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On the double suspension E^2

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Introduction

Concerning the Freudenthal's suspension homomorphism E, we have an exact sequence:

$$\cdots \longrightarrow \pi_i(S^n) \xrightarrow{E} \pi_{i+1}(S^{n+1}) \longrightarrow \pi_i(\Omega(S^{n+1}), S^n) \longrightarrow \cdots,$$

where $\Omega(S^{n+1})$ is the space of loops in S^{n+1}. The group $\pi_i(\Omega(S^{n+1}), S^n)$ is canonically isomorphic to the homotopy group $\pi_{i+1}(S^{n+1}; E_+^{n+1}, E_-^{n+1})$ of the suspension triad. Denote by $\mathcal{C}_p$ the class [12]* of finite abelian groups whose p-primary components vanish. The following isomorphisms are due to James [8].

THEOREM (2·10). *$\pi_i(\Omega(S^{n+1}), S^n)$ and $\pi_{i+1}(S^{2n+1})$ are isomorphic if n is odd.*

(2·10)′. *$\pi_i(\Omega(S^{n+1}), S^n)$ and $\pi_{i+1}(S^{2n+1})$ are $\mathcal{C}_2$-isomorphic if n is even.*

For an odd prime p, (2·10)′ is not true. However we have a $\mathcal{C}_p$-isomorphism [12] between $\pi_i(S^n)$ and $\pi_{i-1}(S^{n-1})+\pi_i(S^{2n-1})$, ($n$: even). Then it becomes more important to treat the double suspension

$$E^2=E\circ E:\ \pi_{i-1}(S^{n-1}) \longrightarrow \pi_i(S^n) \longrightarrow \pi_{i+1}(S^{n+1}) \text{ for even } n.$$

For the space of singular 2-spheres $\Omega^2(S^{n+1})=\Omega(\Omega(S^{n+1}))$, we have an exact sequence:

$$\cdots \longrightarrow \pi_{i-1}(S^{n-1}) \xrightarrow{E^2} \pi_{i+1}(S^{n+1}) \longrightarrow \pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1}) \longrightarrow \pi_{i-2}(S^{n-1}) \longrightarrow \cdots .$$

Let $S_k^n=S^n \cup e^{2n} \cup \cdots \cup e^{kn}$ be a reduced product [7] of n-sphere S^n relative its point e_0. S_k^n is canonically imbedded in $\Omega(S^{n+1})$, and the injection induces isomorphisms $\pi_i(S_k^n)\approx\pi_i(\Omega(S^{n+1}))$ for $i<(k+1)n-1$. We consider the following exact sequenence involving the group $\pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1})$;

$$\cdots \longrightarrow \pi_{i+1}(\Omega(S^{n+1}), S_{p-1}^n) \longrightarrow \pi_{i-1}(\Omega(S_{p-1}^n), S^{n-1}) \longrightarrow \pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1}) \longrightarrow \cdots .$$

Then the main results of this paper are the followings.

THEOREM (2·11). *For even n and a prime p, the groups $\pi_{i+1}(\Omega(S^{n+1}), S_{p-1}^n)$ and $\pi_{i+2}(S^{pn+1})$ are $\mathcal{C}_p$-isomorphic.*

THEOREM (7·6). *For even n and a prime p, the groups $\pi_{i-1}(\Omega(S_{p-1}^n), S^{n-1})$ and $\pi_i(S^{pn-1})$ are $\mathcal{C}_p$-isomorphic.*

Denote by $\pi_i(X;p)$ and $\pi_i(X,A;p)$ the p-primary components of $\pi_i(X)$ and $\pi_i(X,A)$ respectively.

THEOREM (8·3) *For even n and for an odd prime p, we have an exact sequence*

$$\cdots \longrightarrow \pi_{i+2}(S^{pn+1};p) \xrightarrow{\Delta} \pi_i(S^{pn-1};p) \longrightarrow \pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1};p) \longrightarrow \cdots \textit{ for } i>pn-1,$$

* Numbers in brackets refer to the references at the end of the paper.

where the homomorphism Δ satisfies the relation $\Delta \circ E^2 = f_{p*}$ for a map $f_p: S^{pn-1} \longrightarrow S^{pn-1}$ of degree p.

Let $S^{pn-1}_{f_p}$ be a mapping-cylinder of the map f_p, then

THEOREM (8·7). *we have an exact sequence*:

$$\cdots \longrightarrow \pi_i(\Omega^2(S^{pn+1}), S^{pn-1}; p) \longrightarrow \pi_i(S^{pn-1}_{f_p}, S^{pn-1}) \longrightarrow \pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1}; p) \longrightarrow \cdots,$$

for even n and for an odd prime p.

As a corollary, we have an isomorphism

$$(8\cdot7)' \qquad \pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1}; p) \approx \pi_i(S^{pn-1}_{f_p}, S^{pn-1}) \text{ for } i < p^2 n - 2.$$

James has pointed out that the naturality (8·4) for the exact sequence (8·3) implies the following relations:

THEOREM (8·10) $E^2(\pi_{i-1}(S^{n-1}; p)) \supset p^2(\pi_{i+1}(S^{n+1}; p))$

and

$$(8\cdot11) \qquad p^n(\pi_{i+1}(S^{n+1}; p)) = 0$$

for even n, an odd prime p and for all i.

In § 1, we consider a space X which has the same cohomological structure as the product space $Y \times A$, with a map $h: X \longrightarrow Y$ which carries a subset A of X to a single point. Then h induces isomorphisms of homotopy groups $\pi_i(X, A) \approx \pi_i(Y)$. This is also true for the cohomology mod. p, and h induces $\mathcal{C}_p$-isomorphisms of homotopy groups. In § 2, this isomorphism theorem is applied to a map $h_p: (S^n_\infty, S^n_{p-1}) \longrightarrow (S^{pn}_\infty, e_0)$, so-called the combinatorial extension [7] of a shrinking map $h_p': (S^n_p, S^n_{p-1}) \longrightarrow (S^{pn}, e_0)$. Then the theorems (2·10), (2·10)' and (2·11) are verified. In § 3, we calculate the cohomology of the loop-space $\Omega(S^n_{p-1})$ and some other spaces. In § 4, we prove that the group $\pi_i(S^2_{p-1})$ is $\mathcal{C}_p$-isomorphic to the group $\pi_i(S^{2p-1})$ for $i>2$. This means that there is a map $g: S^{2p-1} \longrightarrow S^2_{p-1}$ such that the correspondence $(\alpha, \beta) \longrightarrow E\alpha + g_*(\beta)$ defines a $\mathcal{C}_p$-isomorphism of $\pi_{i-1}(S^1) + \pi_i(S^{2p-1})$ to $\pi_i(S^2_{p-1})$. Conversely, for even n, if there exists a map $g: S^{pn-1} \longrightarrow S^n_{p-1}$ such that the correspondence $(\alpha, \beta) \longrightarrow E\alpha + g_*(\beta)$ defines a $\mathcal{C}_p$-isomorphism of homotopy groups, then $n = 2p^r$ for an integer r. § 5 and § 6 are devoted to the preliminaries for § 7 and § 8 in which the theorems (7·6), (8·3), (8·7) and (8·10) are proved. In appendix, we list several values of $\pi_i(S^{n+1}; p)$ for unstable cases from results for stable cases.

1. A theorem for a map $h: (X, A) \longrightarrow (Y, y_0)$.

Let X, A and x_0 be a topological space, its subspace and a point of A. Let $I = [0.1]$ be the unit interval. Denote by $\Omega(X, A, x_0)$, or simply by $\Omega(X, A)$, the space of paths in X which start in A and end at x_0. i.e.,

$$\Omega(X, A) = \{f: I \longrightarrow X \mid f(0) \in A, f(1) = x_0\},$$

where the topology in $\Omega(X, A)$ is the compact-open topology.

Let $I^i = I \times \cdots \times I$ be the unit i-cube which is regarded as a face $I^i \times (0)$ of $I^{i+1} = I^i \times I$. Denote by $\dot{I}^{i+1}$ the boundary of I^{i+1} and by J^i the complementary face of

$I^i : J^i = \dot{I}^{i+1} - (I^i - \dot{I}^i)$. The homotopy group $\pi_{i+1}(X, A, x_0)$ is the set of the homotopy classes of maps $g: (I^{i+1}, \dot{I}^{i+1}, J^i) \longrightarrow (X, A, x_0)$. For the map g, we associate a map $\Omega g: (I^i, \dot{I}^i) \longrightarrow (\Omega(X, A, x_0), f_0)$ by the formula $\Omega g(t_1, \cdots, t_i)(t) = g(t_1, \cdots, t_i, t)$, then the correspondence $g \longrightarrow \Omega g$ is one-to-one and we have an isomorphism $(f_0(I) = x_0)$

(1・1) $\Omega : \pi_{i+1}(X, A, x_0) \approx \pi_i(\Omega(X, A, x_0), f_0),\ i > 0.$

A map $h : (X, A, x_0) \longrightarrow (Y, B, y_0)$ defines a map of path-spaces which is denoted by

$$\Omega h : \Omega(X, A, x_0) \longrightarrow \Omega(Y, B, y_0).$$

For the homomorphisms h_* and Ωh_* induced by the maps h and Ωh, we have the commutative diagram:

(1・2)
$$\begin{tikzcd} \pi_{i+1}(X, A) \arrow[r, "h_*"] \arrow[d, "\Omega"] & \pi_{i+1}(Y, B) \arrow[d, "\Omega"] \\ \pi_i(\Omega(X, A)) \arrow[r, "\Omega h_*"] & \pi_i(\Omega(Y, B)). \end{tikzcd}$$

If $A = x_0$, $\Omega(X, x_0, x_0)$ is the space of loops in X and it is denoted by $\Omega(X, x_0)$ or simply by $\Omega(X)$. For a map $g: (I^{i+1}, \dot{I}^{i+1}, J^i) \longrightarrow (X, A, x_0)$ we associate a map $\Omega' g$: $(I^i, \dot{I}^i, J^{i-1}) \longrightarrow (\Omega(X, x_0), \Omega(A, x_0), f_0)$ by the formula $\Omega' g(t_1, \cdots, t_{i-1}, t_i)(t) = g(t_1, \cdots, t_{i-1}, t, t_i)$, then the correspondence $g \longrightarrow \Omega' g$ is one-to-one and we have an isomorphism

(1・1)′ $\Omega' : \pi_{i+1}(X, A, x_0) \approx \pi_i(\Omega(X, x_0), \Omega(A, x_0), f_0)$ for $i > 1$.

For a map $h: (X, A, x_0) \longrightarrow (Y, B, y_0)$ and the induced map $\Omega h: (\Omega(X), \Omega(A), f_0) \longrightarrow (\Omega(Y), \Omega(B), f_0')$, we have the following commutative diagram:

(1・2)′
$$\begin{tikzcd} \pi_{i+1}(X, A) \arrow[r, "h_*"] \arrow[d, "\Omega'"] & \pi_{i+1}(Y, B) \arrow[d, "\Omega'"] \\ \pi_i(\Omega(X), \Omega(A)) \arrow[r, "\Omega h_*"] & \pi_i(\Omega(Y), \Omega(B)). \end{tikzcd}$$

Define a map (projection) $p : \Omega(X, A) \longrightarrow A$ by $p(f) = f(0), f \in \Omega(X, A)$, then we have the commutative diagram:

(1・3)
$$\begin{tikzcd} \pi_{i+1}(X, A) \arrow[r, "\partial"] \arrow[d, "\Omega"] & \pi_i(A) \\ \pi_i(\Omega(X, A)) \arrow[ur, "p_*"'] & \end{tikzcd}$$

where ∂ is the boundary homomorphism.

Now we define a sort of mapping-cylinder Z of the map h as follows: the space

$$Z = (X - x_0) \times [0,1) \cup Y$$

is the image of $X \times I \cup Y$ under the identification $\eta : X \times I \cup Y \longrightarrow Z$ which is defined by $\eta(x, 1) = \eta(h(x)), x \in X$ and $\eta(x_0, t) = \eta(y_0), t \in I$. Define two injections $i_X : X \longrightarrow Z$ and $i_Y : Y \longrightarrow Z$ by $i_X(x) = \eta(x, 0)$ and $i_Y(y) = \eta(y)$, then $X \cap Y = x_0 = y_0 \in Z$. As is easily seen that Y is a deformation retract of Z and the retraction $r : Z \longrightarrow Y$ is given by $r(\eta(x, t)) = \eta(x, 1) = \eta(h(x))$ and $r(\eta(y)) = \eta(y)$, then the composition $r \circ i_X : X \longrightarrow Z \longrightarrow Y$ is the map h. Consider the following diagram

$$\begin{tikzcd} \cdots \arrow[dr] & & \pi_{i+1}(Z) \arrow[r, "j"] \arrow[d, shift left, "r_*"] & \pi_{i+1}(Z, X) \arrow[d, "\Omega"] \arrow[dr, "\partial"] & & \cdots \\ & \pi_{i+1}(X) \arrow[ur, "i_{X*}"] \arrow[dr, "h_*"'] & & & \pi_i(X) \arrow[ur] \arrow[dr] & \\ \cdots \arrow[ur] & & \pi_{i+1}(Y) \arrow[r, "\Delta"] \arrow[u, shift left, "i_{Y*}"] & \pi_i(\Omega(Z, X)) \arrow[ur, "p_*"'] & & \cdots, \end{tikzcd}$$

where $\Delta = \Omega \circ j \circ i_{Y*}$. The commutativity of the diagram is easily verified. Since i_{Y*}, r_* and Ω are isomorphisms, the exacteness of the upper sequence implies that of the lower sequence.

Next suppose that h maps A to the point y_0. For a point x of A, we associate a path $f_x \in \Omega(Z, X)$ by the formula $f_x(t) = \eta(x, t)$, $t \in I$, then we have an injection

$$i_A : A \longrightarrow \Omega(Z, X), \quad i_A(x) = f_x,$$

such that the composition $p \circ i_A : A \longrightarrow \Omega(Z, X) \longrightarrow X$ is the injection of A into X. In the diagram

$$(1\cdot4) \qquad \begin{tikzcd}
\cdots \arrow[dr] & & \pi_{i+1}(X, A) \arrow[r, "\partial"] \arrow[d, "h_*"] & \pi_i(A) \arrow[d, "i_{A*}"] \arrow[dr, "i_*"] & & \cdots \\
& \pi_{i+1}(X) \arrow[ur, "j"] \arrow[dr, "h_*"'] & & & \pi_i(X) \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] & & \pi_{i+1}(Y) \arrow[r, "\Delta"] & \pi_i(\Omega(Z, X)) \arrow[ur, "p_*"'] & & \cdots,
\end{tikzcd}$$

the upper and lower sequences are exact and the first and third triangles are commutative. For the second square, the anti-commutativity holds:

$$(1\cdot4)' \qquad \Delta \circ h_* = -(i_{A*} \circ \partial).$$

Proof. By the isomorphism Ω of $(1\cdot1)$, it is sufficient to prove that $\Omega^{-1}(i_{A*}(\partial\beta))$ equals to $-\Omega^{-1}(\Delta(h_*(\beta))) = -\Omega^{-1}(\Omega(j(i_{Y*}(h_*\beta)))) = j(i_{Y*}(h_*(-\beta)))$ for arbitrary $\beta \in \pi_{i+1}(X, A)$. Let $g: (I^{i+1}, \dot{I}^{i+1}, J^i) \longrightarrow (X, A, x_0)$ be a map of β, then the element $\Omega^{-1}(i_{A*}(\partial\beta)) \in \pi_{i+1}(Z, X)$ is represented by a map $G: (I^{i+1}, \dot{I}^{i+1}, J^i) \longrightarrow (Z, X, x_0)$ which is given by the formula

$$G(t_1, \cdots, t_i, t_{i+1}) = \eta(g(t_1, \cdots, t_i, 0), t_{i+1}).$$

Next the element $j(i_{Y*}(h_*(-\beta)))$ is represented by a map G' given by the formula

$$G'(t_1, \cdots, t_i, t_{i+1}) = \eta(h(g(t_1, \cdots, t_i, 1-t_{i+1}))) = \eta(g(t_1, \cdots, t_i, 1-t_{i+1}), 1).$$

Define a homotopy $G_t: (I^{i+1}, \dot{I}^{i+1}, J^i) \longrightarrow (Z, X, x_0)$ for $0 \leqq t \leqq 1$ by the formula

$$G_t(t_1, \cdots, t_{i+1}) = \begin{cases} \eta(g(t_1, \cdots, t_i, t-2t_{i+1}), 0), & 0 \leqq t_{i+1} \leqq \dfrac{t}{2}, \\ \eta(g(t_1, \cdots, t_i, 0), \dfrac{2t_{i+1}-t}{2-t}), & \dfrac{t}{2} \leqq t_{i+1} \leqq 1, \end{cases}$$

and for $1 \leqq t \leqq 2$ by the formula

$$G_t(t_1, \cdots, t_{i+1}) = \begin{cases} \eta(g(t_1, \cdots, t_i, \dfrac{t-2t_{i+1}}{t}), t-1), & 0 \leqq t_{i+1} \leqq \dfrac{t}{2}, \\ \eta(g(t_1, \cdots, t_i, 0), 2t_{i+1}-1), & \dfrac{t}{2} \leqq t_{i+1} \leqq 1, \end{cases}$$

then we have that $G = G_0$ is homotopic to $G' = G_2$ and that $\Omega^{-1}(i_{A*}(\partial\beta)) = j(i_{Y*}(h_*(-\beta)))$. Therfore the formula $(1\cdot4)'$ is proved. q. e. d.

Let $\mathcal{C}$ be a class of abelian groups in the sense of Serre [12]. The five lemma is applicable to the diagram $(1\cdot4)$, and we have that

LEMMA $(1\cdot5)$ *the following two conditions are equivalent.*

i) $i_{A*} : \pi_i(A) \longrightarrow \pi_i(\Omega(Z, X))$ *is* $\mathcal{C}$*-isomorphic for* $i \leqq N$ *and* $\mathcal{C}$*-onto for* $i = N+1$;

ii) $h_* : \pi_{i+1}(X, A) \longrightarrow \pi_{i+1}(Y)$ *is* $\mathcal{C}$*-isomorphic for* $i \leqq N$ *and* $\mathcal{C}$*-onto for* $i = N+1$.

In the following, we suppose that a coefficient ring R is one of the ring of integers Z and the field of p elements Z_p(p: prime). We denote by $\mathcal{C}_R$ the class

$\mathcal{C}_Z=\mathcal{C}_0$ (when $R=Z$) of the trivial group, or the class $\mathcal{C}_{Z_p}=\mathcal{C}_p$ (when $R=Z_p$) which consists of finite groups with vanishing p-components. Then we recall a generalization of J. H. C. Whitehead's theorem from [12, Ch. III]:

(1·6) *Let X and Y be arcwise connected and simply connected spaces and let $f: X \longrightarrow Y$ be a map such that $f_*: \pi_2(X) \longrightarrow \pi_2(Y)$ is onto. If $H_i(X)$ and $H_i(Y)$ have finite numbers of generators for all i, then the following two conditions are equivalent*:

i) $f_*: \pi_i(X) \longrightarrow \pi_i(Y)$ *is a $\mathcal{C}_R$-isomorphism for $i \leqq N$ and $\mathcal{C}_R$-onto for $i=N+1$;*

ii) $f^*: H^i(Y, R) \longrightarrow H^i(X, R)$ *is an isomorphism for $i \leqq N$ and an isomorphism into for $i=N+1$.*

Now consider the following conditions (1·7) for a map $h: (X, A) \longrightarrow (Y, y_0)$.

