Ma, J.
Osaka J. Math.
51 (2014), 823–828

SELF-BUMPINGS ON KLEINIAN ONCE-PUNCTURED TORUS GROUPS

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(Received July 27, 2012, revised January 21, 2013)

Abstract

We give a new proof of the characterization of self-bumping points on the space of Kleinian once-punctured torus groups of Ito [8], based on some recent work of Bromberg [4].

1. Introduction

Let F be a finite type hyperbolic surface, $AH(F)$ be the set of discrete faithful representations of $\pi_1(F)$ into $PSL_2(\mathbb{C})$ up to conjugacy, under the condition that each loop around a cusp of F is mapped to a parabolic element in $PSL_2(\mathbb{C})$. Then $AH(F)$ admits a natural topology, named the algebraic topology. The interior of $AH(F)$ is parameterized by the product of two Teichmüller spaces, i.e., the quasi-Fuchsian space $\mathcal{QF}(F) = \mathcal{T}(F) \times \mathcal{T}(F)$, due to works of Ahlfors and Bers, see [10]. But the boundary of $AH(F)$ is fairly complicated. For example, there are self-bumping points in the boundary of $AH(F)$ [2, 12, 5], and further more, there are non-locally-connected points in the boundary of $AH(F)$ [4, 9]. We refer the reader to Canary [6] for a survey on the pathological phenomena of the deformation spaces of hyperbolic structures on general hyperbolic 3-manifolds.

In this note, we consider the simplest case, $F = F_{1,1}$, the once-punctured torus. A point $\rho \in \partial AH(F_{1,1})$ is *self-bumping* if for any neighborhood U of ρ in $AH(F_{1,1})$, there is a small neighborhood V of ρ contained in U, such that $V \cap (\mathcal{T}(F_{1,1}) \times \mathcal{T}(F_{1,1}))$ is disconnected.

There is a one-to-one correspondence between the set of simple closed curves in $F_{1,1}$ and $\mathbb{Q} \cup \infty$. We fix two oriented curves α (corresponding to ∞) and β (corresponding to 0) in $F_{1,1}$ which intersect in one point, they give us a marking of $\pi_1(F_{1,1})$. Then $AH(F_{1,1})$ is also the deformation space of marked hyperbolic structures on $F_{1,1} \times (-1, 1)$, with a rank-one cusp corresponding to the cusp of $F_{1,1}$.

Take $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B_\mu = \begin{pmatrix} i\mu & i \\ i & 0 \end{pmatrix}$ in $PSL_2(\mathbb{C})$, then A and B_μ generate a free group $G_\mu = \langle A, B_\mu \rangle$ in $AH(F_{1,1})$ for suitable $\mu \in \mathbb{C}$, where $AB_\mu A^{-1} B_\mu^{-1}$ is a parabolic element which corresponds to the cusp of $F_{1,1}$.

2010 Mathematics Subject Classification. 30F40, 57M50.

The author was supported in part by NSFC 10901038 and NSFC 11371094.

The set

$$\{\mu \in \mathbb{C} \mid \Im(\mu) > 0,\ \langle A, B_\mu \rangle \in AH(F_{1,1})\} \tag{1.1}$$

is called the *Maskit slice*, we denote it by $\mathscr{M}$, which is homeomorphic to $\overline{\mathbb{H}^2} - \infty$ (see [13]), but its boundary in $\mathbb{C}$ is highly complicated. In particular, the boundary is fractal-like. By [15], if $\Im(\mu) \geq 2$, then $\mu \in \mathscr{M}$.

The set

$$\{\mu \in \mathbb{C} \mid \mu = (p+1)x + py,\ x, y \in \mathscr{M}\} \tag{1.2}$$

is denoted by $\mathscr{M}(p)$ for $p \in \mathbb{N}$, then from a simple calculation, we have $\mathscr{M}(p) \subseteq \mathscr{M}(1) \subset \mathscr{M}$, see [8].

