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THE L^p THEORY OF STANDARD HOMOMORPHISMS

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Suppose that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous nonzero homomorphism between weighted convolution algebras on R^+, and let ϕ also designate the extension of this map to the corresponding measure algebras $M(\omega_1)$ and $M(\omega_2)$. For $1 < p < \infty$, we prove: (a) the semigroup $\mu_t = \phi(\delta_t)$ acts as a strongly continuous semigroup on $L^p(\omega_2)$; (b) Whenever $L^1(\omega_1) * f$ is dense in $L^1(\omega_1)$, then $L^p(\omega_2) * \phi(f)$ is dense in $L^p(\omega_2)$; (c) Each h in $L^p(\omega_2)$ can be factored as $h = \phi(f) * g$; (d) ϕ is continuous from the strong operator topology of $M(\omega_1)$ acting on $L^1(\omega_1)$ to the strong operator topology of $M(\omega_2)$ acting on $L^p(\omega_2)$.

1. Introduction. In this paper we show that the L^p analogue of a number of questions we have studied ([**10**], [**8**], [**11**], [**7**]) involving homomorphisms and semigroups on weighted L^1 spaces on $R^+ = [0, \infty)$ all have positive answers when $1 < p < \infty$. If $\omega(t) > 0$ is a Borel function on R^+ which is locally bounded and locally bounded away from 0 and if $1 \le p < \infty$, we let $L^p(\omega)$ be the Banach space of (equivalence classes of) measurable functions on R^+ with $f\omega$ in $L^p(R^+)$, with the inherited norm

$$\|f\| = \|f\|_{\omega,p} = \|f\omega\|_p = \left(\int_0^\infty |f(t)\omega(t)|^p \, dt\right)^{1/p}.$$

We are particularly interested in the case that $L^1(\omega)$ is a Banach algebra and all $L^p(\omega)$ are $L^1(\omega)$-modules under the usual convolution multiplication $f * g(x) = \int_0^x f(x-t)g(t)\, dt$. Therefore we will usually assume that $\omega(t)$ is an *algebra weight*, that is $\omega(t)$ satisfies:
(1) $\omega(x+y) \le \omega(x)\omega(y)$;
(2) $\omega(x)$ is right continuous;
(3) $\omega(0) = 1$.
(1), (2), and (3) are just normalizations and are essentially equivalent to $L^1(\omega)$ being an algebra in which case $L^p(\omega)$ is an $L^1(\omega)$-module [**9**], where the module action is convolution. The most

important cases are the classical case $\omega(t) \equiv 1$, so that $L^1(\omega) = L^1(R^+)$, and the case $\lim_{t\to\infty} \omega(t)^{\frac{1}{t}} = 0$, so that $L^1(\omega)$ is a radical Banach algebra.

When $\omega(t)$ is an algebra weight, the space $M(\omega)$ of locally finite Borel measures satisfying $\|\mu\| = \|\mu\|_\omega = \int_{R^+} \omega(t)\ d|\mu|(t) < \infty$ is also a Banach algebra under convolution and each $L^p(\omega)$ is an $M(\omega)$ Banach module. We will usually identify the measure μ in $M(\omega)$ with the linear operator of convolution by μ on $L^p(\omega)$, so that $M(\omega)$ has a strong operator topology for its action on each $L^p(\omega)$. Particularly important is the fact that convolution by the point mass δ_a is right translation by a, so that the set of all δ_t for $t \geq 0$ is identified with the strongly continuous right translation semigroup on $L^p(\omega)$ for $1 \leq p < \infty$.

Suppose now that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous algebra homomorphism, with $\omega_1(t)$ and $\omega_2(t)$ algebra weights. This homomorphism has a unique extension to a homomorphism, which we also call ϕ, from $M(\omega_1)$ to $M(\omega_2)$ [**10**, Theorem 3.4, p. 596], so that $\mu_t = \phi(\delta_t)$ is a semigroup in $M(\omega_2)$. The following definition lists the properties we would like to prove for ϕ.

DEFINITION (1.1). We say the above homomorphism ϕ is *standard for* p, where $1 \leq p < \infty$ is fixed, if the following properties all hold.

(a) The semigroup $\mu_t = \phi(\delta_t)$ is strongly continuous on $L^p(\omega_2)$; that is $\lim_{t\to 0} \mu_t * g = g$ for all g in $L^p(\omega_2)$.