Hypotheses (1·7), i) *X, A and Y are arcwise connected and simply connected,* $\pi_2(X, A)=\pi_2(Y)=0$ *and* $h_*: \pi_3(X, A) \longrightarrow \pi_3(Y)$ *is onto*;

ii) $H_i(X), H_i(A)$ *and* $H_i(Y)$ *have finite numbers of generators for all i, and* $H_*(Y, R)$ *is R-free*;

iii) *there exists subgroups B and F of $H^*(X, R)$ such that the cup-product induces an isomorphism* $B \otimes F \approx H^*(X, R)$;

iv) *the injection homomorphism* $i^*: H^*(X,R) \longrightarrow H^*(A, R)$ *maps F isomorphically onto* $H^*(A, R)$;

v) *the induced homomorphism* $h^*: H^*(Y, R) \longrightarrow H^*(X, R)$ *maps $H^*(Y, R)$ isomorphically onto B.*

The main purpose of this § is to prove that

Theorem (1·8) *if the hypotheses* (1·7), i)—v) *are filfulled, then the homomorphisms* $h_*: \pi_{i+1}(X, A) \longrightarrow \pi_{i+1}(Y)$ *and* $i_{A_*}: \pi_i(A) \longrightarrow \pi_i(\Omega(Z, X))$ *are $\mathcal{C}_R$-isomorphisms for all i.*

Proof. Let E be the space of paths in Z which start in X, i.e.,

$$E=\{f: I \longrightarrow Z \mid f(0) \in X\}.$$

We regard X as a subset of E whose points are paths $f: I \longrightarrow x \in X$, then X is a deformation retract of E. Let $p: E \longrightarrow Z$ be a projection defined by $p(f)=f(1)$, then (E, p, Z) is a fibre-space with the fibre $\Omega(Z, X)$. The composition $X \xrightarrow{i} E \xrightarrow{p} Z \xrightarrow{r} Y$ is the map h. By the homotopy equivalences $i: X \longrightarrow E$ and $r: Z \longrightarrow Y$, the conditions (1·7), ii)—v) are rewritten as the followings:

(1·7)′, ii) $H_i(E)$, $H_i(A)$ and $H_i(Z)$ have finite numbers of generators for all i, and $H_*(Z, R)$ is R-free;

iii) there exist subgroups B and F of $H^*(E, R)$ such that the cup-product induces an isomophism $B \otimes F \approx H^*(E, R)$;

iv) the injection $i_A: A \longrightarrow \Omega(Z, X) \subset E$ induces a homomorphism $i_A{}^*: H^*(E, R) \longrightarrow H^*(A, R)$ which maps F isomorphically onto $H^*(A, R)$;

v) the (projection) homomorphism $p^*: H^*(Z, R) \longrightarrow H^*(E, R)$ maps $H^*(Z, R)$

isomorphically onto B.

Applying (1·5) for the case $\mathcal{C}=\mathcal{C}_0$ and $N=1$, we have from (1·7), i) that

(1·7)', i) E, A, Z and $\Omega(Z, X)$ are arcwise and simply connected and $i_{A_*}: \pi_2(A) \longrightarrow \pi_2(\Omega(Z, X))$ is onto.

Let $(E_r^{p,q})$ be the cohomological spectral sequence over the coefficient ring R associated with (E, p, Z). From (1·7)', i) and ii), we have isomorphisms (cf. [10, Ch. II, Prop. 8])

$$E_2^{p,q} \approx H^p(Z, R) \otimes H^q(\Omega(Z, X), R).$$

The isomorphism of (1·7)', iv) is divided into the composition: $F \subset H^*(E, R) \longrightarrow E_\infty^{0,*} \subset E_2^{0,*} \approx H^*(\Omega(Z, X), R) \longrightarrow H^*(A, R)$. Therefore F is a direct factor of $H^*(\Omega(Z, X), R)$ and $d_r(1 \otimes F)=0$ for $2 \leqq r < \infty$. Then $d_r(B \otimes F)=0$ for $2 \leqq r$. Consider the image of $B \otimes F \subset E_2^*$ into E_∞^* which is a graded ring over $H^*(E, R) \approx B \otimes F$. Since $1 \otimes F \subset E_\infty^{0,*}$ and $B \otimes 1 \subset E_\infty^{*,0}$, we have that $E_\infty^* \approx B \otimes F \approx E_\infty^{*,0} \otimes E_\infty^{0,*}$. Set $F^q = F \cap H^q(\Omega(Z, X), R)$ and suppose that $H^q(\Omega(Z, X), R)=F^q$ for $q<n$, then $E_2^{p,q}=E_\infty^{p,q}$ for $q<n$ and the boundary operator $d_r: E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q}$ has to be trivial for $q<n$ and $n \geqq 2$. Therefore $d_r(E_r^{0,n})=0$, $E_r^{0,n}=H(E_r^{0,n}) = E_{r+1}^{0,n}$ and $H^n(\Omega(Z, X), R) \approx E_2^{0,n}=E_\infty^{0,n} \approx F^n$. The induction on the demension n implies that $i_A^*: H^*(\Omega(Z, X), R)=F \approx H^*(A, R)$.

Since $\pi_i(Z)$, $\pi_i(X)$ and $\pi_i(Z, X) \approx \pi_{i-1}(\Omega(Z, X))$ have finite numbers of generators, $H_i(\Omega(Z, X))$ has also a finite number of generators for all i. Applying (1·6) to the injection $i_A: A \longrightarrow \Omega(Z, X)$, we have that $i_A: \pi_i(A) \longrightarrow \pi_i(\Omega(Z, X))$ is a $\mathcal{C}_R$-isomorphism for all i. Then the theorem follows from the lemma (1·5). q. e. d.

2. Reduced product of sphere and the group $\pi_i(\Omega(S^{n+1}), S_{p-1}^n)$

According to [7] we denote by S_∞^n the reduced product of the unit sphere $S^n = \{(t_1, \cdots, t_{n+1}) \mid t_i: \text{real numbers}, \sum t_i^2=1\}$ relative to its point $e_0=(1,0,\cdots,0)$, i.e., an *FM*-complex generated by the points of S^n-e_0 in the sense of [15]. S_∞^n has the free semi-group structure with the unit e_0, and its point x is represented by a product $x=x_1 \cdots x_r$ for some $x_i \in S^n$, $i=1, \cdots, r$. Denote that $S_k^n=\{x_1 \cdots x_k \mid x_i \in S^n, i=1, \cdots, k\}$ and that $e^{kn}=S_k^n-S_{k-1}^n$, then e^{kn} is an open kn-cell and $S_\infty^n=e_0 \cup e^n \cup e^{2n} \cup \cdots$ is a *CW*-complex. Note that $S_0^n=e_0$ and $S_1^n=S^n$. Define a map

(2·1) $d_n: (S^n \times I, S^n \times \dot{I} \cup e_0 \times I) \longrightarrow (S^{n+1}, e_0)$ *which maps* $(S^n-e_0) \times (I-\dot{I})$ *homeomorphically onto* $S^{n+1}-e_0$, $n \geqq 0$,

by the formulas $d_n((t_1, \cdots, t_{n+1}), t)=(1-2t(1-t_1), 2tt_2, \cdots, 2tt_{n+1}, 2(t(1-2t)(1-t_1))^{\frac{1}{2}})$ and $d_n((t_1, \cdots, t_{n+1}), 1-t)=(1-2t(1-t_1), 2tt_2, \cdots, 2tt_{n+1}, -2(t(1-2t)(1-t_1))^{\frac{1}{2}})$ for $0 \leqq t \leqq \frac{1}{2} (\frac{1}{2} \leqq 1-t \leqq 1)$.

Define

(2·2) *an extension* $\bar{d}_n: (S_\infty^n \times I, S_\infty^n \times \dot{I}) \longrightarrow (S^{n+1}, e_0)$ *of* $d_n=\bar{d}_n \mid S_1^n \times I$,

by the formula $d_n(x_1 \cdots x_k, (t-\lambda_{i-1})/(\lambda_i-\lambda_{i-1}))$ for $\lambda_{i-1} \leqq t \leqq \lambda_i$ and for $i=1, \cdots, k$, where $x_i \in S^n$, $x_1 \cdots x_k \in S^n_k \subset S^n_\infty$, $\lambda_0 = 0$ and $\lambda_i = \sum_{j=1}^{i} \rho(x_j, e_0) / \sum_{j=1}^{k} \rho(x_j, e_0)$ for the distance function ρ. The map $\bar{d}_n$ defines a continuous map

(2·2)′ $\tilde{i}: S^n_\infty \longrightarrow \Omega(S^{n+1}) = \Omega(S^{n+1}, e_0)$.

As is easily seen $\tilde{i}$ is one-to-one into and then $\tilde{i}$ is an injection on S^n_k, $k < \infty$. We define *a suspension homomorphism* E by setting

(2·2)″ $E = \Omega^{-1} \circ i_* : \pi_i(S^n) \longrightarrow \pi_i(\Omega(S^{n+1})) \approx \pi_{i+1}(S^{n+1})$.

By [7] and [15] we have that $\tilde{i}$ induces isomorphisms of homology, cohomology and homotopy groups:

(2·3) $\tilde{i}^* : H^*(\Omega(S^{n+1})) \approx H^*(S^n_\infty)$ and $\tilde{i}_* : \pi_i(S^n_\infty) \approx \pi_i(\Omega(S^{n+1}))$.

It is easy to see that

(2·3)′ $\tilde{i}_* : \pi_i(S^n_\infty, S^n_k) \approx \pi_i(\Omega(S^{n+1}), S^n_k)$.

From the relations in the cohomology ring $H^*(\Omega(S^{n+1}))$ [10, Ch. IV, Prop. 18], (2·4) *for suitably chosen generators* $e_i \in H^{in}(S^n_\infty)$ *we have the following relations (cup-product) in them*:

i) *if* n *is even, then* $e_i \cdot e_j = \binom{i+j}{i} e_{i+j}$ *and* $e_1{}^j = j!\, e_j$,

ii) *if* n *is odd, then* $e_{2i} \cdot e_{2j} = \binom{i+j}{i} e_{2(i+j)}$, $e_1^2 = 0$, $e_1 \cdot e_{2i} = e_{2i} \cdot e_1 = e_{2i+1}$ *and* $e_2{}^j = j!\, e_{2j}$.

Next we introduce James' combinatorial extension from [7]:

(2·5) *a map* $f: (S^n_k, S^n_{k-1}) \longrightarrow (S^m_\infty, e_0)$ *can be extended over the whole of* S^n_∞ *and we have a combinatorial extension* $\bar{f}: S^n_\infty \longrightarrow S^m_\infty$ *of the map* f *such that* $\bar{f} | S^n_k = f$. *If* f_t *is a homotopy, then* $\bar{f}_t$ *is also a homotopy.*

The map $\bar{f}$ is defined briefly as follows. First we remark that $f(S^n_k) \subset S^m_j$ for some $j < \infty$. For a point $x = x_1 \cdots x_t$ ($x_i \in S^n$, $i = 1, \cdots, t$) of S^n_∞, we define its image $\bar{f}(x)$ by the formula $\bar{f}(x) = \bar{f}(x_1 \cdots x_t) = \prod_\sigma f(x_{\sigma(1)} \cdots x_{\sigma(k)})$, where σ is a monotone increasing function of $(1, \cdots, k)$ into $(1, \cdots, t)$ and the order of the multiplication Π is an order of $\{\sigma\}$ such that $\sigma < \sigma'$ if and only if $\sigma(i) = \sigma'(i)$, $i = 1, \cdots, k' - 1$ and $\sigma(k') < \sigma'(k')$ for an integer $k' \leqq k$. $\bar{f}(x)$ is independent of the representation $x = x_1 \cdots x_t$ and $\bar{f}$ is continuous.

For a given map $f: (S^n, e_0) \longrightarrow (S^m, e_0)$, we difine its suspension $Ef: (S^{n+1}, e_0) \longrightarrow (S^{m+1}, e_0)$ by the formula $Ef(d_n(x, t)) = d_m(f(x), t)$, $x \in S^n$, $t \in I$. The combinatorial extension of f is a homomorphism: $\bar{f}(x_1 \cdots x_t) = \bar{f}(x_1) \cdots \bar{f}(x_t)$. Then

(2·6) *the compositions* $\Omega(Ef) \circ \tilde{i} : S^n_\infty \longrightarrow \Omega(S^{n+1}) \longrightarrow \Omega(S^{m+1})$ *and* $\tilde{i} \circ \bar{f}: S^n_\infty \longrightarrow S^m_\infty \longrightarrow \Omega(S^{m+1})$ *are homotopic to each other.*

Proof. Define a homotopy $F_\theta : (S^n_\infty \times I, S^n_\infty \times \dot{I}) \longrightarrow (S^{m+1}, e_0)$ by the formula $F_\theta : (x_1 \cdots x_k, t) = d_m(f(x_i), (t - \lambda^\theta_{i-1}) / (\lambda^\theta_i - \lambda^\theta_{i-1}))$ for $\lambda^\theta_{i-1} \leqq t \leqq \lambda^\theta_i$ and for $i = 1, \cdots, k$, where $x_i \in S^n$, $x_1 \cdots x_k \in S^n_k \subset S^n_\infty$, $\lambda^\theta_0 = 0$ and $\lambda_i{}^\theta = (1-\theta)(\sum_{j=1}^{i} \rho(x_j, e_0) / \sum_{j=1}^{k} \rho(x_j, e_0)) + \theta (\sum_{j=1}^{i} \rho(f(x_j), e_0) / \sum_{j=1}^{k} \rho(f(x_j), e_0))$. Then F_θ defines a homotopy $f_\theta : S^n_\infty \longrightarrow \Omega(S^{m+1})$ such that $f_0 = \Omega(Ef) \circ \tilde{i}$ and $f_1 = \tilde{i} \circ \bar{f}$. q. e. d.

Define a map

(2·7) $\psi_n:(I^n, \dot{I}^n) \longrightarrow (S^n, e_0), \qquad n\geqq 1,$

by setting $\Psi_1(t)=d_0(-1, t)$ and $\psi_n(t_1, \cdots, t_{n-1}, t_n)=d_{n-1}(\psi_{n-1}(t_1, \cdots, t_{n-1}), t_n)$ for $n\geqq 2$. Then ψ_n maps $I^n-\dot{I}^n$ homeomorphically onto S^n-e_0. Define a map

(2·7)′ $h_k' : (S_k^n, S_{k-1}^n) \longrightarrow (S^{kn}, e_0)$

by the formula $h_k'(\psi_n(t_1 \cdots t_n) \cdots \psi_n(t_{(k-1)n+1}, \cdots, t_{kn})) = \psi_{kn}(t_1, \cdots, t_{kn})$ then h_k' maps $e^{kn}=S_k^n-S_{k-1}^n$ homeomorphiclally onto $S^{kn}-e_0$. Let

(2·7)″ $h_k: (S_\infty^n, S_{k-1}^n) \longrightarrow (S_\infty^{kn}, e_0)$

be the combinatrial extension of h_k'.

For a given map $f: (S^n, e_0) \longrightarrow (S^m, e_0)$, we define a map

(2·8) $(f)^k: (S^{kn}, e_0) \longrightarrow (S^{km}, e_0)$

such that the diagram

$$\begin{array}{ccc} S_k^n & \xrightarrow{h_k'} & S^{kn} \\ \downarrow \bar{f} & & \downarrow (f)^k \\ S_k^m & \xrightarrow{h_k'} & S^{km} \end{array}$$

is commutative. Obviously such a map $(f)^k$ is determined uniquely and continuous.

(2·8)′ If $\alpha \in \pi_n(S^m)$ *is represented by* f, *then* $(f)^k$ *represents* $(-1)^{\varepsilon}E^{(k-1)m}(\alpha \circ E^{n-m}\alpha \circ \cdots \circ E^{(k-1)(n-m)}\alpha) \in \pi_{kn}(S^{km})$, *where* $\varepsilon=\frac{1}{2}kn(n+m)(k-1)$ *and we orient the sphere* S^r *such that the map* ψ_r *preserves the orientations.*

Proof. Define a map

(2·8)″ $\phi_{n,r}: (S^n\times S^r, S^n\vee S^r) \longrightarrow (S^{n+r}, e_0)$

by the formula $\phi_{n,r}(\psi_n(t_1, \cdots, t_n), \psi_r(u_1, \cdots, , u_r))=\psi_{n+r}(t_1, \cdots, t_n, u_1, \cdots, u_r)$, $(t_1, \cdots, t_n) \in I^n$, $(u_1, \cdots, u_r) \in I^r$, then $\phi_{n,r}$ maps $(S^n-e_0)\times(S^r-e_0)$ homeomorphically onto $S^{n+r}-e_0$. Then we have the following commutative diagram

$$\begin{array}{ccc} S^n\times S^{(k-1)n} & \xrightarrow{f\times (f)^{k-1}} & S^m\times S^{(k-1)m} \\ \downarrow \phi_{n,(k-1)n} & & \downarrow \phi_{m,(k-1)m} \\ S^{kn} & \xrightarrow{(f)^k} & S^{km} \end{array}$$

where $(f\times(f)^{k-1})(x,y) = (f(x), (f)^{k-1}(y))$. In the notation of [2], the diagram shows that the class α_k of $(f)^k$ is the reduced join of α and α_{k-1}. By [2, Th. 3·2.], $\alpha_k=(-1)^{(k-1)n(n+m)}E^{(k-1)m}\alpha\circ E^n\alpha_{k-1}$, and then (2·8)′ is proved by the induction on k.

q. e. d.

Let F be the combinatorial extension of $(f)^k$, then from the definition of the combinatorial extension it is easily verified that the diagram

$$\begin{array}{ccc} S_\infty^n & \xrightarrow{h_k} & S_\infty^{kn} \\ \downarrow \bar{f} & & \downarrow F \\ S_\infty^m & \xrightarrow{h_k} & S_\infty^{km} \end{array}$$

is commutative. From (1·1), (2·3) and (2·6), we have the following commutative diagram

$$(2\cdot 9)\qquad \begin{array}{ccccccc} \pi_{i-1}(S^{n+1}) & \underset{\approx}{\xleftarrow{\Omega^{-1}\circ\tilde{i}_*}} & \pi_i(S_\infty^n) & \longrightarrow & \pi_i(S_\infty^n, S_{k-1}^n) & \xrightarrow{H_k'} & \pi_{i+1}(S^{kn+1}) \\ \downarrow Ef_* & & \downarrow \bar{f}_* & & \downarrow \bar{f}_* & & \downarrow E(f)_*^k \\ \pi_{i+1}(S^{m+1}) & \underset{\approx}{\xleftarrow{\Omega^{-1}\circ\tilde{i}_*}} & \pi_i(S_\infty^m) & \longrightarrow & \pi_i(S_\infty^m, S_{k-1}^m) & \xrightarrow{H_k'} & \pi_{i+1}(S^{km+1}), \end{array}$$

where $H'_k = \Omega^{-1}\circ\tilde{i}_*\circ h_{k_*} : \pi_i(S^r_\infty, S^r_{k-1}) \longrightarrow \pi_i(S^{kr}_\infty) \longrightarrow \pi_i(\Omega(S^{kr+1})) \longrightarrow \pi_{i+1}(S^{kr+1})$ for $r=n, m$ and H'_k is equivalent to h_{k_*}.

The following theorem is due to James [8].

THEOREM (2·10). *If n is odd, then $\pi_i(\Omega(S^{n+1}), S^n)$ and $\pi_{i+1}(S^{2n+1})$ are isomorphic.*

Proof. If $n=1$, this follows from the fact that $\pi_{i+1}(S^2)\approx\pi_{i+1}(S^3)+\pi_i(S^1)$ and $\pi_j(S^1)=0$ for $j>1$. Now suppose $n\geqq 3$. By the isomorphisms of (1·1) and (2·3)′, it is sufficient to prove that $h_{2_*} : \pi_i(S^n_\infty, S^n) \longrightarrow \pi_i(S^{2n}_\infty)$ is an isomorphism for all i. The conditions i) and ii) of (1·7) are easily verified for the map $h_2 : (S^n_\infty, S^n) \longrightarrow (S^{2n}_\infty, e_0)$. Take generators $e_i \in H^{in}(S^n_\infty)$ and $e'_i \in H^{2in}(S^{2n}_\infty)$ such as in (2·4) and such that $h^*_2(e'_1)=e_2$. From the relatinos in (2·4), h_2 satisfies iii) and iv) of (1·7) when we set $F=\{e_0, e_1\}$ and $B=\{e_0, e_2, e_4, \cdots\}$. We have that $h^*_2(e'_i)=e_{2i}$ since $(i!)h^*_2(e'_i) =h^*_2((e'_1)^i)=(h^*_2(e'_1))^i=(e_2)^i=(i\,!)e_{2i}$. Then the map h_2 satisfies all the conditions of (1·7) for $R=Z$, and the theorem (1·8) implies that $h_{2_*} : \pi_i(S^n_\infty, S^n) \longrightarrow \pi_i(S^{2n}_\infty)$ is an isomorphism for all $i>1$. q. e. d.

