see $\theta(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in \pi$.

According to the Proposition 1.4, the element $g = (d, D)$ is the most general form. The matrix D is uniquely determined and the translation part d can vary only in two dimensions.

Proof of Theorem 1.1. Let $\Gamma = \pi \cap G$, $\Gamma' = \pi' \cap G$. As the example shows, the characteristic subgroup Γ may not go into Γ' by the homomorphism θ. Let $\Lambda = \Gamma \cap \theta^{-1}(\Gamma')$. Then Λ is a normal subgroup of π and has a finite index. Let $Q = \pi/\Lambda$.

Consider the homomorphism $\Lambda \to \Gamma' \hookrightarrow G$, where the first map is the restriction of θ. Since Λ is a lattice of G, by Malćev's work, any such a homomorphism extends uniquely to a continuous homomorphism $C : G \to G$, cf. [**5**, 2.11]. Thus, $\theta|_\Lambda = C|_\Lambda$, where $C \in \mathrm{Endo}(G)$; and hence, $\theta(z, 1) = (Cz, 1)$ for all $z \in \Lambda$ (more precisely, $(z, 1) \in \Lambda$).

Let us denote the composite homomorphism $\pi \to \pi' \to G \rtimes \mathrm{Aut}(G) \to \mathrm{Aut}(G)$ by $\overline{\theta}$; and define a map $f : \pi \to G$ by

$$\theta(w, K) = (Cw \cdot f(w, K), \overline{\theta}(w, K)). \tag{1}$$

For any $(z,1) \in \Lambda$ and $(w,K) \in \pi$, apply θ to both sides of $(w,K)(z,1)(w,K)^{-1} = (w \cdot Kz \cdot w^{-1}, 1)$ to get $Cw \cdot f(w,K) \cdot \overline{\theta}(w,K)(Cz) \cdot f(w,K)^{-1} \cdot (Cw)^{-1} = \theta(w \cdot Kz \cdot w^{-1})$. However, $w \cdot Kz \cdot w^{-1} \in \Lambda$ since Λ is normal in π, and the latter term equals to $C(w \cdot Kz \cdot w^{-1}) = Cw \cdot CKz \cdot (Cw)^{-1}$ since $C : G \to G$ is a homomorphism. From this we have

$$\overline{\theta}(w,K)(Cz) = f(w,K)^{-1} \cdot CKz \cdot f(w,K). \tag{2}$$

This is true for all $z \in \Lambda$. Note that $\overline{\theta}(w,K)$ and K are automorphisms of the Lie group G; and $C : G \to G$ is an endomorphism. By the uniqueness of extension of a homomorphism $\Lambda \to G$ to an endomorphism $G \to G$, as mentioned above, *the equality (2) holds true for all* $z \in G$. It is also easy to see that $f(zw,K) = f(w,K)$ for all $z \in \Lambda$ so that $f : \pi \to G$ does not depend on Λ. Thus, f factors through $Q = \pi/\Lambda$. Moreover, $\overline{\theta} : \pi \to \mathrm{Aut}(G)$ also factors through Q since Λ maps trivially into $\mathrm{Aut}(G)$. We still use the notation (w,K) to denote elements of Q and $\overline{\theta}$ to denote the induced map $Q \to \mathrm{Aut}(G)$.

CLAIM. *With the Q-structure on G via $\overline{\theta} : Q \to \mathrm{Aut}(G)$, $f \in Z^1(Q;G)$; that is $f : Q \to G$ is a crossed homomorphism.*

Proof. We shall show

$$f((w,K) \cdot (w',K')) = f(w,K) \cdot \overline{\theta}(w,K) f(w',K')$$

for all (w,K), $(w',K') \in \pi$. (Note that we are using the elements of π to denote the elements of Q.) Apply θ to both sides of $(w,K)(w',K') = (w \cdot Kw', KK')$ to get $Cw \cdot f(w,K) \cdot \overline{\theta}(w,K)[Cw' \cdot f(w',K')] = C(w \cdot Kw') \cdot f((w,K)(w',K'))$. From this it follows that

$$\begin{aligned} f((w,K)(w',K')) = (CKw')^{-1} \cdot f(w,K) \\ \cdot \overline{\theta}(w,K)(Cw') \cdot \overline{\theta}(w,K) f(w',K'). \end{aligned}$$