For any $\mu \in \operatorname{int}\mathscr{M}$, the manifold $M_{G_\mu} = (\mathbb{H}^3 \cup \Omega_{G_\mu})/G_\mu$ has two conformal boundaries, where Ω_{G_μ} is the set of discontinuity of G_μ on $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$. The thrice-punctured sphere boundary of M_{G_μ} is quasi-conformally rigid, but the once-punctured torus boundary is quasi-conformally flexible, and every marked conformal structure on $F_{1,1}$ is realizable by a unique $\mu \in \operatorname{int}\mathscr{M}$. M_{G_μ} is a hyperbolic manifolds in $AH(F_{1,1})$ which has one more rank-one cusp than manifolds in $\operatorname{int}AH(F_{1,1})$, the new cusp corresponds to the parabolic element A. In other words, we can view $\mathscr{M} \subset \partial AH(F_{1,1})$ if we choose the marking on $F_{1,1}$ suitably, i.e., when $\rho(\alpha) = A$ and $\rho(\beta) = B_\mu$.

Our aim here is to reprove the following theorem of Ito [8]:

Theorem 1.1. *$\mu \in \mathscr{M}$ corresponds to a self-bumping point on $AH(F_{1,1})$ if and only if $\mu \in \mathscr{M}(1)$.*

Given Theorem 1.1, it is easy to give a complete characterization of self-bumping points on the space of $AH(F_{1,1})$ as what have done by Ito [8], see Theorem 2.4 in next section.

Comparing to the proof in [8], where Ito emphasizes on exotic convergence sequences of [2], our proof here depends on the result in [4], i.e., the local homeomorphism between the deformation space and the (nearly) product of a sub-manifold of the Maskit slice and a sub-space of $\hat{\mathbb{C}}$. See also [14] for the generalization of Ito's result to high genus Kleinian surface groups.

2. Proof of the theorem

For a point $x \in \mathscr{M}$, we denote by $\mathscr{S}^{\circ}_{x,k}$ the solution space of

$$\{z \in \mathbb{C} \mid x - kz \in \operatorname{int}\mathscr{M}, x - (k+1)z \in \operatorname{int}\mathscr{M}_-\} \tag{2.1}$$

for $k \in \mathbb{Z}_{\geq 0}$, and we denote by $\mathscr{S}_{x,k}$ the solution space of

$$\{z \in \mathbb{C} \mid x - kz \in \mathscr{M},\ x - (k+1)z \in \mathscr{M}_-\} \tag{2.2}$$

for $k \in \mathbb{Z}_{\geq 0}$, where $\mathscr{M}_- = -\mathscr{M}$ as a subset of $\mathbb{C}$. Note that $\mathscr{S}_{x,i} \cap \mathscr{S}_{x,j} = \emptyset$ for any $0 \leq i < j$. If $\mathscr{S}^\circ_{x,k}$ is non-empty, then the closure of it is contained in $\mathscr{S}_{x,k}$. A priori, it is possible that $\mathscr{S}^\circ_{x,k} = \emptyset$ but $\mathscr{S}_{x,k} \neq \emptyset$, for example, if $\mathscr{S}_{x,k}$ is a sequence of discrete points in $\mathbb{C}$.

Proposition 2.1. *A point $x \in \mathscr{M}$ corresponds to a self-bumping point of $AH(F_{1,1})$ if and only if $\mathscr{S}_{x,k}$ is non-empty for some $k \in \mathbb{N}$.*

Proof. We first assume that $\mathscr{S}^\circ_{x,k}$ is non-empty for some $k \in \mathbb{N}$. Since for any $y \in \partial\mathscr{M}$, we have $1 \leq \Im(y) \leq 2$ (see [15]). Then a simple calculation shows that $\Im(x) \geq 3$ provided $\mathscr{S}^\circ_{x,k}$ is non-empty for some $k \in \mathbb{N}$, thus we have $x \in \operatorname{int}\mathscr{M}$. We note that $\mathscr{S}^\circ_{x,0}$ is non-empty, which is a translation of $\operatorname{int}\mathscr{M}$, so it is homeomorphic to an open half-space of $\mathbb{C}$. It is easy to see that for any $y \in \mathbb{C}$, we have $y \in \mathscr{M}$ if and only if $(y+2) \in \mathscr{M}$.