(b) Whenever $L^1(\omega_1) * f$ is dense in $L^1(\omega_1)$, then $L^p(\omega_2) * \phi(f)$ is dense in $L^p(\omega_2)$.

(c) For each h in $L^p(\omega_2)$, we can write $h = \phi(f) * g$ for some $f \in L^1(\omega_1)$ and $g \in L^p(\omega_2)$.

(d) Whenever $\{\lambda_n\}$ is a net in $M(\omega_1)$ for which $\lim_{n\to\infty} \lambda_n * f = \lambda * f$ for all f in $L^1(\omega_1)$ then $\lim_{n\to\infty} \phi(\lambda_n) * g = \phi(\lambda) * g$ for all g in $L^p(\omega_2)$; that is, ϕ is continuous from the strong operator topology on $M(\omega_1)$ acting on $L^1(\omega_1)$ to the strong operator topology of $M(\omega_2)$ acting on $L^p(\omega_2)$.

The main result of this paper, Theorem (3.1), is that ϕ is always standard for p when $1 < p < \infty$. When $p = 1$, we have previously shown , in joint work with Peter McClure [**7**, Theorem (2.2), p. 280], that conditions (a), (b), (c) and (d) are equivalent, but when $p = 1$, we have so far only been able to prove that these conditions hold for a special class of weights [**8**, Theorem (3.4), p. 284], the regulated weights of Bade and Dales [**1**].

In §2, we will compare various types of convergence in $L^p(\omega)$ and show that a class of semigroups, which will include all $\mu_t = \phi(\delta_t)$, are strongly continuous when $p > 1$. §3 will be devoted to proving that all ϕ are standard for all $p > 1$. In §4, we observe that, for sequences which do not come from semigroups $\mu_t = \phi(\delta_t)$, the best convergence results require that the weights be regulated, just as when $p = 1$ [**7**].

2. Types of convergence. Many of the parts of the definition of standard homomorphisms involve comparing various convergence properties for a bounded sequence or net $\{\lambda_n\}$ in $M(\omega)$, just as in the case $p = 1$ (see [**10**], [**7**]). For $1 \le p < \infty$, the dual space of $L^p(\omega)$ is $L^q(1/\omega)$, where q is the conjugate exponent to p, under the usual duality $\langle f, h\rangle = \int_0^\infty f(t)h(t)\,dt$. Also, when $\omega(t)$ is an algebra weight, $M(\omega)$ is the dual space of $C_0(1/\omega)$ under the analogous duality [**10**, Theorem 2.2, p. 592]. Here $C_0(1/\omega)$ is the subspace of $L^\infty(1/\omega)$ composed of continuous functions $h(t)$ for which $\lim_{t\to\infty} h(t)/\omega(t) = 0$. The following is our basic convergence result.

LEMMA (2.1). *Suppose that $\omega(t)$ is an algebra weight and that $\{\lambda_n\}$ is a bounded net in $M(\omega)$. If either*

(i) *There is a $\nu \neq 0$ in $M(\omega)$ for which $\lambda_n * \nu \to \lambda * \nu$ weak* in $M(\omega) = C_0(1/\omega)^*$*

or

(ii) *There is a $g \neq 0$ in some $L^p(\omega)$ with $1 < p < \infty$, for which $\text{weak} - \lim(\lambda_n * g) = \lambda * g$ in $L^p(\omega)$, then $\lambda_n * \nu \to \lambda * \nu$ weak* for all ν in $M(\omega)$ and $\lambda_n * g \to \lambda * g$ weakly in $L^p(\omega)$ for all g in all $L^p(\omega)$ with $1 < p < \infty$.*

The most important ν in $M(\omega)$ is the point mass δ_0, which is the identity for convolution, so that the assertion of convergence for this ν just says $\lambda_n \to \lambda$ weak* in $M(\omega)$. We could have considered only sequences instead of bounded nets in the above lemma, since, when restricted to bounded sets, the weak*- topology on $M(\omega)$ and the weak topologies on $L^p(\omega)$ for finite p are metrizable.