From (2) we have $\overline{\theta}(w,K)Cw' = f(w,K)^{-1} \cdot CKw' \cdot f(w,K)$ so that $f((w,K) \cdot (w',K')) = f(w,K) \cdot \overline{\theta}(w,K) f(w',K')$. □

In [**4**], it was proved that $H^1(Q;G) = 0$ whenever Q is a finite group and G is a connected and simply connected nilpotent Lie group. The proof uses induction on the nilpotency of G together with the fact that $H^1(Q;G) = 0$ for a finite group Q and a real vector group G. This means that any crossed homomorphism is "principal". In other words, there exists $d \in G$ such that

$$f(w,K) = d \cdot \overline{\theta}(w,K)(d^{-1}). \tag{3}$$

Let $D = \mu(d^{-1}) \circ C$ and $g = (d,D) \in \mathrm{aff}(G)$, and we check that θ is "conjugation" by g. Using (1), (2) and (3), one can show $\overline{\theta}(w,K) \circ \mu(d^{-1}) \circ C = \mu(d^{-1}) \circ C \circ K$. Thus, for any $(w,K) \in \pi$,

$$\begin{aligned}
&\theta(w,K) \cdot (d,D) \\
&\quad = (Cw \cdot f(w,K),\ \overline{\theta}(w,K)) \cdot (d,\ \mu(d^{-1}) \circ C) \\
&\quad = (Cw \cdot f(w,K) \cdot \overline{\theta}(w,K)(d),\ \overline{\theta}(w,K) \circ \mu(d^{-1}) \circ C) \\
&\quad = (Cw \cdot d \cdot \overline{\theta}(w,K)(d^{-1}) \cdot \overline{\theta}(w,K)(d),\ \overline{\theta}(w,K) \circ \mu(d^{-1}) \circ C) \\
&\quad = (Cw \cdot d,\ \mu(d^{-1}) \circ C \circ K) \\
&\quad = (d,D) \cdot (w,K).
\end{aligned}$$

This finishes the proof of theorem. □

Proof of Corollary 1.2. We start with the homomorphism $h_\# : \pi_1(M) \to \pi_1(M)$, induced from h, as our θ in the Theorem 1.1 and obtain $\tilde{g} = (d,D)$ satisfying

$$h_\#(\alpha) \circ \tilde{g} = \tilde{g} \circ \alpha.$$

Let $g : M \to M$ be the induced map. Then $h_\# = g_\#$. Since any two continuous maps on a closed aspherical manifold inducing the same homomorphism on the fundamental group (up to conjugation by an element of the fundamental group) are homotopic to each other, h is homotopic to g. This completes the proof of the corollary. □

Proof of Proposition 1.4. Let $g = (d,D)$, $\gamma = (c,C)$. Since $\theta(\alpha) \cdot g = g \cdot \alpha$ holds when $\alpha = (z,1) \in \Lambda$, we have $Dz = d^{-1}z'd$, where $\theta(z,1) = (z',1)$. Similarly, $Cz = c^{-1}z'c$. Thus $Cz = \mu(c^{-1}d)Dz$ for all $z \in \Lambda$. Since Λ is a lattice, this equality holds on G. Consequently, $C = \mu(c^{-1}d)D$. Now $\gamma = (c,C) = (c,\ \mu(c^{-1}d)D) = (d^{-1}c,\ \mu(c^{-1}d))(d,D) = (h,\mu(h^{-1}))(d,D)$, if we let $h = d^{-1}c$. Set $\xi = (h, \mu(h^{-1}))$. Then $\gamma = \xi \cdot g$. Now we shall observe that $h \in G^{\Psi'}$. Let $\theta(\alpha) = (b, B)$. Then $\theta(\alpha)\xi g = \theta(\alpha)\gamma = \gamma\alpha = \xi g \alpha = \xi\theta(\alpha)g$ yields $Bh = h$ for all $(b, B) = \theta(\alpha)$. Clearly then $B \in \Psi'$ by definition. For a Bieberbach group π, note that rank $H^1(\pi; \mathbb{Z}) = \dim G^{\Phi}$. □

2. Application to Fixed-point theory. Let M be a closed manifold and let $f : M \to M$ be a continuous map. The *Lefschetz number* $L(f)$ of f is defined by

$$L(f) := \sum_k \operatorname{trace}\{(f_*)_k : H_k(M; \mathbb{Q}) \to H_k(M; \mathbb{Q})\}.$$