It is also proved in [8] that

(2·10)′ *if n is even $\pi_i(\Omega(S^{n+1}), S^n)$ and $\pi_{i+1}(S^{2n+1})$ are $\mathcal{C}_2$-isomorphic.*

This is, however, contained in the following theorem as the case $p=2$.

THEOREM (2·11) *If n is even and p is a prime, then $\pi_i(\Omega(S^{n+1}), S^n_{p-1})$ and $\pi_{i+1}(S^{pn+1})$ are $\mathcal{C}_p$-isomorphic.*

Proof. By (1·1), (2·3) and (2·3)′, it is sufficient to prove that $h_{p_*} : \pi_i(S^n_\infty, S^n_{p-1}) \longrightarrow \pi_i(S^{pn}_\infty)$ is a $\mathcal{C}_p$-isomorphism for all i. Since h_p is of degree 1 on e^{pn}, we may take generators $e_i\in H^{in}(S^n_\infty)$ and $e'_i\in H^{ipn}(S^{pn}_\infty)$ such as in (2·4) and such that $h^*_p(e'_1) =e_p$. Set $B=\{e_0, e_p, e_{2p}, \cdots\}\otimes Z_p$ and $F=\{e_0, \cdots, e_{p-1}\}\otimes Z_p$. Then the conditions i), ii) and iv) are easily verified for the map $h_p : (S^n_\infty, S^n_{p-1}) \longrightarrow (S^{pn}_\infty, e_0)$ and for the coefficient field $R=Z_p$. By i) of (2·4), $e_{jp}\cdot e_i=\binom{jp+i}{i}e_{jp+i}$ and $\binom{jp+i}{i}\equiv 1$ (mod. p) for $0\leqq i<p$. Next, in the integral coefficient, we have that $h^*_p(e'_j)=\frac{1}{j!}\binom{2p}{p}\cdots\binom{jp}{p}e_{jp}$ since $(j!)h^*_p(e'_j)=h^*_p((e'_1)^j)=(h^*_p(e'_1))^j=(e_p)^j=\binom{2p}{p}\cdots\binom{jp}{p}e_{jp}$. Since $\frac{1}{j!}\binom{2p}{p}\cdots\binom{jp}{p}=\binom{2p-1}{p-1}\cdots\binom{jp-1}{p-1}\equiv 1$ (mod. p), we have an isomorphism $h^*_p : H^*(S^{pn}_\infty, Z_p)\approx B$. Therefore the map h_p satisfies the conditions of (1·7), and we have from the theorem (1·8) that $h_{p_*} : \pi_i(S^n_\infty, S^n_{p-1}) \longrightarrow \pi_i(S^{pn}_\infty)$ is a $\mathcal{C}_p$-isomorphism for $i>1$. Then the theorem is proved. q. e. d.

COROLLARY (2·12). *We have an isomorphism of p-primary components:*

$$H_p=H'_p\circ\tilde{i}^{*-1}=\Omega^{-1}\circ\tilde{i}_*\circ h_{p_*}\circ\tilde{i}^{*-1} : \pi_i(\Omega(S^{n+1}), S^n_{p-1}\,;p)\approx\pi_i(S^n_\infty, S^n_{p-1}\,;p)$$
$$\approx\pi_i(S^{pn}_\infty;p)\approx\pi_i(\Omega(S^{pn+1})\,;p)\approx\pi_{i+1}(S^{pn+1}\,;p)$$

3. Cohomology of some path spaces

In this §, we suppose that n is even.

First we calculate the cohomology ring of the space $\Omega(S^n_{k-1})$ of loops in S^n_{k-1}. The path-space $\Omega(S^n_{k-1}, S^n_{k-1})$ is a fibre-space with the base S^n_{k-1} and the fibre

$\Omega(S^n_{k-1})$. Let $(E^{p,q}_r, d_r)$ be the (integral) cohomological spectral sequence associated with this fibering. Since $\Omega(S^n_{k-1}, S^n_{k-1})$ is contractible, $E^{p,q}_\infty=0$ for $p+q>0$. Since S^n_{k-1} is simply connected and since $H_*(S^n_{k-1})$ is free, $E^{p,q}_2 \approx H^p(S^n_{k-1}) \otimes H^q(\Omega(S^n_{k-1}))$. If an element α of $E^{p,q}_r$ is d_s-cocycle for $r \leqq s < t$, we denote by $\kappa^r_t\,\alpha$ its cohomology class in $E^{p,q}_t$. If $p \neq 0, n, \cdots\cdots, (k-1)n$, then $E^{p,q}_2=E^{p,q}_r=0$ for $r \geqq 2$ and $d_r (r \geqq 2)$ is trivial when $r \neq n, 2n, \cdots\cdots, (k-1)n$. Therefore we have isomorphisms $\kappa^2_n: E^*_2 = E^*_n$, $\kappa^{in+1}_{i(n+1)}: E^*_{in+1}=E^*_{i(n+1)}$, $i=1, 2, \cdots\cdots, k-1$ and $\kappa^{(k-1)n+1}_\infty: E^*_{(k-1)n+1}=E^*_\infty$. Consider elements $a \in H^{n-1}(\Omega(S^n_{k-1}))$ and $b_j \in H^{j(kn-2)}(\Omega(S^n_{k-1}))$, $j=0, 1, 2, \cdots\cdots$, such that

(3・1), i) $d_n(\kappa^2_n(1\otimes a))=\kappa^2_n(e_1 \otimes 1)$,

ii) $b_0=1$,

iii) $d_{(k-1)n}(\kappa^2_{(k-1)n}(1\otimes b_j))=\kappa^2_{(k-1)n}(e_{k-1}\otimes a\cdot b_{j-1})$.

Then

(3・2), i) *the elements a and b_j are uniquely determined by* (3・1),

ii) *and* $b_i \cdot b_j = \binom{i+j}{i} b_{i+j}$.

Proof. Since $E^{0,n-1}_{n+1}=E^{0,n-1}_\infty=0$ and $E^{n,0}_{n+1}=E^{n,0}_\infty=0$, the sequence $0 \longrightarrow E^{0,n-1}_n \xrightarrow{d_n} E^{n,0}_n \longrightarrow 0$ is exact. Since κ^2_n are always isomorphisms, we have an isomorphism $(\kappa^2_n)^{-1}\circ d_n^{-1}\circ \kappa^2_n: E^{n,0}_2 \longrightarrow E^{n,0}_n \longrightarrow E^{0,n-1}_n \longrightarrow E^{0,n-1}_2$. Then a is determined uniquely by (3・1), i). Since $E^{p,q}_{(k-1)n+1}=E^{p,q}_\infty=0$ if $p+q \neq 0$, the boundary homomorphisms $d_{(k-1)n}$ are always isomorphisms. Since d_r: $0=E^{-r,q+r-1}_r \longrightarrow E^{0,q}_r$ is trivial, $E^{0,q}_{r+1} \subset E^{0,q}_r$ and $(\kappa^2_{(k-1)n})^{-1}: E^{0,q}_{(k-1)n} \longrightarrow E^{0,q}_2$ is defined and an isomorphism into. Since $d_r(E^{(k-1)n,q}_r)=0$, κ^2_r is defined on the whole of $E^{(k-1)n,q}_2$. Therefore a homomorphism $(\kappa^2_{(k-1)n})^{-1}\circ d^{-1}_{(k-1)n}\circ \kappa^2_{(k-1)n}$: $E^{(k-1)n,q}_2 \longrightarrow E^{(k-1)n,q}_{(k-1)n} \longrightarrow E^{0,q+(k-1)n-1}_{(k-1)n} \longrightarrow E^{0,q+(k-1)n-1}_2$ is defined. Then $1\otimes b_j$ is the image of $e_{k-1}\otimes a\cdot b_{j-1}$ under this homomorphism and b_j is determined by ii) and iii) of (3・1) uniquely. Next we prove the formula ii) of (3・2) by the induction on the total dimension $i+j$. Obviously $b_0\cdot b_i=b_i\cdot b_0=b_i$. Suppose that $i, j>0$, $b_{i-1}\cdot b_j=\binom{i+j-1}{i-1} b_{i+j-1}$ and $b_i\cdot b_{j-1}=\binom{i+j-1}{i} b_{i+j-1}$. Since $1\otimes b_i$ and $1\otimes b_j$ are d_r-cocycles for $2 \leqq r < (k-1)n$, their product $(1\otimes b_i)\cdot(1\otimes b_j)=(1\otimes b_i\cdot b_j)$ is also d_r-cocycle for $2\leqq r<(k-1)n$. Then

$$
\begin{aligned}
&d_{(k-1)n}(\kappa^2_{(k-1)n}(1\otimes b_i\cdot b_j))\\
&=d_{(k-1)n}(\kappa^2_{(k-1)n}(1\otimes b_i)\cdot\kappa^2_{(k-1)n}(1\otimes b_j))\\
&=\kappa^2_{(k-1)n}((e_{k-1}\otimes a\cdot b_{i-1})\cdot(1\otimes b_j))+\kappa^2_{(k-1)n}((1\otimes b_i)\cdot(e_{k-1}\otimes a\cdot b_{j-1}))\\
&=\kappa^2_{(k-1)n}(e_{k-1}\otimes a\cdot(b_{i-1}\cdot b_j+b_i\cdot b_{j-1}))\\
&=\kappa^2_{(k-1)n}(e_{k-1}\otimes(\tbinom{i+j-1}{i-1}+\tbinom{i+j-1}{i})a\cdot b_{i+j-1})\\
&=d_{(k-1)n}(\kappa^2_{(k-1)n}(1\otimes\tbinom{i+j}{i}b_{i+j})).
\end{aligned}
$$

By operating the homomorphism $(\kappa^2_{(k-1)n})^{-1}\circ d^{-1}_{(k-1)n}$, we have that $1\otimes b_i\cdot b_j=1\otimes\binom{i+j}{i} b_{i+j}$ and then the formula ii) of (3・2) is verified. e. q. d.

Let $P^*(a, b_j)$ be a subring of $H^*(\Omega(S^n_{k-1}))$ generated by the elements a, b_1, b_2, $\cdots\cdots$, and set $P^t(a, b_j)=H^t(\Omega(S^n_{k-1}))\cap P^*(a, b_j)$. Then

LEMMA (3·3). *the elements b_j and $a\cdot b_j$, $j=0, 1, 2, \cdots\cdots$, are of infinite orders, and $H^t(\Omega(S^n_{k-1}))/P^t(a, b_j) \in \mathcal{C}_p$ for a prime $p \geqq k$ and for all t (n : even).*

As a corollary,

(3·3)′ $\qquad H^*(\Omega(S^n_{k-1}), Z_p) \approx P^*(a, b_j) \otimes Z_p$ for a prime $p \geqq k$.

Proof of (3·3). Obviously $H^0(\Omega(S^n_{k-1}))=P^0(a, b_j)=b_0$, so that we prove (3·3) by the induction on the dimension $t>0$. Suppose that (3·3) is true for dim. $<t$, $t>0$.

i), *The case $t \neq j(kn-2)+in-1$ and $t \neq j(kn-2)+(i+1)n-2$ for $j \geqq 0$ and for $1 \leqq i < k$.* In this case, $t-in+1 \neq j(kn-2)$ and $t-in+1 \neq j(kn-2)+n-1$, then $E_2^{in,\, t-in+1} \approx H^{t-in+1}(\Omega(S^n_{k-1})) \in \mathcal{C}_p$ and $E_{in}^{in,t-in+1} \in \mathcal{C}_p$. Since the coboundaries in $E_{in}^{0,t}$ are trivial, $E_2^{0,t}=E_n^{0,t} \supset E_{2n}^{0,t} \supset \cdots \supset E_{kn}^{0,t}=E_\infty^{0,t}=0$ and d_{in} maps $E_{in}^{0,t}/E_{(i+1)n}^{0,t}$ isomorphically into $E_{in}^{in,\, t-in+1} \in \mathcal{C}_p$. Then $E_{(i+1)n}^{0,t} \in \mathcal{C}_p$ implies $E_{in}^{0,t} \in \mathcal{C}_p$ for $1 \leqq i < k$. Therefore we have that $E_2^{0,t} \approx H^t(\Omega(S^n_{k-1})) \in \mathcal{C}_p$.

ii) *The case $t=j(kn-2)+n-1$.* In this case $t-in+1=(j-1)(kn-2)+(k-i+1)n-2$ and $E_2^{in,\, t-in+1} \approx H^{t-in+1}(\Omega(S^n_{k-1})) \in \mathcal{C}_p$ for $1<i<k$. Similarly to the case i), $E_{in}^{0,t}/E_{(i+1)n}^{0,t} \in \mathcal{C}_p$ for $1<i<k$ and this implies thet $E_{2n}^{0,t} \in \mathcal{C}_p$. Since $d_n(\kappa_n^2(1 \otimes a\cdot b_j)) = d_n(\kappa_n^2(1 \otimes a)) \cdot \kappa_n^2(1 \otimes b_j) = \kappa_n^2(e_1 \otimes b_j)$, the sequence: $E_{2n}^{0,t} \longrightarrow E_n^{0,t}/\{\kappa_n^2(1 \otimes a\cdot b_j)\} \xrightarrow{d_n} E_n^{n,t-n+1}/\{\kappa_n^2(e_1 \otimes b_j)\} \approx H^{t-n+1}(\Omega(S^n_{k-1}))/\{b_j\}) \in \mathcal{C}_p$ is exact and $\kappa_n^2(1 \otimes a\cdot b_j)$ has to be of infinite order. Then $E_n^{0,t}/\kappa_n^2(1 \otimes a\cdot b_j) \approx H^t(\Omega(S^n_{k-1}))/\{a\cdot b_j\} \in \mathcal{C}_p$ and $a\cdot b_j$ is of infinite order.

iii) *The case $t=(j+1)(kn-2)$, $j \geqq 0$.* In this case $t-in+1=j(kn-2)+(k-i-1)n+n-1$ and $E_2^{in,\, t-in+1} \approx H^{t-in+1}(\Omega(S^n_{k-1})) \in \mathcal{C}_p$ for $1 \leqq i < k-1$. Similarly to the case i), $E_{in}^{0,t}/E_{(i+1)n}^{0,t} \in \mathcal{C}_p$ for $1 \leqq i < k-1$ and this implics that $E_2^{0,t}/E_{(k-1)n}^{0,t} \in \mathcal{C}_p$. Since $d_{in}(E_{in}^{(k-1)n,\, t-(k-1)n+1})=0$, the sequence $E_{in}^{(k-i-1)n,\, t-(k-i-1)n} \xrightarrow{d_{in}} E_{in}^{(k-1)n,\, t-(k-1)n+1} \xrightarrow{\kappa_{(i+1)n}^{in}} E_{(i+1)n}^{(n-1)n,t-(k-1)n+1} \longrightarrow 0$ is exact. Since $t-(k-i-1)n=j(kn-2)+(i+1)n-2$, $E_2^{(k-i-1)n,\, t-(k-i-1)n} \approx H^{t-(k-i-1)n}(\Omega(S^n_{k-1})) \in \mathcal{C}_p$ and $E_{in}^{(k-i-1)n,\, t-(k-i-1)n} \in \mathcal{C}_p$ for $1 \leqq i < k-1$, and then $\kappa_{(i+1)n}^{in}$ is a $\mathcal{C}_p$-isomorphism for $1 \leqq i < k-1$. Therefore the homomorphism $(\kappa^2_{(k-1)n})^{-1} \circ d^{-1}_{(k-1)n} \circ \kappa^2_{(k-1)n}$: $E_2^{(k-1)n,\, t-(k-1)n+1} \longrightarrow E_{(k-1)n}^{(k-1)n,\, t-(k-1)n+1} \longrightarrow E_{(k-1)n}^{0,t} \longrightarrow E_2^{0,t}$ is a $\mathcal{C}_p$-isomorphism, since $d_{(k-1)n}$ is an isomorphism. This isomorphism maps $e_{k-1} \otimes a\cdot b_j$ to $1 \otimes b_{j+1}$, then $H^{t-(k-1)n+1}(\Omega(S^n_{k-1}))/\{a\cdot b_j\} \in \mathcal{C}_p$ implies that $H^t(\Omega(S^n_{k-1}))/\{b_{j+1}\} \in \mathcal{C}_p$ and the element b_{j+1} has an infinite order.

iv) *The case $t=j(kn-2)+in-1$ or $t=j(kn-2)+in-2$ for $j \geqq 0$ and for $1<i<k$.* Since $d_n(\kappa_n^2(e_{i-1} \otimes a\cdot b_j)) = d_n(\kappa_n^2(1 \otimes a)) \cdot \kappa_n^2(e_{i-1} \otimes b_j) = \kappa_n^2((e_1 \otimes 1)\cdot(e_{i-1} \otimes b_j)) = \kappa_n^2(ie_i \otimes b_j)$ by (2·4), i), the boundary homomorphism d_n: $E_n^{(i-1)n,\, j(kn-2)+n-1} \longrightarrow E_n^{in,\, j(kn-2)}$ is a $\mathcal{C}_p$-isomorphism for $1 \leqq i < k$. Then we have that $E_r^{(i-1)n,\, j(kn-2)+n-1} \in \mathcal{C}_p$ and $E_r^{in,\, j(kn-1)} \in \mathcal{C}_p$ for $r>n$ and for $1 \leqq i < k$. It is easily seen that in this case the image of d_{in}: $E_{in}^{0,t} \longrightarrow E_{in}^{in,\, t-in+1}$ belongs to $\mathcal{C}_p$ for $1 \leqq i < k$. Then by the same argument as the case i), we have that $H^t(\Omega(S^n_{k-1})) \in \mathcal{C}_p$.

Consequently, for all dimension $t>0$, $H^s(\Omega(S^n_{k-1}))/P^s(a, b_j) \in \mathcal{C}_p$ for $s<t$ implies $H^t(\Omega(S^n_{k-1}))/P^t(a, b_j) \ni \mathcal{C}_p$, and the lemma (3·3) is proved by the induction.

Next we prove that

(3・4) $H^i(\Omega(S^n_{k-1}, S^n_{k-2})) \approx H^{i-(k-1)n+1}(\Omega(S^n_{k-1}))$ *and* $H_i(\Omega(S^n_{k-1}, S^n_{k-2})) \approx H_{i-(k-1)n+1}(\Omega(S^n_{k-1}))$ *for* $i>0$.

Proof. Let $E_r^{p,q}$ be the spectral sequence associated with the fibering $(\Omega(S^n_{k-1}, S^n_{k-1}), \Omega(S^n_{k-1}, S^n_{k-2})) \longrightarrow (S^n_{k-1}, S^n_{k-2})$ whose fibre is $\Omega(S^n_{k-1})$. Since the pair (S^n_{k-1}, S^n_{k-2}) is acyclic, the spectral sequence $E_r^{p,q}$ is trivial for $r \geqq 2$. Then $H^{(k-1)n+q}(\Omega(S^n_{k-1}, S^n_{k-1}), \Omega(S^n_{k-1}, S^n_{k-2})) = E_\infty^{(k-1)n,q} = E_2^{(k-1)n,q} \approx H^{(k-1)n}(S^n_{k-1}, S^n_{k-2}) \otimes H^q(\Omega(S^k_{n-1})) \approx H^q(\Omega(S^n_{k-1}))$. Since the space $\Omega(S^n_{k-1}, S^n_{k-1})$ is contractible, the coboundary homomorphism $\delta: H^{(k-1)n+q-1}(\Omega(S^n_{k-1}, S^n_{k-2})) \longrightarrow H^{(k-1)n+q}(\Omega(S^n_{k-1}, S^n_{k-1}), \Omega(S^n_{k-1}, S^n_{k-2}))$ is an isomorphism for $(k-1)n+q-1>0$. Therefore, by setting $i=(k-1)n+q-1$, we have the isomorphism (3・4). For the homology the proof is similar. q. e. d.