Proof of Lemma (2.1). The key to passing between weak* and weak convergence for bounded nets or sequences is the observation that if f belongs to $L^1(\omega) \cap L^p(\omega) \subseteq M(\omega) \cap L^p(\omega)$ for some $1 < p < \infty$, then

(2.2)
$$\lambda_n * f \to \lambda * f \text{ weak* in } M(\omega) \Leftrightarrow \lambda_n * f \to \lambda * f \text{ weakly in } L^p(\omega).$$

Formula (2.2) holds because both weak* and weak convergence are equivalent to $\lim\langle \lambda_n * f, h\rangle = \langle \lambda * f, h\rangle$ for all continuous h with compact support in $R^+ = [0, \infty)$, since the continuous functions with compact support are dense in $C_0(1/\omega)$ and in all $L^q(1/\omega)$ for $1 \leq q < \infty$.

Now suppose hypothesis (i) holds; then $\lambda_n * \nu \to \lambda * \nu$ weak* in $M(\omega)$ for all ν in $M(\omega)$ by [**10**, Lemma 3.2, p. 595]. Thus formula (2.2) shows that $\lambda_n * f \to \lambda * f$ weakly for all f in $L^1(\omega) \cap L^p(\omega)$. Since $L^1(\omega) \cap L^p(\omega)$ is dense in $L^p(\omega)$ when $p < \infty$, this proves that the weak limit of $\lambda_n * f$ is $\lambda * f$ for all f in all $L^p(\omega)$ with $1 < p < \infty$.

Now we suppose that hypothesis (ii) holds and verify that $\lambda_n \to \lambda$ weak* in $M(\omega)$, which is hypothesis (i) for $\nu = \delta_0$. Since $\{\lambda_n\}$ is a bounded net, it follows from weak* compactness that there is a submet $\{\lambda'_n\}$ which converges weak* to some λ' in $M(\omega)$. To complete the proof we show that we must have $\lambda' = \lambda$. The subsequence $\{\lambda'_n\}$ satisfies hypothesis (i), so $\lambda'_n * g \to \lambda' * g$ weakly in $L^p(\omega)$. But $\{\lambda'_n * g\}$ is a subnet of the weakly convergent net $\{\lambda_n * g\}$, so that $\lambda * g = \lambda' * g$ and $g \neq 0$. Since it follows from the Titchmarsh convolution theorem [**3**] that the collection of locally finite measures on R^+ is an integral domain under convolution, we have $\lambda = \lambda'$ as required. This completes the proof. ☐

We can now prove that the convolution semigroups we need to consider are strongly continuous on $L^p(\omega)$.

THEOREM (2.3). *Suppose that $\{\mu_t\}$ is a convolution semigroup in $M(\omega)$ with $\|\mu_t\|$ bounded as $t \to 0^+$. Then $\{\mu_t\}$ is a strongly continuous semigroup on $L^p(\omega)$ for all p with $1 < p < \infty$ if any of the following conditions hold.*

(i) weak* $- \lim_{t\to 0^+} \mu_t = \delta_0$.

(ii) *There is some $\nu \neq 0$ in $M(\omega)$ for which $\mu_t * \nu$ is* weak*-*continuous from the right at some $t \geq 0$.*

(iii) *There is some $g \neq 0$ in some $L^p(\omega)$ with $1 < p < \infty$ for* *which $\mu_t * g$ is weakly continuous from the right in $L^p(\omega)$ at some $t \geq 0$.*

Proof. It follows from Lemma (2.4) that any of conditions (i), (ii) and (iii) implies that μ_t acts as a weakly continuous semigroup on all $L^p(\omega)$ with $1 < p < \infty$. But it is a standard result [**12**, Theorem 10.6.5, p. 324] that a weakly continuous semigroup is strongly continuous. □

For $p = 1$, the conditions in the above theorem imply that μ_t is strongly continuous on $L^1(\omega)$ for $t > 0$ and that all $\mu_t * \nu$ are weak*-continuous for $t \geq 0$ [**11**, Theorem (2.1), p. 160]. For all such semigroups in $M(\omega)$ to be strongly continuous on $L^1(\omega)$ at $t = 0$ is completely equivalent to all continuous homomorphisms from some $L^1(\omega_1)$ to $L^1(\omega)$ being standard for $p = 1$ [**11**, Theorem 2.9, p. 164].

3. Standard homomorphisms. We are now ready for our major result, verifying that all homomorphisms satisfy the conditions of Definition (1.1) when $p > 1$.