To define the *Nielsen number* $N(f)$ of f, we define an equivalence relation on $\operatorname{Fix}(f)$ as follows: For $x_0, x_1 \in \operatorname{Fix}(f)$, $x_0 \sim x_1$ if and only if there exists a path c from x_0 to x_1 such that c is homotopic to $f \circ c$ relative to the end points. An equivalence class of this relation is called a *fixed point class* (=FPC) of f. To each FPC F, one can assign an integer $\operatorname{ind}(f, F)$. A FPC F is called *essential* if $\operatorname{ind}(f, F) \neq 0$. Now,

$$N(f) := \text{the number of essential fixed point classes.}$$

These two numbers give information on the existence of fixed point sets. If $L(f) \neq 0$, every self-map g of M homotopic to f has a non-empty fixed point set. The Nielsen number is a lower bound for the number of components of the fixed point set of all maps homotopic to f. Even though $N(f)$ gives more information than $L(f)$ does, it is harder to calculate. If M is an infra-nilmanifold, and f is homotopically periodic, then it is known that $L(f) = N(f)$.

LEMMA 2.1. *Let $B \in \mathrm{GL}(n, \mathbb{R})$ with a finite order. Then* $\det(I - B) \geq 0$.

Proof. Since B has finite order, it can be conjugated into the orthogonal group $O(n)$. Since all eigenvalues are roots of unity, there exists $P \in \mathrm{GL}(n, \mathbb{R})$ such that PBP^{-1} is a block diagonal matrix, with each block being a 1×1, or, a 2×2-matrix. All 1×1 blocks must be $D = [\pm 1]$, and hence $\det(I - D) = 0$ or 2. For a

2×2 block, it is of the form $\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. Consequently, each 2×2-block D has the property that $\det(I-D) = (1-\cos t)^2 + \sin^2 t = 2(1-\cos t) \geq 0$. $\square$

THEOREM 2.2. *Let $f : M \to M$ be a continuous map on an infra-nilmanifold $M = \pi\backslash G$. Let $g = (d, D) \in \operatorname{aff}(G)$ be a homotopy lift of f by Corollary 1.2. Then $L(f) = N(f)$ (resp., $L(f) = -N(f)$) if and only if $\det(I - D_*A_*) \geq 0$ (resp., $\det(I - D_*A_*) \leq 0$) for all $A \in \Phi$, the holonomy group of M.*

Proof. Since $L(f)$ and $N(f)$ are homotopy invariants, we may assume that $f = g$. Let $\Gamma = \pi \cap G$, and let $\Lambda = \Gamma \cap f_\#^{-1} f_\#(\Gamma \cap f_\#^{-1}(\Gamma))$. Then Γ is a normal subgroup of π, of finite index. Moreover, $f_\# : \pi \to \pi$ maps Λ into itself. Therefore, f induces a map on the finite-sheeted regular covering space $\Lambda\backslash G$ of $\pi\backslash G$.

Let $\tilde{f}$ be a lift of f to $\Gamma\backslash G$. Then

$$L(f) = \frac{1}{[\pi : \Lambda]} \Sigma \ \operatorname{ind}(f, p_\Lambda \operatorname{Fix}(\alpha\tilde{f}))$$

$$N(f) = \frac{1}{[\pi : \Lambda]} \Sigma \ |\operatorname{ind}(f, p_\Lambda \operatorname{Fix}(\alpha\tilde{f}))|$$

where the sum ranges over all $\alpha \in \pi/\Lambda$. See, [**2**, III 2.12]. However, each $\alpha\tilde{f}$ is a map on the nilmanifold $\Lambda\backslash G$, and hence $\operatorname{ind}(f, p_\Lambda \operatorname{Fix}(\alpha\tilde{f}))$ is determined by $\det(I - (\alpha f)_*)$. It is not hard to see that, for any $\alpha \in \operatorname{Inn}(G)$, α_* has eigenvalue only 1. Therefore, it is enough to look at elements with non-trivial holonomy. Now the hypothesis guarantees that $\det(I - (\alpha f)_*) = \det(I - D_*A_*) \geq 0$ or ≤ 0 always. Consequently, $L(f) = N(f)$ or $L(f) = -N(f)$.