If the solution space $\mathscr{S}^\circ_{x,k}$ is non-empty for some $k \in \mathbb{N}$, by [15] and (2.1), for any $z \in \mathscr{S}^\circ_{x,k}$, we have

$$\frac{\Im(x+1)}{k+1} \leq \Im(z) \leq \frac{\Im(x-1)}{k}. \tag{2.3}$$

So $\mathscr{S}^\circ_{x,k}$ is contained in an infinite length horizontal strip in $\mathbb{C}$ of bounded width, $\mathscr{S}^\circ_{x,k}$ is invariant under the translation $z \to z+2$, and which is a 2-dimensional manifold (may be disconnected) in $\mathbb{C}$. In particular, $\mathscr{S}^\circ_{x,0}$ and $\mathscr{S}^\circ_{x,k}$ approach to $\infty \in \hat{\mathbb{C}}$ simultaneously, this gives us the 2-dimensional bumping of all solution spaces corresponding to x.

Now we can use a result of Bromberg (Theorem 4.13 of [4]), which said that there is a small neighborhood U of x in $\mathscr{M}$, such that there is a local homeomorphism Φ between

$$\{(y, w) \mid y \in U,\ w \in \overline{\sqcup_{k\in\mathbb{Z}_{\geq 0}}\mathscr{S}_{y,k}}\} \tag{2.4}$$

and $AH(F_{1,1})$ at (x,∞), where $\overline{\sqcup_{k\in\mathbb{Z}_{\geq 0}}\mathscr{S}_{y,k}}$ is the one-point compactification of $\sqcup_{k\in\mathbb{Z}_{\geq 0}}\mathscr{S}_{y,k}$ by the point $\infty = \infty_y$ for each y. The local homeomorphism Φ maps each (y, ∞_y) to the manifold M_{G_y}, maps each (y, w) to a manifold $\Phi((y, w))$ in $AH(F_{1,1})$, which is obtained from a preferred Dehn filling of a manifold $M_{y,w}$. The holonomy group of $M_{y,w}$ is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} iy & i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$. $M_{y,w}$ is a manifold with a rank-two cusp, the Dehn filling can be performed if $|w|/\sqrt{2\Im w}$ is large enough. $\Phi((y, w))$ lies in $\operatorname{int} AH(F_{1,1})$ if $w \in \mathscr{S}^\circ_{y,k}$, and $\Phi((y, w))$ lies in $\partial AH(F_{1,1})$ if $w \in \mathscr{S}_{y,k} - \mathscr{S}^\circ_{y,k}$.

The solution space $\mathscr{S}^\circ_{y,k}$ is continuous on y, so if $\mathscr{S}^\circ_{x,k}$ is non-empty for some $k \in \mathbb{N}$, then $\mathscr{S}^\circ_{y,k}$ is non-empty for any y which is near to x enough. Then from the 2-dimensional bumping of all solution spaces corresponding to y, it is easy to see the self-bumping of $\Phi((x, \infty)) = M_{G_x}$ on $AH(F_{1,1})$.

If $\mathscr{S}^\circ_{x,k}$ is empty for any $k \in \mathbb{N}$, but $\mathscr{S}_{x,k}$ is non-empty for some $k \in \mathbb{N}$, we claim that for any small neighborhood U of x in $\mathscr{M}$, there is a 2-manifold V in U, such that x lies in the boundary of V, and for any y in $\operatorname{int} V$, $\mathscr{S}^\circ_{y,k}$ is non-empty. This is true from the Bers–Sullivan–Thurston density theorem in our case: $AH(F_{1,1})$ is the closure of $\operatorname{int} AH(F_{1,1})$, which is from Minsky [13]. Now, if $\mathscr{S}^\circ_{y,k}$ is empty for any y in U which is near to x, then $\Phi((x, w))$ is a manifold in $\partial AH(F_{1,1})$ which can not be approached by a sequence of quasi-Fuchsian manifolds for $w \in \mathscr{S}_{x,k}$ when $\Re(w)$ is large enough (due to the fact that $w \in \mathscr{S}_{x,k}$ if and only if $(w+2) \in \mathscr{S}_{x,k}$, we can assume $\Re(w)$ is large enough, and then $|w|/\sqrt{2\Im w}$ is large enough). Since $\operatorname{int} AH(F_{1,1})$ is 4-dimensional, $\operatorname{int}\mathscr{M}$ is 2-dimensional, and $\mathbb{C}$ is 2-dimensional, our claim holds. Then, similar to the case that $\mathscr{S}^\circ_{x,k}$ is non-empty, since $\mathscr{S}^\circ_{y,k}$ is non-empty for each $y \in \operatorname{int} V$, we have x corresponds to a self-bumping point of $\Phi((x, \infty)) = M_{G_x}$ on $AH(F_{1,1})$.