Let $i: \Omega(S^n_{k-1}) \longrightarrow \Omega(S^n_{k-1}, S^n_{k-2})$ be the injection, then

(3・5) *the injection homomorphism* $i^*: H^{kn-2}(\Omega(S^n_{k-1}, S^n_{k-2})) \longrightarrow H^{kn-2}(\Omega(S^n_{k-1}))$ *is a* $\mathcal{C}_p$*-isomorphism for a prime* $p \geqq k \geqq 2$.

Proof. Let $E_r^{s,q}$ be the spectral sequence mod. p associated with the fibering $\Omega(S^n_{k-1}, S^n_{k-2}) \longrightarrow S^n_{k-2}$, then $E_2^{s,q} \approx H^s(S^n_{k-2}, Z_p) \otimes H^q(\Omega(S^n_{k-1}), Z_p)$. By (3・3)′, $E_2^{s,q}=0$ when $kn-2 \leqq s+q \leqq kn-1$, $(s,q) \neq (0, kn-2)$ and $(s,q) \neq (0, (k+1)n-3)$. Then the operator d_r is trivial in $E_r^{s,q}$ if $s+q=kn-2$, thus $E_2^{0,kn-2}=E_\infty^{0,kn-2}$ and $E_\infty^{s,kn-2-s}=0$ if $s>0$. Since the injection homomorphism i^* is represented by the composition $H^{kn-2}(\Omega(S^n_{k-1}, S^n_{k-2}), Z_p) \longrightarrow E_\infty^{0,kn-2} \subset E_2^{0,kn-2} \approx H^{kn-2}(\Omega(S^n_{k-1}), Z_p)$, i^* is an isomorphism with the coefficient Z_p. By (3・3) and (3・4) the groups $H^*(\Omega(S^n_{k-1}))$ and $H^*(\Omega(S^n_{k-1}, S^n_{k-2}))$ have no p-torsions. Then the injection homomorphism $i^*: H^{kn-2}(\Omega(S^n_{k-1}, S^n_{k-2})) \longrightarrow H^{kn-2}(\Omega(S^n_{k-1}))$ is a $\mathcal{C}_p$-isomorphism. q. e. d.

REMARK. It may be proved that the homomorphism $(\kappa^2_{(k-1)n})^{-1} \circ d^{-1}_{(k-1)n} \circ \kappa^2_{(k-1)n}: E_2^{(k-1)n,q} \longrightarrow E_2^{0,q+(k-1)n-1}$ is equivalent to the homomorphism $i^* \circ \delta^{-1}: H^{(k-1)n+q}(\Omega(S^n_{k-1}, S^n_{k-1}), \Omega(S^n_{k-1}, S^n_{k-2})) \approx H^{(k-1)n+q-1}(\Omega(S^n_{k-1}, S^n_{k-2})) \longrightarrow H^{(k-1)n+q-1}(\Omega(S^n_{k-1}))$. Then the isomorphism of (3・4) maps b_{j+1} to $a \cdot b_j$ and, in (3・5), b_1 is the image of a generator of $H^{kn-2}(\Omega(S^n_{k-1}, S^n_{k-2}))$.

Consider a complex $K'=S^n_{k-1} \cup e^{kn}$. Then

(3・6) *the following two conditions are equivalent* $(p \geqq k)$

i) $e_1^k \neq 0$ *in* $H^{kn}(K', Z_p)$.

ii) $H^{kn-2}(\Omega(K'), Z_p)=0$.

Proof. $e_1 \cdot e_{k-1} = te_k$ for an integer t and for a generator e_k of $H^{kn}(K')$. Since $e_1^{k-1}=(k-1)!\,e_{k-1}$ and $(k-1)! \not\equiv 0 \pmod{p}$, the condition i) is equivalent to

i)′ $t \not\equiv 0 \pmod{p}$.

Let $E_r^{s,q}$ be the cohomological spectral sequence mod. p associated with the fibering $\Omega(K', K') \longrightarrow K'$, then $E_2^{s,q} \approx H^s(K', Z_p) \otimes H^q(\Omega(K'), Z_p)$ and E_∞^* is trivial. As the proof of (3・3), we see that $H^i(\Omega(K'), Z_p)=0$ for $n-1<i<kn-2$. Since $d_r: E_r^{0,kn-2} \longrightarrow E_r^{r,kn-r-1}$ is trivial for $2 \leqq r < (k-1)n$, we have an isomorphism $\kappa^2_{(k-1)n}$: $E_2^{0,kn-2} \approx E_{(k-1)n}^{0,kn-2}$. Since $E_{(k-1)n+1}^{0,kn-2} = E_{(k-1)n+1}^{(k-1)n,n-1}=0$, we have an isomorphism $d_{(k-1)n}$

$E^{0,kn-2}_{(k-1)n} \approx E^{(k-1)n,n-1}_{(k-1)n}$. Since the coboundary operator d_r is trivial in $E_r^{(k-1)n,n-1}$ for $n < r < (k-1)n$, we have an isomorphism $\kappa^{n+1}_{(k-1)n} : E^{(k-1)n,n-1}_{n+1} \approx E^{(k-1)n,n-1}_{(k-1)n}$. Since $d_r(E_n^{(k-1)n,2n-2}) = 0$ if $k > 2$, the sequence $0 \longrightarrow E^{(k-1)n,n-1}_{n+1} \longrightarrow E_n^{(k-1)n,n-1} \longrightarrow E_n^{kn,0}$ is exact if $k > 2$. If $k = 2$ the sequence $0 \longrightarrow E_n^{0,2n-2} \longrightarrow E_n^{n,n-1} \longrightarrow E_n^{2n,0}$ is exact. Consequently we have that $H^{kn-2}(\Omega(K'), Z_p) \approx E_2^{0,kn-2} = 0$ if and only if d_n: $E_n^{(k-1)n,n-1} \longrightarrow E_n^{kn,0}$ is an isomorphism. Since $d_n(\kappa_n^2(e_{k-1} \otimes a)) = \kappa_n^2(e_1 \cdot e_{k-1} \otimes 1) = t\kappa_n^2(e_k \otimes 1)$ and since κ_n^2 are isomorphisms, the conditions i)$'$ and ii) are equivalent.

q. e. d.

4. The group $\pi_i(S^n_{k-1})$, n: even

In this § we suppose that n is even.

First we consider the case $n = 2$. Let M_{k-1} be the $(k-1)$-dimensional complex projective space. There is a fibre bundle (S^{2k-1}, p_0, M_{k-1}) with the fibre S^1. Then we have an isomorphism $\pi_i(M_{k-1}) \approx \pi_i(S^{2k-1}) + \pi_{i-1}(S^1)$ for all i and $\pi_i(M_{k-1}) = 0$ for $2 < i < 2k-1$. Since the dimension of S^2_{k-1} is $2k-2$, the identity on S^2 is extended over a map

$$f:\ S^2_{k-1} \longrightarrow M_{k-1}$$

and these maps f are homotopic to each other.

Theorem (4·1) *The map f induces a $\mathcal{C}_p$-isomorphism $f_* : \pi_i(S^2_{k-1}) \longrightarrow \pi_i(M_{k-1}) \approx \pi_i(S^{2k-1}) + \pi_{i-1}(S^1)$ for a prime $p \geqq k \geqq 2$.*

Proof. By (1·6), it is sufficient to prove that

$$f^* : H^*(M_{k-1}, Z_p) \approx H^*(S^2_{k-1}, Z_p).$$

Let e_i, $i = 1, \cdots, k-1$, be generators of $H^{2i}(S^2_{k-1})$ such as in (2·4). From the definition of f, there is a generator e of $H^2(M_{k-1})$ such that $f^*(e) = e_1$. As is well known, e^i is a generator of $H^{2i}(M_{k-1})$ for $0 \leqq i \leqq k-1$. Since $f^*(e^i) = (f^*(e))^i = (e_1)^i = i!\, e_i$ and since $i! \not\equiv 0$ (mod. p) for $0 \leqq i \leqq k-1 < p$, we have that f^* is a $\mathcal{C}_p$-isomorphism for all dimensions. Then f^* is an isomorphism of mod. p. q. e. d.

(4·2) *There is a map g: $S^{2k-1} \longrightarrow S^2_{k-1}$ whose class in $\pi_{2k-1}(S^2_{k-1})$ is not divisible by p for a prime $p \geqq k$. Then g_*: $\pi_i(S^{2k-1}) \longrightarrow \pi_i(S^2_{k-1})$ is a $\mathcal{C}_p$-isomorphism for $i > 2$. ($k \geqq 2$).*

Proof. The first part of (4·2) is easily verified from (4·1). Since $p_{0*}: \pi_{2k-1}(S^{2k-1}) \longrightarrow \pi_{2k-1}(M_{k-1})$ is an isomorphism, there is a map g': $S^{2k-1} \longrightarrow S^{2k-1}$ such that the compositions $p_0 \circ g'$ and $f \circ g$ are homotopic to each other. Then the degree of g' is not divisible by p. Since g' induces $\mathcal{C}_p$-isomorphisms of the cohomology groups, g' induces $\mathcal{C}_p$-isomorphisms of the homotopy groups by (1·6). Then $p_{0*} \circ g'_* = f_* \circ g_*$ is a $\mathcal{C}_p$-isomorphism. Since p_{0*} is an isomorphism for $i > 2$ and since f_* and g'_* are $\mathcal{C}_p$-isomorphisms, g_* is a $\mathcal{C}_p$-isomorphism for $i > 2$. q. e. d.

(4·3) *There is a map g_0: $S^{2k-2} \longrightarrow \Omega(S^2_{k-1})$ such that ($p \geqq k$)*

$$g_0^* : H^{2k-2}(\Omega(S^2_{k-1}), Z_p) \approx H^{2k-2}(S^{2k-2}, Z_p).$$

Proof. Let T be the universal covering space of $\Omega(S^2_{k-1})$ and let $\sigma: T \longrightarrow$

$\Omega(S^2_{k-1})$ be the projection. Then we have that $H^i(\Omega(S^2_{k-1}), Z_p) \approx H^i(T, Z_p)+H^{i-1}(T, Z_p)$ and $\sigma^*: H^{2k-2}(\Omega(S^2_{k-1}), Z_p) \approx H^{2k-2}(T, Z_p)$ (cf. [10, Ch. I, Prop. 4, Cor. 1]). By $(3\cdot 3)'$, $H^i(T, Z_p) \approx H^i(S^{2k-2}, Z_p)$ for $i<4k-4$. Then there is a map $g': S^{2k-2} \longrightarrow T$ such that $g'^*: H^{2k-2}(T, Z_p) \approx H^{2k-2}(S^{2k-2}, Z_p)$. Then (4·3) is proved by setting $g_0 = \sigma \circ g'$. q. e. d.

PROPOSITION (4·4). *Let n be even and let p be a prime $\geqq k \geqq 2$. If there is a map $g: S^{kn-1} \longrightarrow S^n_{k-1}$ such that $(\Omega g)^*: H^{kn-2}(\Omega(S^n_{k-1}), Z_p) \approx H^{kn-2}(\Omega(S^{kn-1}), Z_p)$, then the correspondence $(\alpha, \beta) \longrightarrow E\alpha + g_*\beta$ defines a $\mathcal{C}_p$-isomorphism $\pi_{i-1}(S^{n-1})+\pi_i(S^{kn-1}) \longrightarrow \pi_i(S^n_{k-1})$, $i>1$.*

Proof. First consider the case $n=2$ and consider the universal covering space T of $\Omega(S^2_{k-1})$ as in the proof of (4·3). Then there exists a map $g': S^{2k-2} \longrightarrow T$ such that $p\circ g' = \Omega g | S^{2k-2}$. Since $H^i(T, Z_p) \approx H^i(S^{2k-2}, Z_p)$ for $i<4k-4$, $g'_*: \pi_{2k-2}(S^{2k-2}) \longrightarrow \pi_{2k-2}(T)$ is a $\mathcal{C}_p$-isomorphism. Therefore the class of $\Omega g | S^{2k-2}$ in $\pi_{2k-2}(\Omega(S^2_{k-1}))$ is not divisible by p. Since $\pi_{i-1}(S^1)=0$ for $i>2$, $\pi_2(S^{2k-1})=0$ and $\pi_1(S^1) \approx \pi_2(S^2_{k-1})$, (4·4) is proved, for $n=2$, by (4·2).

Next consider the case $n \geqq 4$. Define a map $G: S^{n-1} \times \Omega(S^{kn-1}) \longrightarrow \Omega(S^n_{k-1})$ by the formula $G(x, y) = i(x) * \Omega g(y)$ where $*$ indicates the product in loops and i is the injection of $(2\cdot 2)'$. It is easy to see that the induced homomorphism $G_*: \pi_{i-1}(S^{n-1} \times \Omega(S^{kn-1})) \longrightarrow \pi_{i-1}(\Omega(S^n_{k-1}))$ is equivalent to the correspondence given in (4·4). By (1·6), it is sufficient to prove that $G^*: H^*(\Omega(S^n_{k-1})) \longrightarrow H^*(S^{n-1} \times \Omega(S^{kn-1})) = H^*(S^{n-1}) \otimes H^*(\Omega(S^{kn-1}))$ is a $\mathcal{C}_p$-isomorphism. Let e_j be generators of $H^{j(kn-2)}(\Omega(S^{kn-1}))$ such as in (2·4) and let a and b_j be the elements of (3·2). We may set $G^*(b_j) = 1 \otimes g^*(b_j) = 1 \otimes t_j e_j$ for some integers t_j. By the assumption of (4·4), $t_1 \not\equiv 0 \pmod{p}$. By the assertion of (2·4), and (3·2), ii), we have that $1 \otimes t_1^j j! e_j = 1 \otimes (t_1 e_1)^j = (G^*(b_1))^j = G^*(b_1^j) = G^*(j!b_j) = 1 \otimes t_j j! e_j$. Therefore $t_j = t_1^j \not\equiv 0 \pmod{p}$. Obviously $G^*(a)$ is a generator $a' \otimes 1$ of $H^{n-1}(S^{n-1} \times \Omega(S^{kn-1}))$ and $G^*(a \cdot b_j) = t_j(a' \otimes e_j)$. Then it follows from (3·3) that G^* is a $\mathcal{C}_p$-isomorphism. q. e. d.

LEMMA (4·5) *Let n be even $\geqq 4$ and let p be a prime $\geqq k \geqq 2$. Then the following two conditions are equivalent;*

i) *there is a map $g: S^{kn-1} \longrightarrow S^n_{k-1}$ such that $(\Omega g)^*: H^{kn-2}(\Omega(S^n_{k-1}), Z_p) \approx H^{kn-2}(\Omega(S^{kn-1}), Z_p)$,*

ii) *there is a complex $K = S^n_{k-1} \cup e^{kn}$ such that $e_1^k \neq 0$ in $H^{kn}(K, Z_p)$.*

Proof. Consider a map $g': S^{kn-1} \longrightarrow S^n_{k-1}$ and a complex $K' = S^n_{k-1} \cup e^{kn}$ in which e^{kn} is attached by the map g'. Let $G': S^{kn-2} \longrightarrow \Omega(S^n_{k-1})$ be the restriction of $\Omega g'$ on $S^{kn-2} \subset \Omega(S^{kn-1})$. Define a space $\Omega' = \Omega(S^n_{k-1}) \cup e^{kn-1}$ by attaching a cell e^{kn-1} with the map G'. Let $d: \Omega(S^n_{k-1}) \times I \longrightarrow S^n_{k-1}$ be defined by setting $d(x, t) = x(t)$, then d is extendable over $d: \Omega' \times I \longrightarrow K'$ such that d maps $e^{kn-1} \times (I - \dot{I})$ homeomorphically onto e^{kn}. This shows that Ω' is imbedded in $\Omega(K')$. Let $\phi_1: \Omega' \longrightarrow S^{kn-1}$ and $\phi_2: K' \longrightarrow S^{kn}$ be maps which pinch $\Omega(S^n_{k-1})$ and S^n_{k-1} respectively. In the diagram

$$\begin{array}{ccccc} \pi_i(\Omega', \Omega(S^n_{k-1})) & \xrightarrow{i_*} & \pi_i(\Omega(K'), \Omega(S^n_{k-1})) & \underset{\approx}{\xleftarrow{\Omega'}} & \pi_{i+1}(K', S^n_{k-1}) \\ \downarrow \phi_{1*} & & & & \downarrow \phi_{2*} \\ \pi_i(S^{kn-1}) & & \xrightarrow{\quad E \quad} & & \pi_{i+1}(S^{kn}) \end{array}$$

the commutativity holds, where i is the injection. By making use of [3, Th II], the homomorphisms ϕ_{1*}, ϕ_{2*}, E and also i^* are isomorphisms for $i \leqq (k+1)n-4$. Then $i_*: \pi_i(\Omega') \longrightarrow \pi_i(\Omega(K'))$ is an isomorphism for $i \leqq (k+1)n-4$. By (1·6), $i^*: H^i(\Omega(K')) \approx H^i(\Omega')$ for $i < (k+1)n-4$. In the exact sequence $\cdots \longrightarrow H^i(\Omega', Z_p) \longrightarrow H^i(\Omega(S^n_{k-1}), Z_p) \xrightarrow{\delta} H^{i+1}(\Omega', \Omega(S^k_{n-1}), Z_p) \longrightarrow \cdots$ the coboundary homomorphism δ is equivalent to the homomorphism $(\Omega g')^*: H^i(\Omega(S^n_{k-1}), Z_p) \longrightarrow H^i(\Omega(S^{kn-1}), Z_p)$ for $i < (k+1)n-4$ since we have the following commutative diagram

$$\begin{array}{ccccc} & H^i(\Omega(S^n_{k-1}), Z_p) & \xrightarrow{\delta} & H^{i+1}(\Omega', \Omega(S^n_{k-1}), Z_p) & \\ \swarrow (\Omega g')^* & \downarrow G'^* & & \wr\!\!\!\downarrow \bar{G}'^* & \nwarrow \phi_1^* \\ H^i(\Omega(S^{kn-1}), Z_p) & & & & H^{i+1}(S^{kn-1}, Z_p) \\ \searrow i^* \approx & & & & \swarrow \approx \\ & H^i(S^{kn-2}, Z_p) & \underset{\approx}{\xrightarrow{\delta'}} & H^{i+1}(E^{kn-1}, S^{kn-2}, Z_p) & \end{array}$$

where $\bar{G}'$ is a characteristic map of e^{kn-1} such that $G' = \bar{G}' | S^{kn-2}$ and the map i is the injection. Therefore $H^{kn-2}(\Omega(K'), Z_p) = H^{kn-2}(\Omega', Z_p) = 0$ if and only if $(\Omega g')^*: H^{kn-2}(\Omega(S^n_{k-1}), Z_p) \longrightarrow H^{kn-2}(\Omega(S^{kn-1}), Z_p)$ is an isomorphism. Then (3·6) implies the lemma (4·5). q. e. d.

THEOREM (4·6). i) *If* $p > k$, *then there is a map* $g: S^{kn-1} \longrightarrow S^n_{k-1}$ *such that the correspondence* $(\alpha, \beta) \longrightarrow E\alpha + g_*(\beta)$ *induces a* $\mathcal{C}_p$*-isomorphism* : $\pi_{i-1}(S^{n-1}) + \pi_i(S^{kn-1}) \longrightarrow \pi_i(S^n_{k-1})$ *for all* $i > 1$ (n: *even*).

If the assertion of i) *is true for the case* $p = k$, *then* $n/2$ *has to be a power of* p.