Conversely, suppose $L(f) = N(f)$ (resp. $L(f) = -N(f)$). Let $\alpha = (a, A) \in \pi$. If $\operatorname{Fix}(g \circ \alpha) = \varnothing$, then clearly $\det(I - D_*A_*) = 0$. Otherwise, $\operatorname{Fix}(g \circ \alpha)$ is isolated, and the indices at these fixed points are $\det(I - D_*A_*)$. By the formula above relating $L(f)$, $N(f)$ with the ones on covering spaces, all $\det(I - D_*A_*)$ must have the same sign. This proves the theorem. $\square$

COROLLARY 2.3 [**3**]. *Let $f : M \to M$ be a homotopically periodic map on an infra-nilmanifold. Then $N(f) = L(f)$.*

Proof. Here is an argument which is completely different from the one in [**3**]. Let $\Gamma = \pi \cap G$, and $\Phi = \pi/\Gamma$, the holonomy group.

Let $g = (d, D) \in G \rtimes \mathrm{Aut}(G)$ be a homotopy lift of f to G. Let E be the lifting group of the action of $\langle g \rangle$ to G. That is, E is generated by π and g. Then E/Γ is a finite group generated by Φ and D. For every $A \in \Phi$, DA lies in E/Γ, and has a finite order. By Lemma 2.1, $\det(I - DA) \geq 0$ for all $A \in \Phi$. By Theorem 2.2, $L(f) = N(f)$. □

Let S be a connected, simply connected solvable Lie group and H be a closed subgroup of S. The coset space $H\backslash S$ is called a solvmanifold.

COROLLARY 2.4 [**7**]. *Let $f : M \to M$ be a homotopically periodic map on an infra-solvmanifold. Then $N(f) = L(f)$.*

Proof. In [**5**], the statement for solvmanifolds was proved. We needed a subgroup invariant under $f_{\#}$. To achieve this, a new model space M' which is homotopy equivalent to M, together with a map $f' : M' \to M'$ corresponding to f was constructed. The new space M' is a fiber bundle over a torus with fiber a nilmanifold; and f' is fiber-preserving. Moreover, we found a fully invariant subgroup Λ of π of finite index (so, is invariant under $f'_{\#}$). Now we can apply the same argument as in the proof of Theorem 2.2. □

EXAMPLE 2.5. Let π be an orientable 3-dimensional Bieberbach group with holonomy group $\mathbb{Z}_2$. More precisely, $\pi \subset \mathbb{R}^3 \rtimes O(3) = E(3)$ is generated by $(e_1, I), (e_2, I)$, (e_3, I) and (a, A), where $a = (1/2, 0, 0)^t$, A is a diagonal matrix with diagonal entries $1, -1$ and -1. Note that $(a, A)^2 = (e_1, I)$. Let $M = \mathbb{R}^3/\pi$ be the flat manifold. Consider the endomorphism $\theta : \pi \to \pi$ which is defined by the conjugation by $g = (d, D)$, where

$$d = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

Let $f : M \to M$ be the map induced from g. There are only two conjugacy classes of g; namely, g and αg. $\mathrm{Fix}(g) = (0, 0, 0)^t$ and $\mathrm{Fix}(\alpha g) = (1/4, 0, 0)^t$. Since $\det(I - D) = \det(I - AD) = +2$, $L(f) = N(f) = 2$.

The Lefschetz number can be calculated from homology groups also.

(1) $H_0(M; \mathbb{R}) = \mathbb{R}$; f_* is the identity map.

(2) $H_1(M;\mathbb{R}) = \mathbb{R}$, which is generated by the element (e_1, I). f_* is multiplication by 3 (the (1,1)-entry of D).

(3) $H_2(M;\mathbb{R}) = \mathbb{R}$; f_* is multiplication by $\det \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = -2$.

(4) $H_3(M;\mathbb{R}) = \mathbb{R}$; f_* is multiplication by $\det(D) = -6$.

Therefore, $L(f) = \Sigma(-1)^i \mathrm{trace} f_* = 1 - 3 + (-2) - (-6) = 2$. Note that f has infinite period, and this example is not covered by Corollary 2.3.

EXAMPLE 2.6. Let π be same as in Example 2.5. This time $g = (d, D)$, is given by

$$d = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Let $f : M \to M$ be the map induced from g. There are six conjugacy classes of g; namely, g and αg, $\alpha t_1 g$, $\alpha t_1^2 g$, $\alpha t_1^3 g$, and $\alpha t_1^4 g$. Each class has exactly one fixed point. Clearly, $\det(I - D) = +2$ and $\det(I - AD) = -10$. Therefore, the first fixed point has index $+1$ and the rest have index -1. Consequently, $L(f) = -4$, while $N(f) = 6$.

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Received July 14, 1992.

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