THEOREM (3.1). *If $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous nonzero homomorphism, then ϕ is standard for $1 < p < \infty$.*

Proof. Recall that we always extend the homomorphism to a homomorphism $\phi : M(\omega_1) \to M(\omega_2)$, and that ϕ has the same norm on $L^1(\omega_1)$ and $M(\omega_1)$ [**10**, Theorem 3.4, p. 596]. We let $\mu_t = \phi(\delta_t)$ and note that since $||\delta_t|| = \omega_1(t)$ we have

$$||\mu_t|| \leq ||\phi||\omega_1(t), \tag{3.2}$$

where the norm $||\mu_t||$ is taken in $M(\omega_2)$.

We first prove (a) of Definition (1.1). Choose some f in $L^1(\omega_1)$ with $\phi(f) \neq 0$. Then $\mu_t * \phi(f) = \phi(\delta_t * f)$ is norm continuous, and hence weak*-continuous in $L^1(\omega_2) \subseteq M(\omega_2)$. Formula (3.2) shows that $||\mu_t||$ is bounded as $t \to 0^+$, so it follows from Theorem (2.3) that $\{\mu_t\}$ acts as a strong continuous semigroup on all $L^p(\omega_2)$ with $1 < p < \infty$.

Now use (a) to prove (b) in Definition (1.1). For simplicity we normalize to the case that $\lim_{t\to\infty} \omega_1(t)^{1/t} < 1$. This normalization is accomplished by replacing $\omega_1(t)$ by some $e^{-rt}\omega_1(t)$ and recalling that the map $f(t) \to f(t)e^{-rt}$ is an isometric isomorphism from $L^1(e^{-rt}\omega_1(t))$ onto $L^1(\omega_1)$.

By our normalization, $u(t) \equiv 1$ belongs to $L^1(\omega_1)$ and, using formula (3.2), we see that $\lim_{t\to\infty} \|\mu_t\|^{1/t} < 1$. Thus if we let $-A$ be the generator of the semigroup $\{\mu_t\}$ on $L^p(\omega_2)$, we have that A is a one-one closed operator from a dense subspace of $L^p(\omega_2)$ onto $L^p(\omega_2)$ with

$$A^{-1}(g) = \int_0^\infty \mu_t * g \, dt \tag{3.3}$$

as a Bochner integral in $L^p(\omega_2)$ whenever $g \in L^p(\omega_2)$ [**5**, pp. 620-622]. (Actually, since $\mu_t * g$ is continuous, the integral is just a vector-valued improper Rieman integral.) The main part of the proof of (b) is proving that if we let $v = \phi(u)$ then

$$v * g = \int_0^\infty \mu_t * g \, dt \text{ for all } g \text{ in } L^p(\omega_2). \tag{3.4}$$

Formula (3.4) will imply that $L^p(\omega_2) * v = \mathrm{Range}(A^{-1}) = \mathrm{Dom}(A)$, which is dense in $L^p(\omega_2)$.

The first step in verifying (3.4) for $p > 1$ is to prove it when $p = 1$. In $L^1(\omega_2)$ we have $\mu_t * g$ strongly measurable, in fact continuous for $t > 0$ [**10**, Theorem 3.6, p. 599], and $\|\mu_t * g\| \leq \|\phi\|\|g\|\omega_1(t)$ which is integrable by our normalization. Thus the integral in formula (3.4) defines a bounded linear operator on $L^1(\omega_2)$. Since each μ_t is a multiplier of $L^1(\omega_2)$, so is this bounded operator. Hence [**10**, Theorem (2.2) (E), p. 592] there is a measure λ in $M(\omega_2)$ for which $\lambda * g = \int_0^\infty \mu_t * g \, dt$ for g in $L^1(\omega_2)$. Now choose some $g = \phi(f) \neq 0$ in the range of ϕ. The standard convolution formula says $u * f = \int_0^\infty u(t)\delta_t * f \, dt = \int_0^\infty \delta_t * f \, dt$. Applying ϕ to this formula gives

$$v * g = \phi(u * f) = \int_0^\infty \phi(\delta_t * f) \, dt = \int_0^\infty \mu_t * g \, dt = \lambda * g.$$

Since locally finite measures on R^+ form an integral domain, it follows that $v = \lambda$, so that formula (3.4) holds for $p = 1$.