Proof. i) is proved from (4·5), (4·4) and (2·3) by seiting $k = S^n_k$. If $n = 2$, i) is true for the case $p = k$ since (4·2). Suppose that $n \geqq 4$ and $k = p$. Consider the map G which is defined in the proof of (4·4). Then from the assertion of i), G_* induces $\mathcal{C}_p$-isomorphisms of the homotopy groups. By (1·6), we have an isomorphism $G^*: H^*(\Omega(S^n_{p-1}), Z_p) \approx H^*(S^{n-1} \times \Omega(S^{pn-1}), Z_p)$. Then $(\Omega g)^*: H^{pn-2}(\Omega(S^n_{p-1}), Z_p) \approx H^{pn-2}(\Omega(S^{pn-1}), Z_p)$. By (4·5), there is a complex $K = S^n_{p-1} \cup e^{pn}$ such that $e_1^p \neq 0$ in $H^{pn}(K, Z_p)$. For the Steenrod's operation $P^{n/2}$, we have that $P^{n/2}: H^n(K, Z_p) \approx H^{pn}(K, Z_p)$ since $P^{n/2} e_1 = e_1^p$. Consider the map $\bar{d}_n: (S^n_{k-1} \times I, S^n_{k-1} \times \dot{I}) \longrightarrow (S^{n+1}, e_0)$ of (2·2). We construct a complex $L = S^{n+1} \cup e^{pn+1}$ and a map $D': (K \times I, K \times \dot{I}) \longrightarrow (L, e_0)$ such that $\bar{d}_n = D' | S^n_{k-1} \times I$ and that D' maps $e^{pn} \times (I - \dot{I})$ homeomorphically onto e^{pn+1}. Identifying the subset $K \times \dot{I} \cup e_0 \times I$ of $K \times I$ to a single point, we obtain a suspension EK of K, and the identification defines a map $D: EK \longrightarrow L$ such that $D^*: H^i(L, Z_p) \approx H^i(EK, Z_p)$ for $i = n+1$ and $i = pn+1$. From the commutativity of the Steenrod's operation $P^{n/2}$ with D^* and the suspension homomorphism (isomorphism), we have that $P^{n/2}: H^{n+1}(L, Z_p) \approx H^{pn+1}$

(L, Z_p). If $n/2$ is not a power of p, then by the Adem's relations in P^j [1], [6], $P^{n/2}$ is a linear combination of iterations of P^j for $0<j<n/2$. Since $P^j(H^{n+1}(L, Z_p)) \subset H^{n+2j(p-1)}(L, Z_p)=0$ for $0<j<n/2$, P^j and hence $P^{n/2}$ are trivial. This contradicts with the fact that $P^{n/2}$ is an isomorphism. Therefore $n/2$ has to be a power of p. q. e. d.

REMARK We have that (4·6) is true for the case $p=k$ and $n=2p$. This follows from the fact that the cokernel of $E^2: \pi_{2p^2-2}(S^{2p-1}; p) \longrightarrow \pi_{2p^2}(S^{2p+1}; p)$ is Z_p (see appendix).

5. Relative Hopf invariant and applications

Let A and B be spaces and let a_0 and b_0 be points of A and B respectively. Denote by $A \vee B$ the subset $A \times b_0 \cup a_0 \times B$ of $A \times B$. Let $i_1: A \longrightarrow A \times b_0 \subset A \times B$ and $i_2: B \longrightarrow a_0 \times B \subset A \times B$ be the injections and let $p_1: A \vee B \longrightarrow A$ and $p_2: A \vee B \longrightarrow B$ be the projections. It was proved in [16, § 4] that the injection homomorpoisms $i_{1*}: \pi_i(A) \longrightarrow \pi_i(A \vee B)$ and $i_{2*}: \pi_i(B) \longrightarrow \pi_i(A \vee B)$ and the boundary homomorphism $\partial: \pi_{i+1}(A \times B, A \vee B) \longrightarrow \pi_i(A \vee B)$ are isomorphisms into and that we have a direct sum decomposition

(5·1) $\pi_i(A \vee B)=i_{1*}\pi_i(A)+i_{2*}\pi_i(B)+\partial\pi_{i+1}(A \times B, A \vee B)$ *for* $i>1$.

A homomorphism

(5·2) $Q: \pi_i(A \vee B) \longrightarrow \pi_{i+1}(A \times B, A \vee B)$ *given by the formula* $Q(\alpha)=\partial^{-1}(\alpha-i_{1*}(p_{1*}(\alpha))-i_{2*}(p_{2*}(\alpha)))$ *is the projection to the third factor of* (5·1).

It follows from the exactness of the homotopy sequence of the pair $(A \vee B, A)$ that the injection homomorphism $j_*: \pi_i(A \vee B) \longrightarrow \pi_i(A \vee B, A)$ is onto and its kernel is $i_{1*}\pi_i(A)$. Therefore

(5·1)′ $\pi_i(A \vee B, A)=j_*(i_{2*}\pi_i(B))+j_*(\partial\pi_{i+1}(A \times B, A \vee B))$ *for* $i>1$.

Similarly, from the exact squence of the triad $(A \vee B; B, A)$ we have an isomorphism

(5·1)″ $$j'_* \circ j_* \circ \partial: \pi_{i+1}(A \times B, A \vee B) \approx \pi_i(A \vee B; B, A) \qquad \textit{for } i>2,$$

where $j'_*: \pi_j(A \vee B; B, A) \longrightarrow \pi_i(A \vee B; B, A)$ is the injection homomorphism. *Projections*

(5·2)′ $$Q': \pi_i(A \vee B, A) \longrightarrow \pi_{i+1}(A \times B, A \vee B)$$

(5·2)″ *and* $$Q'': \pi_i(A \vee B; B, A) \underset{\approx}{\longrightarrow} \pi_{i+1}(A \times B, A \vee B)$$

are defined such that the diagram

(5·3) $$\begin{tikzcd} \pi_i(A \vee B) \arrow[r, "j_*"] \arrow[dr, "Q"'] & \pi_i(A \vee B, A) \arrow[r, "j'_*"] \arrow[d, "Q'"] & \pi_i(A \vee B; B, A) \arrow[dl, "Q''"] \\ & \pi_{i+1}(A \times B, A \vee B) & \end{tikzcd}$$ 29

is commutative. Then $Q''=(j'_* \circ j_* \circ \partial)^{-1}$, $Q'=Q'' \circ j'_*$ and $Q=Q' \circ j'_* \circ j_*$.

Let K_0 be an $(n-1)$-connected finite cell complex and let e_0 be a vertex, $n \geqq 2$. Attaching an r-cell e^r to K_0 by a characteristic map

$$\mu: (I^r, \dot{I}^r, J^{r-1}) \longrightarrow (K_0 \cup e^r, K_0, e_0), \qquad r \geqq 2,$$

such that μ maps $I^r-\dot{I}^r$ homeomorphically onto e^r, we have a CW-complex $K=K_0\cup e^r$. Denote by I^r_+ and I^r_- subsets of I^r given by $I^r_+=\{(t_1,\cdots,t_r)\in I^r\mid t_r\geqq\frac{1}{2}\}$ and $I^r_-=\{(t_1,\cdots,t_r)\in I^r\mid t_r\leqq\frac{1}{2}\}$. The image of I^r_+ under the map μ is a closed r-cell E^r and its boundary is an $(r-1)$-sphere S^{r-1} containing e_0. Set $\bar{K}_0=K_0\cup\mu(I^r_-)$, then $\bar{K}_0\cup E^r=K$ and $\bar{K}_0\cap E^r=S^{r-1}$. As is easily seen, K_0 is a deformation retract of $\bar{K}_0$ and $\pi_i(K, K_0)\approx\pi_i(K,\bar{K}_0)$ for all i. Define a map

$$\varphi:(K;E^r,\bar{K}_0)\longrightarrow(K\vee S^r;S^r,K)$$

by the formula $\varphi(x)=(x,e_0)$ for $x\in K_0$ and

$$\varphi(\mu(t_1,\cdots,t_{r-1},t_r))=\begin{cases}(\mu(t_1,\cdots,t_{r-1},2t_r),\ e_0), & 0\leqq t_r\leqq\frac{1}{2}\\ (e_0,\ \psi_r(t_1,\cdots,t_{r-1},2t_r-1)), & \frac{1}{2}\leqq t_r\leqq 1\end{cases}$$

for $(t_1,\cdots,t_{r-1},t_r)\in I^r$, where $\psi_r:(I^r,\dot{I}^r)\longrightarrow(S^r,e_0)$ is a map of (2·7). Then φ pinches the sphere S^{r-1} to a single point $e_0\times e_0$ of $K\vee S^r$. Let

$$\phi_r:(K\times S^r,K\vee S^r)\longrightarrow(E^rK,e_0)$$

be a map which maps $(K-e_0)\times(S^r-e_0)$ homeomorphically onto E^rK-e^0. A space EX is called *a suspension* of X rel. $x_0\in X$ if there is a map

$$d:(X\times I,\ X\times\dot{I}\cup x_0\times I)\longrightarrow(EX,x_0)$$

which maps $(X-x_0)\times(I-\dot{I})$ homeomorphically onto $EX-x_0$. The sphere S^{r+1} is a suspension of S^r rel. e_0 by the map d_r of (2·1). The space $E^{r+1}K$ is a suspension of E^rK, if we define a map $d:E^rK\times I\longrightarrow E^{r+1}K$ by setting $d(\phi_r(x,y),t)=\phi_{r+1}(x,d_r(y,t))$, $x\in K$, $y\in S^r$, $t\in I$. Therefore E^rK is an r-fold suspension of K.

Now we define homomorphisms H, H' and H'' by

(5·4) $H=\phi_{r*}\circ Q\circ\varphi_*:\pi_i(K)\longrightarrow\pi_i(K\vee S^r)\longrightarrow\pi_{i+1}(K\times S^r,K\vee S^r)\longrightarrow\pi_{i+1}(E^rK)$,

(5·4)′ $H'=\phi_{r*}\circ Q'\circ\varphi_*:\ \pi_i(K,K_0)\longrightarrow\pi_i(K\vee S^r,K)\longrightarrow\pi_{i+1}(K\times S^r,K\vee S^r)\longrightarrow\pi_{i+1}(E^rK)$,

(5·4)″ $H''=\phi_{r*}\circ Q''\circ\varphi_*:\pi_i(K;E^r,\bar{K}_0)\longrightarrow\pi_i(K\vee S^r;S^r,K)\longrightarrow\pi_{i+1}(K\times S^r,K\vee S^r)\longrightarrow\pi_{i+1}(E^rK)$.

By (5·3) we have a commutative diagram

$$\text{(5·5)}\qquad \begin{tikzcd} \pi_i(K) \arrow[r] \arrow[dr, "H"'] & \pi_i(K,K_0) \arrow[r] \arrow[d, "H'"] & \pi_i(K:E^r,\bar{K}_0) \arrow[dl, "H''"] \\ & \pi_{i+1}(E^rK). & \end{tikzcd}$$

The homomorphism H' is refered as a *relative Hopf homomorphism.*

(5·6) *If n, $r>2$, then the homomorphism $H'':\pi_i(K;E^r,\bar{K}_0)\longrightarrow\pi_{i+1}(E^rK)$ is an isomorphism for $i<$ Min. $(2n+2,n+r,2r)+r-3$ and onto for $i=$ Min. $(2n+2,n+r,2r)+r-3$.*

Proof. Since $(K\times S^r,K\vee S^r)$ is $(n+r-1)$-connected and $K\vee S^r$ is (Min. $(n,r)-1$)-connected, the homomorphism ϕ_{r*} is an isomorphism for $i+1<$ Min. $(n,r)+n+r-1$ and onto for $i+1=$ Min. $(n,r)+n+r-1$ by [3, Th. 2]. Since $(\bar{K}_0,S^{r-1})$,

(E^r, S^{r-1}) and S^{r-1} are Min. $(n, r)-1$, $r-1$ and $(r-2)$-connected respectively, the homomorphism $\varphi_*: \pi_i(K; E^r, \bar{K}_0) \longrightarrow \pi_i(K \vee S^r; S^r, K)$ is an isomorphism for $i <$ Min.$(n, r)+2r-3$ and onto for $i=$Min. $(n, r)+2r-3$ by [9, Cor.(3・5)]. Since Q'' is an isomorphism, (5・6) is proved. q. e. d.

In the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_i(E^r, S^{r-1}) \arrow[r, "i_*"] \arrow[d, "\partial"', "\approx"] & \pi_i(K, \bar{K}_0) \arrow[r] \arrow[dl, "\varphi'_*"'] & \pi_i(K; E^r, \bar{K}_0) \arrow[r] \arrow[dr, "H''"] & \cdots \\
\pi_{i-1}(S^{r-1}) \arrow[r, "E"] & \pi_i(S^r) & \pi_i(K, K_0) \arrow[l, "\varphi'_*"'] \arrow[u, "\approx"'] \arrow[rr, "H'"] & & \pi_{i+1}(E^rK),
\end{tikzcd}$$

the commutativity holds, where $\varphi'=p_2\circ\varphi: K \longrightarrow K \vee S^r \longrightarrow S^r$ and E is a suspension homomorphism. Since E is an isomorphism for $i-1 < 2(r-1)-1$, we have that $\pi_i(K, \bar{K}_0)=$Image i_*+Kernel φ'_* for $i<2r-2$ and that i_* is an isomorphism into and φ'_* is onto for $i<2r-2$. It follows from the exactness of the upper sequence of the above diagram and from (5・6) that

(5・7) $\pi_i(K, K_0) \approx \pi_i(S^r)+\pi_{i+1}(E^rK)$ *for* $i<$Min.$(2n, r)+r-2$ *and the projections to each factors are* φ'_* *and* H'.

The homotopy class of the map μ is a generator of $\pi_r(K, K_0)$ and it is denoted by the same symbol μ. Define a homomorphism

$$P: \pi_{i-r+1}(K_0) \longrightarrow \pi_i(K, K_0)$$

by the formula $P(\alpha)=[\alpha, \mu]$, $\alpha \in \pi_{i-r+1}(K_0)$, where the bracket indicates the generalized Whitehead product [4]. Then

(5・8) *the homomorphisms* P; $\pi_{i-r+1}(K_0) \longrightarrow \pi_i(K, K^0)$ *and* $\mu_*: \pi_i(I^r, \dot{I}^r) \longrightarrow \pi_i(K, K_0)$ *are isomorphisms into for* $i<$Min.$(2n, r)+r-2$ *and we have a direct sum decomposition*

$$\pi_i(K, K_0)=\mu_*\pi_i(I^r, \dot{I}^r)+P\pi_{i-r+1}(K_0) \quad for\ i<\text{Min.}(2n, r)+r-2.$$

Proof. It is sufficient to prove that the compositions $\varphi'_*\circ\mu_*$ and $H'\circ P$ are isomorphisms onto for $i<$Min.$(2n, r)+r-2$. It is easy to see that $\varphi'_*\circ\mu_*$ is equivalent to the suspension homomorphism $E: \pi_{i-1}(S^{r-1}) \longrightarrow \pi_i(S^r)$. Then $\varphi'_*\circ\mu_*$ is an isomorphism for $i<2r-2$. Next consider the homomorphism $H'\circ P=\phi_{r*}\circ Q'\circ\varphi_*\circ P$. For an element $\alpha \in \pi_{i-r+1}(K_0)$, $\varphi_*(P(\alpha))=\varphi_*[\alpha, \mu]=[\varphi_*(\alpha), \varphi_*(\mu)]=[\varphi_*(\alpha), \iota_r]$, where ι_r is the class of ψ_r. Since φ is identical on K_0, $\varphi_*(\alpha)=i'_*(\alpha)$ for the injection $i': K_0 \subset K \vee S^r$. Therefore $\varphi_{r*}(Q'[i'_*(\alpha), \iota_r])=H'(P(\alpha))$. $E^{s+1}K$ is a suspension of E^sK with a map $d: E^sK \times I \longrightarrow E^{s+1}K$ such that $d(\phi_s(x, y), t)=\phi_{s+1}(x, d_s(y, t))$. For a representative $g: (I^i, \dot{I}^i) \longrightarrow (E^sK, e_0)$ of $\beta \in \pi_i(E^sK)$ we associate the class $E\beta \in \pi_{i+1}(E^{s+1}K)$ of a map $Eg: (I^{i+1}, \dot{I}^{i+1}) \longrightarrow (E^{s+1}K, e_0)$ by the formula $Eg(t_1,\cdots, t_{i+1})=d(g(t_1,\cdots,t_i), t_{i+1})$. Then we have a suspension homomorphism $E: \pi_i(E^sK) \longrightarrow \pi_{i+1}(E^{s+1}K)$. Since E^sK is (Min.$(n, r)+s-1$)-connected, E is an isomorphism for $i<2$Min.$(n, r)+2s-2$.

Let $f: (I^{i-r+1}, \dot{I}^{i-r+1}) \longrightarrow (K_0, e_0)$ be a representative of α. Define a map $f\times\psi_s: (I^{i-r+s+1}, \dot{I}^{i-r+s+1}) \longrightarrow (K\times S^s, K \vee S^s)$ by the formula $(f\times\psi_s)(t_1,\cdots, t_{i-r+s+1})=(f(t_1,\cdots, t_{i-r+1}), \psi_s(t_{i-r+2},\cdots, t_{i-r+s+1}))$ and let $\alpha\times\iota_s \in \pi_{i-r+s+1}(K\times S^s, K \vee S^s)$ be the class of $f\times\psi_s$. From the definition of the mappings, we have the formulas $E(\phi_s\circ(f\times\psi_s))=\phi_{s+1}\circ(f\times\psi_{s+1})$ and $E(\phi_{s*}(\alpha\times\iota_s))=\phi_{s+1*}(\alpha\times\iota_{s+1})$. Therefore $\phi_{r*}(\alpha\times\iota_r)$ is an r-fold suspention $E^r(i_*(\alpha))$ for the injection homomorphism $i_*: \pi_{i-r+1}(K_0)\longrightarrow\pi_{i-r+1}(K)$. For the boundary homomorphism $\partial: \pi_{i+1}(K\times S^r, K\vee S^r)\longrightarrow\pi_i(K\vee S^r)$, we have that $\partial(\alpha\times\iota_r)=[i'_*(\alpha), \iota_r]$. Then $\alpha\times\iota_r=Q'(\partial(\alpha\times\iota_r))=Q'[i'_*(\alpha), \iota_r]$. Consequently we have the formula $H'\circ P=E^r\circ i_*$. Since the pair (K, K_0) is $(r-1)$-connected, the homomorphism $i_*: \pi_{i-r+1}(K_0)\longrightarrow\pi_{i-r+1}(K)$ is an isomorphism for $i-r+1<r-1$. The r-fold suspension $E^r: \pi_{i-r+1}(K)\longrightarrow\pi_{i+1}(E^rK)$ is an isomorphism for $i-r+1<2\mathrm{Min.}(n, r)-1$. Then $H'\circ P$ is an isomorphism for $i<\mathrm{Min.}(2n, r)+r-2$. q. e. d.