Conversely, if $\mathscr{S}_{x,k}$ is empty for all $k \in \mathbb{N}$, then $\mathscr{S}_{y,k}$ is empty for all $k \in \mathbb{N}$ and y near enough to x, then again by the theorem of Bromberg, we have x does not correspond to a self-bumping point of $AH(F_{1,1})$. □

Proof of Theorem 1.1. If $x \in \mathscr{M}(1)$, then by definition $x = 2u + v$ for some $u, v \in \mathscr{M}$. Let $z = u + v$, then $x - z \in \mathscr{M}$ and $x - 2z \in \mathscr{M}_-$. In other words, $\mathscr{S}_{x,1} \neq \emptyset$, so x corresponds to a self-bumping point by Proposition 2.1.

Conversely, if x corresponds to a self-bumping point, then $\mathscr{S}_{x,k}$ is non-empty for some $k \in \mathbb{N}$, i.e., there is a $z \in \mathbb{C}$ such that $a + bi = x - kz \in \mathscr{M}$ and $c + di = x - (k+1)z \in \mathscr{M}_-$. Now let $u = a + bi \in \mathscr{M}$ and $v = -c - di \in \mathscr{M}$, then $(k+1)u + kv = x$. So $x \in \mathscr{M}(k) \subseteq \mathscr{M}(1)$. □

Proposition 2.2. *If $\rho \in \partial AH(F_{1,1})$ is a self-bumping point, then ρ is geometrically finite and has exactly one accidental parabolic class.*

Proof. Recall that a representation $\rho \in AH(F_{1,1})$ is *non-wrapping* provided that for any sequence of representations ρ_n converging algebraically to ρ and geometrically to a Kleinian group Γ, then there is a compact core for M_ρ which embeds in M_Γ under the covering map.

If ρ is not geometrically finite with exactly one accidental parabolic class, i.e., if ρ is geometrically infinite or ρ is geometrically finite but ∂M_ρ is quasi-conformally rigid, then there is no accidental parabolic class since ρ is an once-punctured torus group. From [1], Corollary B, we have ρ is a non-wrapping group. Then by Theorem 3 of [7], non-wrapping implies non-self-bumping on once-punctured torus Kleinian groups. □

REMARK 2.3. Proposition 2.2 can also be obtained from the proof of Theorem 1.1 and Theorem 1.3 of [3], even in the statement of their theorems the hyperbolic 3-manifold M should be compact with incompressible boundary or an trivial I-bundle over a closed surface S, but the proofs can extend word-by-word to the deformation space of once-punctured torus Kleinian groups. The proofs use some results on hierarchy path of Masur–Minsky [11], which holds in the once-punctured torus case more simply than the general case. Actually, this approach is overkilled.

From Proposition 2.2, if ρ is a self-bumping point, then $\operatorname{int} M_\rho$ is homeomorphic to $F_{1,1} \times (-1, 1)$ and ∂M_ρ has two components, one of them is homeomorphic to $F_{1,1}$ and the other one is a thrice-punctured sphere, which corresponds to cutting $F_{1,1}$ along a curve, we denote it by $\alpha = \infty$. So M_ρ is isometric to M_{G_μ} for some $\mu \in \mathscr{M}$ by the Ending Lamination Theorem of once-punctured torus groups [13] (re-choosing a marking on M_ρ, and the isometry may be orientation-reversing), then from Theorem 1.1 and Proposition 2.2, all self-bumping points in $AH(F_{1,1})$ can be characterized, see also Section 7 of [8] for a more precise formulation:

Theorem 2.4. *$\rho \in \partial AH(F_{1,1})$ is a self-bumping point in $AH(F_{1,1})$ if and only if M_ρ is isometric to M_{G_μ} for some $\mu \in \mathscr{M}(1)$.*

ACKNOWLEDGEMENTS. The author would like to thank Kentaro Ito for conversations on this topic during the program "Geometry, Topology and Dynamics of Character Varieties", Singapore, 2010. The author also would like to thank the referee for many useful comments.

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