Now fix $p > 1$. Since $L^1(\omega_2) \cap L^p(\omega_2)$ is dense in $L^p(\omega_2)$ and both the integral and convolution by v are continuous linear operators on g in $L^p(\omega_2)$, it will be enough to verify formula (3.4) for g in $L^1(\omega_2) \cap L^p(\omega_2)$. Suppose that h is a continuous function with compact support, so that h belongs to $(L^p(\omega_2))^* = L^q(1/\omega_2)$ and to

$L^1(\omega_2)^* = L^\infty(1/\omega_2)$. It then follows from formula (3.4) for $p = 1$ that

$$\langle v * g, h\rangle = \left\langle \int_0^\infty \mu_t * g \, dt, h \right\rangle,$$

when the integral is considered as an integral in $L^1(\omega_2)$. But the scalar integral $\int_0^\infty \langle \mu_t * g, h\rangle \, dt$ equals $\langle \int_0^\infty \mu_t * g \, dt, h\rangle$ whether the vector integral is considered in $L^1(\omega_2)$ or $L^p(\omega_2)$. Thus $\langle v * g, h\rangle = \langle \int_0^\infty \mu_t * g \, dt, h\rangle$, when the integral is an $L^p(\omega_2)$ integral, for all h in a dense subspace of the dual space $(L^p(\omega_2))^*$. This proves formula (3.4), and hence that $L^p(\omega_2) * v$ is dense in $L^p(\omega_2)$. Thus we have proved part (b) of Definition (1.1) for the special case where $f = u$.

Now let f be an arbitrary function in $L^1(\omega_1)$ for which $L^1(\omega_1) * f$ is dense. Then there is a sequence $\{h_n\}$ in $L^1(\omega_1)$ for which $f * h_n \to u$ in $L^1(\omega_1)$, so $\phi(f) * \phi(h_n) \to v$ in $L^1(\omega_2)$. Thus if g belongs to $L^p(\omega_2)$, then $g * v = \lim_{n\to\infty}(g * \phi(h_n)) * \phi(f)$ so that the dense subspace $L^p(\omega_2) * v$ is contained in the closure of $L^p(\omega_2) * \phi(f)$. This proves that $L^p(\omega_2) * \phi(f)$ is dense in $L^p(\omega_2)$, as required.

We now use (b) to prove (c) of Definition (1.1). Notice that $L^p(\omega_2)$ is a Banach module over $L^1(\omega_1)$ under the multiplication $f \cdot g = \phi(f) * g$. Let $\{e_n\}$ be a bounded approximate identity for the Banach algebra $L^1(\omega_1)$ (for instance $e_n = n\chi_{[0,1/n)}$). Then (c) will follow from the factorization theorem for modules [**2**, Theorem 10, p. 61] if we show that $\{e_n\}$ is a module approximate identity.

Choose some f in $L^1(\omega_1)$ for which $L^p(\omega_2) * \phi(f)$ is dense in $L^p(\omega_2)$. Since $\{e_n\}$ is bounded it will be enough to show that $\lim_{n\to\infty} e_n \cdot (\phi(f) * g) = \phi(f) * g$ for g in $L^p(\omega_2)$. But $e_n \cdot (\phi(f) * g) = \phi(e_n) * \phi(f) * g = \phi(e_n * f) * g \to \phi(f) * g$, since $\{e_n\}$ is an approximate identity for $L^1(\omega_1)$. This proves (c).

Finally we use (c) to prove (d) in Definition (1.1). Suppose $\{\lambda_n\}$ is a net which converges in the strong operator topology of $M(\omega_1)$ on $L^1(\omega_1)$ to λ. Let $h = \phi(f) * g$ be an arbitrary element of $L^p(\omega_2)$. Then $\phi(\lambda_n) * h = \phi(\lambda_n * f) * g \to \phi(\lambda * f) * g = \phi(\lambda) * h$, as required. This completes the proof of the theorem. $\square$

In the same way that we proved formula (3.4) above, we could prove the analogous formula for any $f(t)$ in $L^1(\omega_1)$ in place of $u(t) \equiv 1$. Thus we have:

COROLLARY (3.5). *For all f in $L^1(\omega_1)$ and all g in $L^p(\omega_2)$ where*

$1 \leq p < \infty$, we have $\phi(f) * g = \int_0^\infty f(t)\mu_t * g\, dt$, where the integral is a Bochner integral in $L^p(\omega_2)$.

In part (c) of Definition (1.1) if $\alpha(h) = \inf$ (support h), we can require that $\alpha(g) = \alpha(h)$ so that $\alpha(\phi(f)) = 0$. We just consider the functions in $L^p(\omega_2)$ with support in $[\alpha(h), \infty)$ as the $L^1(\omega_1)$ Banach Module.