Analogous discussions are allowable for the case $K-K_0=\bigcup_{j=1}^{k}e_j^r$. We replace E^r by the union $\bigcup_{j=1}^{k}E^r{}_{(j)}$ of k cubes $E^r{}_{(j)}$ having a single point e_0 in common, $K\vee S^r$ by $\varphi(K)=K\vee(\bigcup_{j=1}^{k}S^r{}_{(j)})$ and E^rK by $\phi_r(K\times(\bigcup_{j=1}^{k}S^r{}_{(j)}))=\bigcup_{j=1}^{k}E^r{}_{(j)}K$, where $S^r{}_{(j)}$ and $E^r{}_{(j)}K$ are copies of S^r and E^rK. Then as an analogy of (5・7), we have an isomorphism

$$\begin{aligned}\pi_i(K, K_0)&\approx\pi_i(\bigcup_{j=1}^{k}S^r{}_{(j)})+\pi_{i+1}(\bigcup_{j=1}^{k}E^r{}_{(j)}K)\\&\approx\sum_{j=1}^{k}(\pi_i(S^r{}_{(j)})+\pi_{i+1}(E^r{}_{(j)}K))\end{aligned}$$

for $i<\mathrm{Min.}(2n, r)+r-2$. Let $\mu_j: (I^r, \dot{I}^r)\longrightarrow(K, K_0)$ be characteristic maps of e_j^r and denote by $\mu_j\in\pi_r(K, K_0)$ the class of μ_j. Define homomorphisms

$$P_j: \pi_{i-r+1}(K_0)\longrightarrow\pi_i(K, K_0)$$

by the formula $P_j(\alpha)=[\alpha, \mu_j]$. Then we have that

PROPOSITION (5・8)$'$ *If $i<\mathrm{Min.}(2n, r)+r-1$ and $n, r<2$, then μ_{j*} and P_j are isomorphisms into and we have that*

$$\pi_i(K, K_0)=\sum_{j=1}^{k}(\mu_{j*}\pi_i(I^r, \dot{I}^r)+P_j\pi_{i-r+1}(K_0)).$$

Next consider the case $n=2$. Then

(5・8)$''$ *the formula of* (5・8)$'$ *is also true for the ease $n=2$.*

Proof. If $n=2$, $\mathrm{Min.}(2n, r)+r-2=\mathrm{Min.}(r+2, 2r-2)$. Obviously (5・8)$'$ is true for $i\leqq r$. Then it is sufficient to prove that (5・8)$'$ is true for $i=r+1>4$. The composition $\varphi'_*\circ\mu_{j*}: \pi_{r+1}(I^r, \dot{I}^r)\longrightarrow\pi_{r+1}(K, K_0)\longrightarrow\pi_{r+1}(\bigcup_{j=1}^{k}S^r{}_{(j)})=\sum_{j=1}^{k}\pi_{r+1}(S^r{}_{(j)})$ is an isomorphism into since the suspension homomorphism $E: \pi_r(S^{r-1})\longrightarrow\pi_{r+1}(S^r)$ is an isomorphism for $r>3$. Then μ_{j*} is an isomorphism into and its image is a direct factor of $\pi_{r+1}(K, K_0)$. Similarly to the proof of (5・8), we have a commutative diagram

$$\begin{array}{ccccc}\pi_2(K_0) & \xrightarrow{P_j} & \pi_{r+1}(K, K_0) & \searrow^{H'} & \\ \downarrow i_* & & & & \pi_{r+2}(\bigcup_{j=1}^{k}E^r{}_{(j)}K)\\ \pi_2(K) & \xrightarrow{E^r} & \pi_{r+2}(E^rK) & \nearrow_{\rho_{j*}} & \end{array}$$

where ρ_{j*} is an injection homomorphism onto a direct factor $\pi_{r+2}(E^r{}_{(j)}K)$ of $\pi_{r+2}(\bigcup_{j=1}^{k} E^r{}_{(j)}K)$. Since i_* and E are isomorphisms for $r>3$, P_j is an isomorphism into and its image is a direct factor of $\pi_{r+1}(K, K_0)$. Obviously the images of μ_{j*} and P_j are disjoint and we have a direct sum decomposition

$$\pi_{r+1}(K, K_0)=\sum_{j=1}^{k}(\mu_{j*}\pi_{r+1}(I^r, \dot{I}^r)+P_j\pi_2(K_0))+A$$

for a direct foctor A of $\pi_{r+1}(K, K_0)$. Now consider a path-space $\Omega(K, K_0)$. By (1·1), $\pi_{r+1}(K, K_0)\approx\pi_r(\Omega(K, K_0))$. Similar calculation to (3·4) shows that

$$H_i(\Omega(K, K_0))\approx H_{i+1}(\Omega(K, K), \Omega(K, K_0))\approx H_r(K, K_0)\otimes H_{i-r+1}(\Omega(K)), \qquad i>0.$$

Then there are a CW-complex $L=\bigcup_{j=1}^{k} S^{r-1}_{(j)}+\bigcup_{\alpha} e^{n_\alpha}_{\alpha}(n_\alpha\geqq r)$ and a map $f: L\longrightarrow \Omega(K, K_0)$ such that f induces isomorphisms of the homology and homotopy groups. Set $L_0=\bigcup_{j=1}^{k} S^{r-1}_{(j)}$, then $\pi_{r-1}(L_0)\approx\pi_{r-1}(L_0)\approx\pi_{r-1}(L)$ and $\pi_r(L_0)$ is a direct factor of $\pi_r(L)$ which corresponds to the factor $\sum_{j=1}^{k}\mu_{j*}\pi_{r+1}(I^r, \dot{I}^r)$ of $\pi_{r+1}(K, K_0)\approx\pi_r(\Omega(K, K_0))$. From the exactness of the sequence: $\pi_r(L_0)\longrightarrow\pi_r(L)\longrightarrow\pi_r(L, L_0)\longrightarrow\pi_{r-1}(L_0)\longrightarrow\pi_{r-1}(L)$, we have that

$$\pi_r(L)/\pi_r(L_0)\approx\pi_r(L, L_0)\approx H_r(L, L_0)\approx H_r(L)\approx H_r(\Omega(K, K_0))\approx H_r(K, K_0)\otimes H_1(\Omega(K))\approx H_r(K, K_0)\otimes\pi_1(\Omega(K))\approx H_r(K, K_0)\otimes\pi_2(K)\approx H_r(K, K_0)\otimes\pi_2(K_0).$$

Therefore $\sum_{j=1}^{k}P_j\pi_2(K_0)+A$ is isomophic to $H_r(K, K_0)\otimes\pi_2(K_0)$. Since K_0 is a simply connected finite cell complex, $\pi_2(K)$ has a finite number of generators. Then the factor A has to be trivial, i. e.,

$$\pi_{r+1}(K, K_0)=\sum_{j=1}^{k}(\mu_{j*}\pi_{r+1}(I^r, \dot{I}^r)+P_j\pi_2(K_0)).$$

Consequently (5·8)″ is established. q. e. d.

Denote by $(S^n)^k$ the topological product $S^n\times\cdots\times S^n$ of k n-spheres, $k>2$. Define a permutation σ_j: $(S^n)^k\longrightarrow(S^n)^k$, $1\leqq j\leqq k$, by the formula $\sigma_j(x_1,\cdots,x_k)=(x_2,\cdots,x_j, x_1, x_{j+1},\cdots,\cdots x_k)$. Set

$$\begin{aligned}
&e^{kn}=(S^n-e_0)^k,\ e_1^{(k-1)n}=e_0\times(S^n-e_0)^{k-1},\\
&S^n_{(1)}=S^n\times(e_0)^{k-1},\ e_0=(e_0)^k,\\
&e_j^{(k-1)n}=\sigma_j(e_1^{(k-1)n}),\ S^n_{(j)}=\sigma_j(S^n_{(1)}),\\
&K=(S^n)^k-e^{kn} \quad\text{and}\quad K_0=K-\bigcup_{j=1}^{k}e_j^{(k-1)n},
\end{aligned}$$

then K, K_0 and $\cup S^n_{(j)}$ are $(k-1)n$, $(k-2)n$ and n skeletons of $(S^n)^k=(e_0\cup e^n)^k$ respectively. Define a map

$$\psi_n^{(k)}: (I^{kn}, \dot{I}^{kn})\longrightarrow((S^n)^k, K)$$

by setting $\psi_n^{(k)}(t_1,\cdots,t_{kn})=(\psi_n(t_1,\cdots,t_n),\cdots,\psi_n(t_{(k-1)n+1},\cdots,t_{kn}))$. This map $\psi_n^{(k)}$ is a homeomorphism on e^{kn} and then it represents an generator ι of $\pi_{kn}((S^n)^k, K)$. The group $\pi_n(K_0)$ is a free module generated by the classes ι_j of the maps $\sigma_j\circ\psi_n$: $I^n\longrightarrow S^n=S^n_{(1)}\longrightarrow S^n_{(j)}\subset K_0$. We may take characteristic maps μ_j: $(I^{(k-1)n}, \dot{I}^{(k-1)n}, J^{(k-1)n-1})\longrightarrow(K, K_0, e_0)$ such that $\mu_j=\sigma_j\circ\mu_1$ and that μ_1 is homotopic to $\psi_n^{(k-1)}$. Denote by $\mu_j\in\pi_{(k-1)n}(K, K_0)$ the class of μ_j.

PROPOSITION (5·9) *If $n \geqq 2$ and $k>2$, then we have a formula*

$$\partial \iota = \sum_{j=1}^{k} (-1)^{(j-1)n} [\iota_j, \mu_j]$$

for the boundary homomorphism $\partial : \pi_{kn}((S^n)^k, K) \longrightarrow \pi_{kn-1}(K, K_0)$.

Proof. By (5·8)′ and (5·8)″, we have a direct sum decomposition $\pi_{kn-1}(K, K_0) = \sum_{j=1}^{k} (\mu_{j*}\pi_{kn-1}(I^{(k-1)n}, \dot{I}^{(k-1)n}) + P_j \pi_n(K_0))$. Then $\partial \iota$ has a from $\partial \iota = \sum_{j=1}^{k} \mu_{i*}(\alpha_j) + \sum_{i,j=1}^{k} c_{i,j} [\iota_i, \mu_j]$ for some elements $\alpha_j \in \pi_{kn-1}(I^{(k-1)n}, \dot{I}^{(k-1)n})$ and some integers $c_{i,j}$.

Let projections $p_m : (S^n)^k \longrightarrow (S^n)^{k-1}$, $1 \leqq m \leqq k$, be defined by setting $p_1(x_1, \cdots, x_k) = (x_2, \cdots, x_k)$ and $p_m = p_1 \circ \sigma_m^{-1}$. p_m maps $e_m^{(k-1)n}$ homeomorphically onto a cell $e^{(k-1)n} = (S^n - e_0)^{k-1}$ of $(S^n)^{k-1}$ and maps $K - e_m^{(k-1)n}$ onto a subcomplex $L = (S^n)^{k-1} - e^{(k-1)n}$ of $(S^n)^{k-1}$. Then the composition $p_1 \circ \mu_1$ is a characteristic map of $e^{(k-1)n}$ and $p_1 \circ \mu_1 = p_1 \circ \sigma_m^{-1} \circ \sigma_m \circ \mu_1 = p_m \circ \mu_m$. As is easily seen, the elements $p_{m*}(\iota_i)$ for $i \neq m$ form a system of the generators of $\pi_n(L)$ and $p_{m*}(\iota_m) = 0$. From (5·8), $\pi_{kn-1}((S^n)^{k-1}, L) = (p_m \circ \mu_m)_* \pi_{kn-1}(I^{(k-1)n}, \dot{I}^{(k-1)n}) + P\pi_n(L)$ and $p_{m*}[\iota_i, \mu_m]$, $i \neq m$, are linearly independent generators of $P\pi_n(L)$. If $j \neq m$, then $(p_m \circ \mu_j)(I^{(k-1)n}) \subset L$ and thus $(p_m \circ \mu_j)_*(\alpha_j) = 0$ and $p_{m*}(\mu_j) = 0$. From the commutativity of the diagram

$$\begin{array}{ccc}
\pi_{kn}((S^n)^k, K) & \xrightarrow{\partial} & \pi_{kn-1}(K, K_0) \\
\downarrow p_{m*} & & \downarrow p_{m*} \\
0 = \pi_{kn}((S^n)^{k-1}, (S^n)^{k-1}) & \xrightarrow{\partial} & \pi_{kn-1}((S^n)^{k-1}, L),
\end{array}$$

we have that $0 = \partial(p_{m*}(\iota)) = p_{m*}(\partial \iota) = \sum_{j=1}^{k} (p_m \circ \mu_j)_*(\alpha_j) + \sum_{i,j=1}^{k} c_{i,j} p_{m*}[\iota_i, \mu_j] = (p_m \circ \mu_m)_*(\alpha_m) + \sum_{i \neq m} c_{i,m} p_{m*}[\iota_i, \mu_m]$. Then it follows from the above decomposition of $\pi_{kn-1}((S^n)^{k-1}, L)$ that $\alpha_m = 0$ and $c_{i,m} = 0$ for $i \neq m$. Therefore we have that

$$\partial \iota = \sum_{j=1}^{k} c_{j,j} [\iota_j, \mu_j].$$

Next we determine the coefficient $c_{j,j}$. Let a map $\xi' : ((S^n)^{k-1}, L) \longrightarrow (S^{(k-1)n}, e_0)$ be defined such that $\xi' \circ \psi_n^{(k-1)} = \psi_{(k-1)n}$, then ξ' maps $e^{(k-1)n}$ homeomorphically onto $S^{(k-1)n} - e_0$. Define a map $\xi_i : (S^n)^k \longrightarrow S^n \times S^{(k-1)n}$ by setting $\xi_1(x_1, \cdots, x_k) = (x_1, \xi'(x_2, \cdots, x_k))$ and $\xi_j = \xi_1 \circ \sigma_j^{-1}$. Then $\xi_j(K) \subset S^n \vee S^{(k-1)n}$, $\xi_j(K_0) \subset S^n$ and $\xi_j \circ \mu_j = \xi_1 \circ \sigma_j^{-1} \circ \sigma_j \circ \mu_1 = \xi_1 \circ \mu_1$ is a characteristic map of the cell $S^n \vee S^{(k-1)n} - S^n$. Consider the following diagram

$$\begin{array}{ccc}
\pi_{kn}((S^n)^k, K) & \xrightarrow{\partial} & \pi_{kn-1}(K, K_0) \\
\downarrow \xi_{j*} & & \downarrow \xi_{j*} \\
\pi_{kn}(S^n \times S^{(k-1)n}, S^n \vee S^{(k-1)n}) & \xrightarrow{\partial} & \pi_{kn-1}(S^n \vee S^{(k-1)n}, S^n).
\end{array}$$

Obviously $\mu_{1*}(\iota)$ is a generator of $\pi_{kn}(S^n \times S^{(k-1)n}, S^n \vee S^{(k-1)n})$ and its boundary is, as the definition, the Whitehead product of the classes of ψ_n and $\psi_n^{(k-1)}$. Therefore $\partial(\xi_{1*}(\iota)) = [\xi_{1*}(\iota_1), \xi_{1*}(\mu_1)] = \xi_{1*}[\iota_1, \mu_1]$. Since σ_i is a homeomorphism of degree $(-1)^{(i-1)n}$, $\xi_{i*}(\partial \iota) = \partial(\xi_{i*}(\iota)) = \partial(\xi_{1*}(\sigma_{i*}^{-1}(\iota))) = \partial(\xi_{1*}((-1)^{(i-1)n}\iota)) = (-1)^{(i-1)n} \xi_{1*}[\iota_1, \mu_1]$. Since $\xi_i \circ \mu_i = \xi_1 \circ \mu_1$, $\xi_i \circ (\sigma_i \circ i_1) = \xi_1 \circ i_1$ and $(\xi_i \circ \sigma_j \circ i_1)(S^n) = e_0$ for

$j \neq i$, we have that $\xi_{i*}(\mu_i)=\xi_{1*}(\mu_1)$, $\xi_{i*}(\iota_i)=\xi_{1*}(\iota_1)$ and $\xi_{i*}(\iota_j)=0$ for $j \neq i$. Then $(-1)^{(i-1)n}\xi_{1*}[\iota_1, \mu_1]=\xi_{i*}(\partial\iota)=\sum_{j=1}^{k}c_{j,j}[\xi_{i*}(\iota_j), \xi_{i*}(\mu_j)]=c_{i,i}[\xi_{1*}(\iota_1), \xi_{1*}(\mu_1)]=c_{i,i}\xi_{1*}[\iota_1, \mu_1]$. Since $\xi_{1*}[\iota_1, \mu_1]$ is a generator of the infinite cyclic group $P\pi_n(S^n) \subset \pi_{kn-1}(S^n \vee S^{(k-1)n}, S^n)$ (by (5·8)), the equality $(-1)^{(i-1)n}\xi_{1*}[\iota_1, \mu_1]=c_{i,i}\xi_{1*}[\iota_1, \mu_1]$ implies that $c_{i,i}=(-1)^{(i-1)n}$. Consequently we have the formula (5·9). q. e. d.

Now consider the reduced product S^n_k, and define a map

$$\eta_k: (S^n)^k \longrightarrow S^n_k$$

by the formula $\eta_k(x_1,\cdots,x_k)=x_1\cdots x_k$. Obviously $\eta_k(K) \subset S^n_{k-1}$, $\eta_k(K_0) \subset S^n_{k-2}$ and η_k maps $e^{kn}=(S^n)^k-K$ homeomorphically onto $e^{kn}=S^n_k-S^n_{k-1}$. Also each $e_i^{(k-1)n}$, $1 \leqq i \leqq k$, is mapped by η_k homeomorphically onto $e^{(k-1)n}=S^n_{k-1}-S^n_{k-1}$. The composition $\eta_k\circ\psi^{(k)}_n: (I^{kn}, \dot{I}^{kn}) \longrightarrow ((S^n)^k, K) \longrightarrow (S^n_k, S^n_{k-1})$ maps $I^{kn}-\dot{I}^{kn}$ homeomorphically onto $e^{kn}=S^n_k-S^n_{k-1}$ and the composition $h'_k\circ\eta_k\circ\psi^{(k)}_n: (I^{kn}, \dot{I}^{kn}) \longrightarrow (S^{kn}, e_0)$ coincides with the map ψ_{kn}. Conversely a characteristic map $\mu': (I^{kn}, \dot{I}^{kn}) \longrightarrow (S^n_k, S^n_{k-1})$ is homotopic to $\eta_k\circ\psi^{(k)}_n$ if and only if the composition $h'_k\circ\mu'$ has the same degree as that of ψ_{kn}. We take also the composition $\eta_k\circ\mu_1: I^{(k-1)n} \longrightarrow K \longrightarrow S^n_{k-1}$ as a characteristic map of $e^{(k-1)n}=S^n_{k-1}-S^n_{k-2}$ and it is denoted by μ. Then $\mu=\eta_k\circ\mu_i$ for $1 \leqq i \leqq k$ and the composition $h'_{k-1}\circ\mu: (I^{(k-1)n}, \dot{I}^{(k-1)n}) \longrightarrow (S^{(k-1)n}, e_0)$ is homotopic to $\psi_{(k-1)n}$. By applying the homomorphisms induced by the map η_k, it is deduced from (5·9) that

(5·9)′ *if $n \geqq 2$ and $k>2$, we have a formula*

$$\partial\iota=\sum_{j=1}^{k}(-1)^{(j-1)n}[\iota_1, \mu]$$

for the boudary homomorphism $\partial: \pi_{kn}(S^n_k, S^n_{k-1}) \longrightarrow \pi_{kn-1}(S^n_{k-1}, S^n_{k-2})$ and for the classes $\iota \in \pi_{kn}(S^n_k, S^n_{k-1})$, $\mu \in \pi_{(k-1)n}(S^n_{k-1}, S^n_{k-2})$ and $\iota_1 \in \pi_n(S^n_{k-1})$ of $\psi^{(k)}_n$, μ and ψ_n respectively.

In particular

(5·9)″ $\qquad \partial\iota=k[\iota_1, \mu] \qquad$ *if n is even.*

Let a map

$$\varphi_n=\varphi: S^n_{k-1} \longrightarrow S^n_{k-1} \vee S^{(k-1)n}$$

be defined as in the begining of this § by setting $K=S^n_{k-1}$ and $K_0=S^n_{k-2}$ and by using the above characteristic map μ.

(5·10). *Let n be even. Let ι_n and $\iota_{(k-1)n}$ be the classes of the maps ψ_n and $\psi_{(k-1)n}$ respectively. Then we have a formula*

$$\varphi_{n*}(\partial\iota)=i_*(\partial\iota)+k[\iota_n, \iota_{(k-1)n}]$$

for the boundary homomorphism $\partial: \pi_{kn}(S^n_k, S^n_{k-1}) \longrightarrow \pi_{kn-1}(S^n_{k-1})$, the induced homomorphism $\varphi_{n*}: \pi_{kn-1}(S^n_{k-1}) \longrightarrow \pi_{kn-1}(S^n_{k-1} \vee S^{(k-1)n})$ and the injection homomorphism $i_*: \pi_{kn-1}(S^n_{k-1}) \longrightarrow \pi_{kn-1}(S^n_{k-1} \vee S^{(k-1)n})$.