The proof that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) in Theorem (3.1) also carries through if $p = 1$. We didn't even need the tricky arguments comparing Bochner integrals in $L^1(\omega_2)$ and $L^p(\omega_2)$. It is also easy to see that (d) $\Rightarrow$ (a) since δ_t is a strongly continuous semigroup, so, when (d) holds, $\mu_t = \phi(\delta_t)$ must also be strongly continuous. Thus our proof of Theorem (3.1) for $p > 1$ also shows that when $p = 1$, the four conditions of Definition (1.1) are equivalent. This gives a slightly simpler proof of the essential parts of our earlier [**7**, Theorem 2.2, p. 280]. But for $p = 1$, we only know that these conditions hold for a restricted class of weights.

4. Compactness and norm convergence. In our previous studies of the standard homomorphism problem [**8**] and [**7**], for $p = 1$, we were able to prove that the weak*-continuous semigroup $\{\mu_t\}$ was strongly continuous by coming up with a condition on the weight $\omega(t)$ which guaranteed that whenever a sequence $\{\lambda_n\}$ converged weak* to λ in $M(\omega)$, then $\lambda_n * f \to \lambda * f$ in norm in $L^1(\omega)$ for appropriate f. It also turned out [**7**, Theorem (4.1)] that weak convergence of $\lambda_n * f$ in $L^1(\omega)$ implied norm convergence. Since we have shown that when $p > 1$, the semigroup μ_t is always strongly continuous on $L^p(\omega)$, and since $L^p(\omega)$ is reflexive, so that weak* and weak convergence are the same, one would expect that weak convergence would imply norm convergence of $\lambda_n * f$ in $L^p(\omega)$ for a more general class of weights. Surprisingly, for general sequences or bounded nets $\{\lambda_n\}$, the norm convergence results are essentially the same for $L^p(\omega)$ for all $1 \leq p < \infty$.

The appropriate class of weights are the regulated weights of Bade and Dales [**1**, Definition 1.3, p. 81]. We say that the algebra weight $\omega(t)$ is *regulated* at $a \geq 0$, if $\lim_{t\to\infty} \omega(t+b)/\omega(t) = 0$ for all $b > a$. The following two results say that regularity, convergence improvement, and compactness are all equivalent for all $1 \leq p < \infty$. Recall, from Lemma (2.1) above, that the assumption that $\{\lambda_n\}$ converges weak* to λ is the same as assuming that $\lambda_n * f \to \lambda * f$ weakly in some (all) $L^p(\omega)$ with $1 < p < \infty$ for some (all) $f \neq 0$ in $L^p(\omega)$. Recall also that, for $f \in L^1_{loc}(R^+)$, we let $\alpha(f) = \inf$ (support f).

THEOREM (4.1). *If the weight $\omega(t)$ is regulated at a, then for all $1 \leq p < \infty$ and all f in $L^p(\omega)$ with $\alpha(f) \geq a$ we have:*

(a) *Convolution by f is a compact operator from $M(\omega)$ to $L^p(\omega)$.*

(b) *If the sequence or bounded net $\{\lambda_n\}$ converges* weak* *to λ in $M(\omega)$, then $\lambda_n * f \to \lambda * f$ in norm in $L^p(\omega)$.*

THEOREM (4.2). *If the algebra weight $\omega(t)$ is not regulated at a, then there is a sequence $\{\lambda_n\}$ in $M(\omega)$ with* weak*-*limit* 0 *for which, for all $1 \leq p < \infty$ and all f in $L^p(\omega)$ with $\alpha(f) \leq a$, we have:*

(a) *The sequence $\lambda_n * f$ diverges in norm in $L^p(\omega)$.*

(b) *Convolution by f is not a compact operator from either $M(\omega)$ or $L^1(\omega)$ to $L^p(\omega)$.*

We will give relatively simple proofs which reduce the results for $p > 1$ to our previous results [**8**, Theorem (2.3)] [**7**, pp. 283-284] when $p = 1$, instead of giving a proof for all $1 \leq p < \infty$. For a detailed study of compactness of convolution operators from $L^p(\omega)$ to itself, without the assumption that ω is an algebra weight, see Detre's thesis [**4**].