Proof. In the diagram

$$
\begin{tikzcd}
 & & \pi_{kn}(S^n_k,\ S^n_{k-1}) \arrow[dl, "\partial"'] \arrow[dr, "\partial"] & \\
 & \pi_{kn-1}(S^n_{k-1}) \arrow[rr] \arrow[d, "\varphi_{n*}"] & & \pi_{kn-1}(S^n_{k-1},\ S^n_{k-2}) \arrow[d, "\varphi_{n*}"] \\
\pi_{kn-1}(S^n_{k-1}) \arrow[r, "i_*"] & \pi_{kn-1}(S^n_{k-1} \vee S^{(k-1)n}) \arrow[rr, "j_*"] & & \pi_{kn-1}(S^n_{k-1} \vee S^{(k-1)n},\ S^n_{k-1})
\end{tikzcd}
$$

the commutativity holds. As is easily seen $\varphi_{n*}(\mu)=j_*(\iota_{(k-1)n}) \in \pi_{(k-1)n}(S^n_{k-1} \vee S^{(k-1)n},$ $S^n_{k-1})$ and $\varphi_{n*}(\iota_1)=\iota_n$. Then $j_*(\varphi_{n*}(\partial\iota))=j_*(k[\iota_n,\ \iota_{(k-1)n}])$ by (5・9)″. From the exactness of the lower sequence of the diagram, there is an element α of π_{kn-1} (S^n_{k-1}) such that $i_*\alpha=\varphi_{n*}(\partial\iota)-k[\iota_n,\ \iota_{(k-1)n}]$. Let $p_1: S^n_{k-1} \vee S^{(k-1)n} \longrightarrow S^n_{k-1}$ be the projection, then it is easy to see that the composition $p_1 \circ \varphi_n$ is homotopic to the identity of S^n_{k-1}. Also the composition $p_1 \circ i: S^n_{k-1} \longrightarrow S^n_{k-1} \vee S^{(k-1)n} \longrightarrow S^n_{k-1}$ is the identity. Obviously $p_{1*}(\iota_{(k-1)n})=0$ for the homomorphism p_{1*} induced by the map p_1. Then $\alpha=p_{1*}(i_*(\alpha))=p_{1*}(\varphi_{n*}(\partial\iota)-k[\iota_n,\ \iota_{(k-1)n}])=p_{1*}(\varphi_{n*}(\partial\iota))=\partial\iota$. Therefore $\varphi_{n*}(\partial\iota)-k[\iota_n,\ \iota_{(k-1)n}]=i_*(\partial\iota)$. q. e. d.

6. A map $\bar{h}: \Omega(S^n_{k-1},\ S^n_{k-2}) \longrightarrow \Omega^2(S^{kn})$

Let $\Omega^2(X,A)$ be defined by $\Omega^2(X,A)=\Omega(\Omega(X),\Omega(A))$ in the notation of § 1. $\Omega^2(X,A)$ is a space of singular 2-cubes:

$$\Omega^2(X,A)=\{f: I^2 \longrightarrow X \mid f(\dot{I}^2) \subset A, f(J^1)=x_0 \in X\}$$

with the compact open topology. For a map $g: (I^{i+2}, \dot{I}^{i+2}, J^{i+1}) \longrightarrow (X, A, x_0)$, define a map $\Omega^2 g: (I^i, \dot{I}^i) \longrightarrow (\Omega^2(X,A), f_0)$ by setting $\Omega^2 g(t_1, \cdots, t_i)\ (u_1, u_2)=g(t_1,$ $\cdots, t_i, u_1, u_2)$, $(t_1, \cdots t_i) \in I^i$, $(u_1, u_2) \in I^2$. Then the correspondence $g \longrightarrow \Omega^2 g$ induces an isomorphism

(6・1) $$\Omega^2: \pi_{i+2}(X,A) \approx \pi_i(\Omega^2(X,A)), \qquad i>0\text{˙}$$

By an analogy of [16, § 4], we shall define a map

(6・2) $$\bar{Q}: \Omega(A \vee B, A) \longrightarrow \Omega^2(A \times B, A \vee B)$$

such that the diagram

(6・2)′
$$
\begin{tikzcd}
\pi_i(A \vee B,\ A) \arrow[r, "Q'"] \arrow[d, "\Omega"] & \pi_{i+1}(A \times B,\ A \vee B) \arrow[d, "\Omega^2"] \\
\pi_{i-1}(\Omega(A \vee B,\ A)) \arrow[r, "\bar{Q}_*"] & \pi_{i-1}(\Omega^2(A \times B,\ A \vee B))
\end{tikzcd}
$$

is commutative.

Define maps η', $\eta'': I^2 \longrightarrow I$ by the formulas

$$
\eta'(t_1,t_2)=\begin{cases}
1-3t_1, & 0 \leqq 3t_1 \leqq 1-t_2, \\
t_2, & 1-t_2 \leqq 3t_1 \leqq 1+t_2, \\
3t_1-1, & 1+t_2 \leqq 3t_1 \leqq 2, \\
1, & 2 \leqq 3t_1 \leqq 3,
\end{cases}
$$

and $\eta''(t_1, t_2)=1-\eta'(1-t_1, t_2)$, $(t_1, t_2) \in I^2$. For a path f of $\Omega(A \vee B, A)$, we associate a singular 2-cell $\bar{Q}f \in \Omega^2(A \times B, A \vee B)$ which is defined by the formula

$$\overline{Q}f(t_1, t_2) = (p_1(f(\eta'(t_1, t_2))),\ p_2(f(\eta''(t_1, t_2)))),$$

where $p_1 : A \vee B \longrightarrow A$ and $p_2 : A \vee B \longrightarrow B$ and the projections in § 5. The continuity of the maps η', η'', f, p_1 and p_2 implies that of the map $\overline{Q}$. Then

(6・2)″ *the diagram* (6・2)′ *is commutative.*

Proof. Let $i : \Omega(A \vee B) \longrightarrow \Omega(A \vee B, A)$ be the injection, then the diagram

$$\begin{array}{ccc} \pi_i(A \vee B) & \xrightarrow{j_*} & \pi_i(A \vee B, A) \\ \downarrow \Omega & & \downarrow \Omega \\ \pi_{i-1}(\Omega(A \vee B)) & \xrightarrow{i_*} & \pi_{i-1}(\Omega(A \vee B, A)) \end{array}$$

is commutative. First we prove the commutativity of the diagram

$$(6\cdot 2)''' \qquad \begin{array}{ccc} \pi_i(A \vee B) & \xrightarrow{Q} & \pi_{i+1}(A \times B, A \vee B) \\ \downarrow \Omega & & \downarrow \Omega^2 \\ \pi_{i-1}(\Omega(A \vee B)) & \xrightarrow{\overline{Q}_*} & \pi_{i-1}(\Omega^2(A \times B, A \vee B)). \end{array}$$

Let $g : (I^i, \dot{I}^i) \longrightarrow (A \vee B, a_0 \times b_0)$ be a map which belongs an element β of $\pi_i(A \vee B)$. It is calculated directly that $(\partial \circ (\Omega^2)^{-1} \circ \overline{Q}_* \circ \Omega)(\beta)$ is represented by a map $G : (I^i, \dot{I}^i) \longrightarrow (A \vee B, a_0 \times b_0)$ which is given by the formula

$$G(t_1, \cdots, t_{i-1}, t_i) = \begin{cases} (p_1(g(t_1, \cdots, t_{i-1}, 1-3t_i)),\ b_0), & 0 \leqq t_i \leqq 1/3, \\ g(t_1, \cdots, t_{i-1}, 3t_i - 1), & 1/3 \leqq t_i \leqq 2/3, \\ (a_0,\ p_2(g(t_1, \cdots, t_{i-1}, 3-3t_i))), & 2/3 \leqq t_i \leqq 1. \end{cases}$$

Then G represents $-i_{1_*}(p_{1_*}(\beta)) + \beta - i_{2_*}(p_{2_*}(\beta)) = \partial(Q(\beta))$ by (5・2). Since ∂ is an isomorphism into, $\partial \circ (\Omega^2)^{-1} \circ \overline{Q}_* \circ \Omega = \partial \circ Q$ implies that $(\Omega^2)^{-1} \circ \overline{Q}_* \circ \Omega = Q$. Therefore (6・2)‴ is commutative. Since j_* is onto, (6・2)″ follows from (5・1)′ and (5・3).

q. e. d.

From the definition of $\overline{Q}$,

(6・3) $\overline{Q}$ *maps the subset* $\Omega(A, A)$ *of* $\Omega(A \vee B, A)$ *into the subset* $\Omega^2(A, A)$ *of* $\Omega^2(A \times B, A \vee B)$.

For given two maps $f : (A, a_0) \longrightarrow (A', a_0')$ and $g : (B, b_0) \longrightarrow (B', b_0')$, we define maps $f \times g : A \times B \longrightarrow A' \times B'$ and $f \vee g : A \vee B \longrightarrow A' \vee B'$ by $(f \times g)(a, b) = (f(a), g(b))$ and $f \vee g = f \times g | A \vee B$. Then the diagram

$$(6\cdot 4) \qquad \begin{array}{ccc} \Omega(A \vee B, A) & \xrightarrow{\overline{Q}} & \Omega^2(A \times B, A \vee B) \\ \downarrow \Omega(f \vee g) & & \downarrow \Omega(f \times g) \\ \Omega(A' \vee B', A') & \xrightarrow{\overline{Q}} & \Omega^2(A' \times B', A' \vee B') \end{array}$$

is commutative.

Let a map

$$\varphi_n : S^n_{k-1} \longrightarrow S^n_{k-1} \vee S^{(k-1)n}$$

be defined as in § 5 by setting $K = S^n_{k-1}$ and $K_0 = S^n_{k-2}$. Remark that $p_2 \circ \varphi_n$ is homotopic to h'_{k-1} of $(2\cdot 7)'$. Define a map

$$\tilde{\phi}_1 : S^n_{k-1}\times S^1 \longrightarrow S^{n+1}$$

by the formula $\tilde{\phi}_1(x, d_0((-1), t))=x(t)$, $x\in S^n_{k-1}\subset \Omega(S^{n+1})$, $t\in I$. Inductively, by setting $\tilde{\phi}_r(x, d_{r-1}(y, t))=d_{n+r-1}(\tilde{\phi}_{r-1}(x, y), t)$, $x\in S^n_{k-1}$, $y\in S^{r-1}$, we obtain a map

$$\tilde{\phi}_r : (S^n_{k-1}\times S^r,\ S^n_{k-1}\vee S^r) \longrightarrow (S^{n+r}, e_0), \qquad r\geqq 1.$$

Remark that the restriction $\tilde{\phi}_r|S^n\times S^r$ is the map $\phi_{n,r}$ of $(2\cdot 8)''$.

Now we define a map

$$(6\cdot 5) \qquad \bar{h}: \Omega(S^n_{k-1}, S^n_{k-2}) \longrightarrow \Omega^2(S^{kn})$$

by setting $\bar{h}=\Omega^2\tilde{\phi}_{(k-1)n}\circ\bar{Q}\circ\Omega\varphi_n : \Omega(S^n_{k-1}, S^n_{k-2}) \longrightarrow \Omega(S^n_{k-1}\vee S^{(k-1)n}, S^n_{k-1}) \longrightarrow \Omega^2(S^n_{k-1}\times S^{(k-1)n}, S^n_{k-1}\vee S^{(k-1)n}) \longrightarrow \Omega^2(S^{kn})$. The restriction of $\bar{h}$ on $\Omega(S^n_{k-1})$ is also denoted by the same symbol

$$(6\cdot 5)' \qquad \bar{h} : \Omega(S^n_{k-1}) \longrightarrow \Omega^2(S^{kn})$$

From $(6\cdot 3)$, we have easily that

$(6\cdot 6)$ *$\bar{h}$ maps $\Omega(S^n_{k-2})$ to a single point e_0 of $\Omega^2(S^{kn})$.*

Consider the map $\phi_r : (S^n_{k-1}\times S^r, S^n_{k-1}\vee S^r) \longrightarrow (E^r S^n_{k-1}, e_0)$ of § 5. Since ϕ_r maps $S^n_{k-1}\times S^r - S^n_{k-1}\vee S^r$ homeomorphically onto $E^r S^n_{k-1}-e_0$, there is a map ζ_r: $E^r S^n_{k-1} \longrightarrow S^{n+r}$ such that $\tilde{\phi}_r=\zeta_r\circ\phi_r$. Then from $(5\cdot 4)'$, $(1\cdot 2)$ and $(6\cdot 2)''$ we have a commutative diagram

$$\begin{array}{ccccc}
\pi_i(S^n_{k-1}, S^n_{k-2}) & \xrightarrow{H'} & \pi_{i+1}(E^{(k-1)n} S^n_{k-1}) & \xrightarrow{\zeta_{(k-1)n*}} & \pi_{i+1}(S^{kn}) \\
\downarrow \Omega & & & & \downarrow \Omega^2 \\
\pi_{i-1}(\Omega(S^n_{k-1}, S^n_{k-2})) & & \xrightarrow{\quad \bar{h}_* \quad} & & \pi_{i-1}(\Omega^2(S^{kn})).
\end{array}$$

LEMMA $(6\cdot 7)$ *The induced homomorphism $\bar{h}^* : H^{kn-2}(\Omega^2(S^{kn})) \longrightarrow H^{kn-2}(\Omega(S^n_{k-1}, S^n_{k-2}))$ is an isomorphism onto $(n\geqq 2, k>2)$.*

Proof. As is easily seen $H^{kn-2}(\Omega^2(S^{kn}))\approx Z$ and $H^i(\Omega^2(S^{kn}))=0$ otherwise for $0<i<2kn-4$. The similar result is true for the homology. By $(3\cdot 4)$, $H^{kn-2}(\Omega(S^k_{k-1}, S^n_{k-2}))\approx H^n(\Omega(S^n_{k-1}))\approx Z$. By $(5\cdot 8)$ and $(5\cdot 8)''$, $\pi_i(\Omega(S^n_{k-1}, S^n_{k-2}))\approx\pi_{i+1}(S^n_{k-1}, S^n_{k-2})\approx\pi_{i+1}(S^{(k-1)n})+\pi_{i+2}(E^{(k-1)n}S^n_{k-1})$ for $i\leqq kn-2\leqq(k+1)n-4$ and H' gives a projection to the second factor. Let $f: S^{kn-2}\longrightarrow\Omega(S^n_{k-1}, S^n_{k-2})$ be a map whose class in $\pi_{kn-2}(\Omega(S^n_{k-1}, S^n_{k-2}))$ corresponds to a generator of $\pi_{kn}(E^{(k-1)n}S^n_{k-1})$. In the above diagram, $\zeta_{(k-1)n_*}$ and Ω^2 are isomorphisms if $i=kn-1$. Thus the composition $\bar{h}\circ f: S^{kn-2}\longrightarrow\Omega(S^n_{k-1}, S^n_{k-2})\longrightarrow\Omega^2(S^{kn})$ represents a generator of $\pi_{kn-2}(\Omega^2(S^{kn}))$. Then $(\bar{h}\circ f)_* : H_i(S^{kn-2})\longrightarrow H_i(\Omega^2(S^{kn}))$ is an isomorphism for $i<2kn-4$. By the duality, $f^*\circ\bar{h}^* : H^{kn-2}(\Omega^2(S^{kn}))\longrightarrow H^{kn-2}(\Omega(S^n_{k-1}, S^n_{k-2}))\longrightarrow H^{kn-2}(S^{kn-2})$ is an isomorphism. Since these three groups are isomorphic to Z, $\bar{h}^*$ and f^* have to be isomorphisms. q. e. d.

Let $\omega_{p,q} : S^{p+q-1} \longrightarrow S^p\vee S^q$ be a map which represents the Whitehead product

$[\iota_p, \iota_q]$ of the classes ι_p and ι_q of the maps ψ_p and ψ_q. Let $\phi_{p,q}: (S^p\times S^q, S^p\vee S^q) \longrightarrow (S^{p+q}, e_0)$ be a map which is given by (2·8)″, then $\phi_{p,q}$ maps $S^p\times S^q - S^p\vee S^q$ homeomorphically onto $S^{p+q}-e_0$. Define a map $\tau_{p,q}: \Omega(S^{p+q-1}) \longrightarrow \Omega^2(S^{p+q})$ by setting

(6·8) $\tau_{p,q}=\Omega^2\phi_{p,q}\circ\bar{Q}\circ\Omega\omega_{p,q}: \Omega(S^{p+q-1}) \longrightarrow \Omega(S^p\vee S^q) \longrightarrow \Omega^2(S^p\times S^q, S^p\vee S^q) \longrightarrow \Omega^2(S^{p+q})$.

Then

(6·8)′ *the induced homomorphism $\tau_{p,q*}: \pi_i(\Omega(S^{p+q-1})) \longrightarrow \pi_i(\Omega^2(S^{p+q}))$ is equivalent to the suspension homomorphism, that is, the following diagram is commutative*

$$\begin{array}{ccc} \pi_i(S^{q+p-1}) & \xrightarrow{E} & \pi_{i+1}(S^{p+q}) \\ \downarrow \Omega & & \downarrow \Omega^2 \\ \pi_{i-1}(\Omega(S^{p+q-1})) & \xrightarrow{\tau_{p,q*}} & \pi_{i-1}(\Omega^2(S^{p+q})). \end{array}$$

Proof. By (1·2) and (6·2)″, it is sufficient to prove that $\phi_{p,q*}\circ Q\circ\omega_{p,q*}=E$. Let E^{p+q} be a closedcell bounded by S^{p+q-1}, then there is an extension $\tilde{\omega}_{p,q}: E^{p+q} \longrightarrow S^p\times S^q$ of $\omega_{p,q}$ such that the composition $\phi_{p,q}\circ\tilde{\omega}_{p,q}: (E^{p+q}, S^{p+q-1}) \longrightarrow (S^{p+q}, e_0)$ is a map of degree 1. In the diagram

$$\begin{array}{ccccc} \pi_i(S^{p+q-1}) & \xleftarrow{\partial_0} & \pi_{i+1}(E^{p+q}, S^{p+q-1}) & & \\ \downarrow \omega_{p,q*} & & \downarrow \tilde{\omega}_{p,q*} & \searrow \phi_{p,q*}\circ\tilde{\omega}_{p,q*} & \\ \pi_i(S^p\vee S_q) & \xleftarrow{\partial} & \pi_{i+1}(S^p\times S^q, S^p\vee S^q) & \xrightarrow{\phi_{p,q*}} & \pi_{i+1}(S^{p+q}) \end{array}$$

the commutativity holds and $\phi_{p,q*}\circ\tilde{\omega}_{p,q*}\circ\partial_0^{-1}=E$. By (5·1) and (5·2), $Q\circ\partial=$identity. Then $\phi_{p,q*}\circ Q\circ\omega_{p,q*}=\phi_{p,q*}\circ Q\circ\partial\circ\tilde{\omega}_{p,q*}\circ\partial_0^{-1}=E$. q. e. d.

REMARK. Consider an injection $i: S^{p+q-1} \longrightarrow \Omega(S^{p+q})$ of (2·2)′. Then Ωi and $\tau_{p,q}$ induces the same homomorphism. It is, however, a problem whether the maps Ωi and $\tau_{p,q}$ are homotopic to each other or not.

Let $\chi: S^{kn-1} \longrightarrow S^n_{k-1}$ be an attaching map of the cell $e^{kn}=S^n_k-S^n_{k-1}$ such that χ represents the element $\partial\iota$ of (5·10). Let $f_k: S^{kn-1} \longrightarrow S^{kn-1}$ be a map of degree k. Let $\tau: \Omega(S^{kn-1}) \longrightarrow \Omega^2(S^{kn})$ be the map $\tau_{n,(k-1)n}=\Omega^2\phi_{n,(k-1)n}\circ\bar{Q}\circ\Omega\omega_{n,(k-1)n}$. Then

LEMMA (6·9) *the compositions $\tau\circ\Omega f_k$ and $\bar{h}\circ\Omega\chi$ are homotopic to each other, that is to say, the diagram*

$$\begin{array}{ccc} \Omega(S^{kn-1}) & \xrightarrow{\Omega\chi} & \Omega(S^n_{k-1}) \\ \downarrow \Omega f_k & & \downarrow \bar{h} \\ \Omega(S^{kn-1}) & \xrightarrow{\tau} & \Omega^2(S^{kn}) \end{array}$$

is homotopically commutative.