Proof of Theorem (4.1). Part (b) is a standard characterization of compactness (cf. [**7**, Theorem (3.2), p. 284]), so we just need to prove (a). Since $\delta_a * L^p(\omega)$ is dense in $L^p(\omega)_a = \{f \in L^p(\omega) : \alpha(f) \geq a\}$, it is enough to prove (a) for functions $f = \delta_a * g$ with g in $L^p(\omega)$. Since $L^1(\omega)$ has a bounded approximate identity which is also a module approximate identity, it follows from the Cohen factorization theorem that we can write $g = h*k$ with h in $L^1(\omega)$ and k in $L^p(\omega)$. The case $p = 1$ of the theorem [**7**, Lemma (3.1), p. 283] shows that $(\delta_a * h)$ acts compactly from $M(\omega)$ to $L^1(\omega)$. Also, convolution by k is a bounded operator from $L^1(\omega)$ to $L^p(\omega)$. Hence convolution by $f = (\delta_1 * h) * k$ is compact, since it is the composition of a compact operator and a bounded operator. This completes the proof of Theorem (4.1). □

Proof of Theorem (4.2). First notice that for h in $C_0(1/\omega)$ we have $\langle \delta_s/\omega(s), h\rangle = h(s)/\omega(s)$, which approaches 0 as $s \to \infty$ by the definition of $C_0(1/\omega)$. Thus $\delta_s/\omega(s)$, and hence any of its subsequences, approach 0 weak* in $M(\omega)$. Since $\omega(t)$ is not regulated at a, there is a $t_0 > a$ for which $\omega(s+t_0)/\omega(s)$ does not approach 0 as $s \to \infty$. Hence we can find a sequence $s_n \to \infty$ with $\omega(s_n+t_0)/\omega(s_n)$ bounded below. We claim that the sequence $\lambda_n = \delta_{s_n}/\omega(s_n)$, which approaches 0 weak* in $M(\omega)$, satisfies the assertion in Theorem (4.2) (a).

Pick some f in some $L^p(\omega)$ with $\alpha(f) \leq a$. Then

$$(\|\lambda_n * f\|_{p,\omega})^p = \int_0^\infty |f(t)|^p(\omega(s_n+t)/\omega(s_n))^p \, dt.$$

Now the weight $(\omega(t))^p$ is also not regulated and $|f|^p$ belongs to $L^1(\omega(t)^p)$. Also $(\omega(s_n+t)/\omega(s_n))^p$ is bounded away from zero so that, as we showed in our proof of [**8**, Theorem (2.3)], $\|(\delta(s_n)/\omega(s_n)^p) * |f|^p\|_{1,\omega^p} = \int_0^\infty |f(t)|^p|\omega(s_n+t)/\omega(s_n)|^p \, dt$ cannot approach 0 as $n \to \infty$. That is, the $p = 1$ result , for the algebra $L^1(\omega^p)$, which we proved previously, gives the result in Theorem (4.2) (a) for $L^p(\omega)$ for all $1 \leq p < \infty$.

Part (a) shows that convolution by f cannot be a compact operator from $M(\omega)$ to $L^p(\omega)$. We complete the proof by showing that, if convolution by g is compact from $L^1(\omega)$ to $L^p(\omega)$, then it is also compact from $M(\omega)$ to $L^p(\omega)$. The proof we gave for the case $p = 1$, in [**7**, Lemma (3.1), p. 283] carries through without change for all $1 \leq p < \infty$, since a bounded approximate identity $\{e_n\}$ for $L^1(\omega)$ is also a module approximate identity for $L^p(\omega)$. This completes the proof. □

The above two theorems show that for arbitrary sequences or nets, as distinct from semigroups, norm convergence in $L^p(\omega)$ for $p > 1$ is no easier to obtain than for $L^1(\omega)$. In contrast, the following small result does give one sense in which $L^p(\omega)$ convergence is easier.

PROPOSITION (4.3). *Suppose that* $\{\lambda_n\}$ *is a net in* $M(\omega)$. *If* $\lim(\lambda_n * f) = \lambda * f$ *in norm in* $L^1(\omega)$ *for all* f *in* $L^1(\omega)$, *then* $\{\lambda_n * g\}$ *converges in norm to* $\lambda * g$ *for all* g *in* $L^p(\omega)$ *and all* $1 < p < \infty$.

Proposition (4.3) can be viewed as a special case of Theorem (3.1) where the homomorphism is the identity on $L^1(\omega)$. Alternately, and more simply, one can use essentially the same proof of (c) $\Rightarrow$ (d) in Theorem (3.1) by considering $L^p(\omega)$ as an $L^1(\omega)$ module and applying the Cohen factorization theorem.

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