Proof. By the definition of τ and $\bar{h}$, it is sufficient to prove the homotopical commutativity of the following diagram:

$$\begin{tikzcd}[column sep=small]
 & & \Omega(S^{kn-1}) \arrow[dll, "\Omega f_k"'] \arrow[drr, "\Omega\chi"] \arrow[dd, "\Omega W"] & & \\
\Omega(S^{kn-1}) \arrow[dd, "\Omega\omega_{n,(k-1)n}"'] & (1) & & (2) & \Omega(S^n_{k-1}) \arrow[dd, "\Omega\varphi_n"] \\
 & & \Omega(S^{kn-1}\vee S^n\vee S^{(k-1)n}) \arrow[dll, "\Omega(p\vee i)"] \arrow[drr, "\Omega(g\vee i)"] \arrow[dd, "\bar{Q}"] & & \\
\Omega(S^n\vee S^{(k-1)n}) \arrow[dd, "\bar{Q}"'] & (3) & & (4) & \Omega(S^n_{k-1}\vee S^{(k-1)n}) \arrow[dd, "\bar{Q}"] \\
 & & \Omega^2((S^{kn-1}\vee S^n)\times S^{(k-1)n}, S^{kn-1}\vee S^n\vee S^{(k-1)n}) \arrow[dll, "\Omega^2(p\times i)"] \arrow[drr, "\Omega^2(g\times i)"] & & \\
\Omega^2(S^n\times S^{(k-1)n}, S^n\vee S^{(k-1)n}) \arrow[drr, "\Omega^2\phi_{n,(k-1)n}"] & & (5) & & \Omega^2(S^n_{k-1}\times S^{(k-1)n}, S^n_{k-1}\vee S^{(k-1)n}) \arrow[dll, "\Omega^2\tilde{\phi}_{(k-1)n}"'] \\
 & & \Omega^2(S^{kn}) & &
\end{tikzcd}$$

wnere $p: S^{kn-1}\vee S^n \longrightarrow S^n$ is the projection, $g: S^{kn-1}\vee S^n \longrightarrow S^n_{k-1}$ is a map defined by $\chi = g|S^{kn-1}: S^{kn-1} \longrightarrow S^n_{k-1}$ and the injection $g|S^n: S^n \subset S^n_{k-1}$, $i: S^n \longrightarrow S^n$ is the identity and $W: S^{kn-1} \longrightarrow S^{kn-1}\vee S^n \vee S^{(k-2)n}$ is a map which represents a sum $\iota_{kn-1}+k[\iota_n, \iota_{(k-1)n}]$ for the classes ι_{kn-1}, ι_n and $\iota_{(k-1)n}$ of the maps ψ_{kn-1}, ψ_n and $\psi_{(k-1)n}$ respectively.

Since the compositions $\omega_{n,(k-1)n}\circ f_k$ and $(p\vee i)\circ W$ represent the same element $k[\iota_n, \iota_{(k-1)n}]$, they are homotopic to each other. Then we have the homotopical commutativity of the square (1). By (5·10), $\varphi_n\circ\chi$ and $(g\vee i)\circ W$ represent the same element $i_*(\partial\iota)+k[\iota_n, \iota_{(k-1)n}]$ and they are homotopic to each other. Then the homotopical commutativity of (2) holds. By (6·4), the sequares (3) and (4) are commutative. Consider a diagram

$$\begin{tikzcd}
((S^{kn-1}\vee S^n)\times I, e_0\times e_0\times I \cup (S^{kn-1}\vee S^n)\times \dot{I}) \arrow[r, "g\times i'"] \arrow[d, "p\times i'"] & (S^n_{k-1}\times I, e_0\times I\cup S^n_{k-1}\times\dot{I}) \arrow[d, "\bar{d}_n"] \\
(S^n\times I, e_0\times I\cup S^n\times \dot{I}) \arrow[r, "d_n"] & (S^{n+1}, e_0)
\end{tikzcd}$$

where i' is the identity of I and d_n and $\bar{d}_n$ are the maps of (2·1) and (2·2). Since $\chi = g|S^{kn-1}$ is an attaching map of $e^{kn}=S^n_k - S^n_{k-1}$ and since $\bar{d}_n$ can be extended over S^n_k, the composition $\bar{d}_n\circ(g|S^{kn-1}\times I): (S^{kn-1}\times I, e_0\times I\cup S^{kn-1}\times\dot{I}) \longrightarrow (S^{n+1}, e_0)$ is nullhomotopic rel. $e_0\times I\cup S^{kn-1}\times\dot{I}$. Since the compsitions $\bar{d}_n\circ(g\times i')$ and $d_n\circ(p\times i')$ coincide on $S^n\times I$ and since $d_n((p\times i')(S^{kn-1}\times I))=e_0$, the above diagram is homotopically commutative. By making use of the map $d_0: (-1)\times I \longrightarrow S^1$, we see that the following diagram is homotopically commutative when $r=1$:

$$\begin{tikzcd}
((S^{kn-1}\vee S^n)\times S^r, S^{kn-1}\vee S^n\vee S^r) \arrow[r, "g\times i_r"] \arrow[d, "p\times i_r"] & (S^n_{k-1}\times S^r, S^n_{k-1}\vee S^r) \arrow[d, "\tilde{\phi}_r"] \\
(S^n\times S^r, S^n\vee S^r) \arrow[r, "\phi_{n,r}"] & (S^{n+r}, e_0)
\end{tikzcd}$$

where i_r is the identity of S^r. Let $F_t^{(1)}: ((S^{kn-1}\vee S^n)\times S^1, S^{kn-1}\vee S^n\vee S^1)\longrightarrow(S^{n+1}, e_0)$ be a homotopy between $F_0^{(1)}=\tilde{\phi}_1\circ(g\times i_1)$ and $F_1^{(1)}=\phi_{n,1}\circ(p\times i_1)$. Define a homotopy $F_t^{(r)}: ((S^{kn-1}\vee S^n)\times S^r, S^{kn-1}\vee S^n\vee S^r)\longrightarrow(S^{n+r}, e_0)$ inductively by setting $F_t^{(r)}(x, d_{r-1}(y, s))=d_{n+r-1}(F_t^{(r-1)}(x,y), s)$, $x\in S^{kn-1}\vee S^n$, $y\in S^{r-1}$, $s\in I$. We calculate easily that $F_0^{(r)}=\tilde{\phi}_r\circ(g\times i_r)$ and $F_1^{(r)}=\phi_{n,r}\circ(p\times i_r)$. Then we have the homotopical commutativity of the above diagram, and therefore the homotopical commutativity of the sequre ⑤. Consequently the homotopical commutativity of the diagram (6·9) is proved. q. e. d.

Let $f: (S^n, e_0)\longrightarrow(S^m, e_0)$ be the suspension of a map $f': (S^{n-1}, e_0)\longrightarrow(S^{m-1}, e_0)$, i. e., $f=Ef'$, $f(d_{n-1}(x, t))=d_{m-1}(f'(x), t)$. Let $\bar{f}: S_{k-1}^n\longrightarrow S_{k-1}^m$ be the combinatorial extension of f, i. e., $\bar{f}(x_1\cdots x_{k-1})=f(x_1)\cdots f(x_{k-1})$, $x_i\in S^n$, $i=1,\cdots,k-1$. Then we obtain a map

$$\Omega\bar{f}: (\Omega(S_{k-1}^n), S^{n-1})\longrightarrow(\Omega(S_{k-1}^m), S^{m-1}) \tag{6·10}$$

such that $\Omega\bar{f}|S^{n-1}=f': S^{n-1}\longrightarrow S^{m-1}\subset\Omega(S_{k-1}^m)$.

LEMMA (6·11) *For the map $\bar{h}$ of (6·5)′, we have the following homotopically commutative diagram*

$$\begin{array}{ccc}
(\Omega(S_{k-1}^n), S^{n-1}) & \xrightarrow{\Omega\bar{f}} & (\Omega(S_{k-1}^m), S^{m-1}) \\
\downarrow{\scriptstyle\bar{h}} & & \downarrow{\scriptstyle\bar{h}} \\
(\Omega^2(S^{kn}), e_0) & \xrightarrow{\Omega^2(f)^k} & (\Omega^2(S^{km}), e_0)
\end{array}$$

where $(f)^k$ is defined in (2·8).

Proof. We shall prove the homotopical commutativity of three squares in the following diagram

$$\begin{array}{ccc}
(\Omega(S_{k-1}^n), S^{n-1}) & \xrightarrow{\Omega\bar{f}} & (\Omega(S_{k-1}^m), S^{m-1}) \\
\downarrow{\scriptstyle\Omega\varphi_n} & (1) & \downarrow{\scriptstyle\Omega\varphi_m} \\
(\Omega(S_{k-1}^n\vee S^{(k-1)n}), \Omega(S_{k-1}^n)) & \xrightarrow{\Omega(\bar{f}\vee(f)^{k-1})} & (\Omega(S_{k-1}^m\vee S^{(k-1)m}), \Omega(S_{k-1}^m)) \\
\downarrow{\scriptstyle\bar{Q}} & (2) & \downarrow{\scriptstyle\bar{Q}} \\
(\Omega^2(S_{k-1}^n\times S^{(k-1)n}, S_{k-1}^n\vee S^{(k-1)n}), \Omega^2(S_{k-1}^n)) & \xrightarrow{\Omega^2(\bar{f}\times(f)^{k-1})} & (\Omega^2(S_{k-1}^m\times S^{(k-1)m}, S_{k-1}^m\vee S^{(k-1)m}), \Omega^2(S_{k-1}^m)) \\
\downarrow{\scriptstyle\Omega^2\tilde{\phi}_{(k-1)n}} & (3) & \downarrow{\scriptstyle\Omega^2\tilde{\phi}_{(k-1)m}} \\
(\Omega^2(S^{kn}), e_0) & \xrightarrow{\Omega^2((f)^k)} & (\Omega^2(S^{km}), e_0),
\end{array}$$

then the lemma follows from the definition (6·5) of the map $\bar{h}$.

Homotopical commutativity of ①, follows from that of the diagram

$$\begin{array}{ccc}
(S^n_{k-1}, S^n_{k-2}) & \xrightarrow{\quad\bar{f}\quad} & (S^m_{k-1}, S^m_{k-2}) \\
\downarrow \varphi_n & (1)' & \downarrow \varphi_m \\
(S^n_{k-1} \vee S^{(k-1)n}, S^n_{k-1}) & \xrightarrow{\bar{f} \vee (f)^{k-1}} & (S^m_{k-1} \vee S^{(k-1)m}, S^m_{k-1})
\end{array}$$

since $S^{r-1} \subset \Omega(S^r_{k-2})$, $\varphi_r(S^r_{k-2}) \subset S^r_{k-1}$, $r=n$ or m, and $\bar{f}(S^n_{k-2}) \subset S^m_{k-2}$. For a map g: $S^{n-1} \longrightarrow S^{m-1}$, we define a map $E_0 g$: $(S^n, e_0) \longrightarrow (S^m, e_0)$ by setting $E_0 g(\lambda_{n-1}(t, x)) = \lambda_{m-1}(t, g(x))$ where $\lambda_{r-1}: I \times S^{r-1} \longrightarrow S^r$ is a map given by $\lambda_{r-1}(t, t_1, \cdots, t_r)) = (2t-1, 2(t-t^2)^{\frac{1}{2}} t_1, \cdots, 2(t-t^2)^{\frac{1}{2}} t_r)$, $r=n$ or m. $E_0 g$ is a sort of suspension of g and $\rho(x, e_0) = \rho(E_0 g(x), e_0)$ for the distance function. Since f is a suspension, there exists a map g and a homotopy

$$f_t: (S^n, e_0) \longrightarrow (S^m, e_0)$$

such that $f_0 = f = Ef'$ and $f_1 = E_0 g$. Let E^r_{0+} and E^r_{0-} be hemispheres of S^r such that $E^r_{0+} = \{(t_1, \cdots, t_{r+1}) \in S^r | t_1 \geqq 0\}$ and $E^r_{0-} = \{(t_1, \cdots, t_{r+1}) \in S^r | t_1 \leqq 0\}$. Let $E_0^{(k-1)r}$ be a closed cell in $e^{(k-1)r}$ given by $E_0^{(k-1)r} = \{x_1 \cdots x_{k-1} \in S^r_{k-1} | x_i \in E^r_{0-},\ i=1, \cdots, k-1\}$ and let $S_0^{(k-1)r-1}$ be the boundary of $E_0^{(k-1)r}$. Define a homotopy $\theta_t^{(r)}: S^r \longrightarrow S^r$ by the formulas $\theta_t^{(r)}(\lambda_{r-1}(u, x)) = \lambda_{r-1}((1+t)u, x)$ for $0 \leqq u \leqq \frac{1}{2}$ and $\theta_t^{(r)}(\lambda_{r-1}(u, x)) = \lambda_{r-1}(t+u-tu, x)$ for $\frac{1}{2} \leqq u \leqq 1$, and define a homotopy $\Theta_t^{(r)}: S^r_{k-1} \longrightarrow S^r_{k-1}$ by the formula $\Theta_t^{(r)}(x_1 \cdots x_{k-1}) = \theta_t^{(r)}(x_1) \cdots \theta_t^{(r)}(x_{k-1})$, $x_i \in S^r$, $i=1, \cdots, k-1$. Then $\Theta_0^{(r)}$ is the identity, $\Theta_t^{(m)} \circ \bar{f}_1 = \bar{f}_1 \circ \Theta_t^{(n)}$, $\Theta_1^{(r)}(S^r_{k-1} - (E_0^{(k-1)r} - S_0^{(k-1)r-1})) \subset S^r_{k-2}$ and $\Theta_1^{(r)}$ maps $E_0^{(k-1)r} - S_0^{(k-1)r-1}$ homeomorphically onto $e^{(k-1)r} = S^r_{k-1} - S^r_{k-2}$. In defining the map φ_r, we may chose a characteristic map μ: $(I^{(k-1)r}, \dot{I}^{(k-1)r}, J^{(k-1)r-1}) \longrightarrow (S^r_{k-1}, S^r_{k-2}, e_0)$ of $e^{(k-1)r}$ such that $E_0^{(k-1)r} \subset E^{(k-1)r} = \mu(I_+^{(k-1)r})$. Define a map

$$\varphi'_r: S^r_{k-1} \longrightarrow S^r_{k-1} \vee S^{(k-1)r}$$

by setting $\varphi'_r(x) = \varphi_r(x)$ for $x \in \bar{S}^r_{k-2} = S^r_{k-2} \cup \mu(I_-^{(k-1)r})$ and $\varphi'_r(y) = (e_0, h'_{k-1}(\Theta_1^{(r)}(y)))$ for $y \in E^{(k-1)r}$. Then $\varphi'_r(E^{(k-1)r} - E_0^{(k-1)r}) = e_0 \times e_0$ and $\varphi'_m(\bar{f}_1(y)) = (\bar{f}_1 \vee (f_1)^{k-1})(\varphi'_n(y))$ for $y \in E^{(k-1)r}$. Define a homotopy $\xi_t^{(r)}: S^r_{k-1} \longrightarrow S^r_{k-1}$ by the formulas $\xi_t^{(r)}(x) = x$ for $x \in S^r_{k-2}$ and

$$\xi_t^{(r)}(\mu(t_1, \cdots, t_{(k-1)r})) = \begin{cases} \mu(t_1, \cdots, (1+t)t_{(k-1)r}), & t_{(k-1)r} \leqq \frac{1}{2}, \\ \mu(t_1, \cdots, t+(1-t)t_{(k-1)r}), & \frac{1}{2} \leqq t_{(k-1)r}, \end{cases}$$

for $(t_1, \cdots, t_{(k-1)r}) \in I^{(k-1)r}$, then $\xi_0^{(r)}$ is the identity and $\xi_1^{(r)} | \bar{S}^r_{k-2} = \varphi_r | \bar{S}^r_{k-2} = \varphi'_r | \bar{S}^r_{k-2}$. The homotopical commutativity of the diagram

$$\begin{array}{ccc}
(S^n_{k-1}, S^n_{k-2}) & \xrightarrow{\quad\bar{f}_1\quad} & (S^m_{k-1}, S^m_{k-2}) \\
\downarrow \varphi'_n & (1)'' & \downarrow \varphi'_m \\
(S^n_{k-1} \vee S^{(k-1)n}, S^n_{k-1}) & \xrightarrow{\bar{f}_1 \vee (f_1)^{k-1}} & (S^m_{k-1} \vee S^{(k-1)m}, S^m_{k-1})
\end{array}$$

is shown by a homotopy $F_t: (S^n_{k-1}, S^n_{k-2}) \longrightarrow (S^m_{k-1} \vee S^{(k-1)m}, S^m_{k-1})$ which is given by the formulas $F_t(y)=(\varphi'_m(\bar{f}_1(y)))=(\bar{f}_1\vee(f_1)^{k-1})(\varphi'_n(y))$ for $y\in E^{(k-1)n}$ and

$$F_t(x)=\begin{cases}(\varphi'_m\circ\bar{f}_1\circ\xi^{(n)}_{2t})(x), & 0\leqq t\leqq\frac{1}{2},\\ (\xi^{(m)}_{2-2t}\circ\bar{f}_1\circ\varphi'_n)(x), & \frac{1}{2}\leqq t\leqq 1,\end{cases}$$

for $x\in\bar{S}^n_{k-1}$. Let $p_2: S^r_{k-1}\vee S^{(k-1)r}\longrightarrow S^{(k-1)r}$ be the projection. From a homotopy $h'_{k-1}\circ\Theta^{(r)}_t: (S^r_{k-1}, S^r_{k-2})\longrightarrow(S^{(k-1)r}, e_0)$, we see that the maps h'_{k-1} and $p_2\circ\varphi'_r$ are homotopic to each other. Since the map μ is chosen such that $p_2\circ\varphi_r$ is homotopic to h'_{k-1}, the maps φ_r and φ'_r carry $E^{(k-1)r}$ onto $S^{(k-1)r}$ with the same degree. Since φ_r and φ'_r coincide on $\bar{S}^r_{k-2}$, there exists a homotopy

$$\varphi^{(r)}_t: (S^r_{k-1}, S^r_{k-2})\longrightarrow(S^r_{k-1}\vee S^{(k-1)r}, S^r_{k-1})$$

such that $\varphi^{(r)}_0=\varphi_r$ and $\varphi^{(r)}_1=\varphi'_r$. By (2・5), $\bar{f}_t: (S^n_{k-1}, S^n_{k-2})\longrightarrow(S^m_{k-1}, S^m_{k-2})$ is a homotopy. Also $(f_t)^{k-1}$ is a homotopy. Then it follows from homotopies $\varphi^{(m)}_t\circ\bar{f}_t$ and $(\bar{f}_t\vee(f_t)^{k-1})\circ\varphi^{(n)}_t$ that the homotopical commutativity of ①″ implies that of ①′. Consequently the homotopical commutativity of the square ① is established.

The commutativity of ② follows from (6・4).

Homotopical commutativity of ③. By making use of the homotopy f_t, we see that the homotopical commutativity of ③ follows from that of the diagram

$$\begin{array}{ccc}(S^n_{k-1}\times S^{(k-1)n}, S^n_{k-1}\vee S^{(k-1)n}) & \xrightarrow{\bar{f}_1\times(f_1)^{k-1}} & (S^m_{k-1}\times S^{(k-1)m}, S^m_{k-1}\vee S^{(k-1)m})\\ \downarrow\tilde{\phi}_{(k-1)n} & (3)' & \downarrow\tilde{\phi}_{(k-1)m}\\ (S^{kn}, e_0) & \xrightarrow{(f_1)^k} & (S^{km}, e_0).\end{array}$$

For given two maps $g: (S^p, e_0)\longrightarrow(S^q, e_0)$ and $g': (S^{p'}, e_0)\longrightarrow(S^{q'}, e_0)$, we define a reduced join [2] $g*g': (S^{p+p'}, e_0)\longrightarrow(S^{q+q'}, e_0)$ by the following commutative diagram

$$\begin{array}{ccc}S^p\times S^{p'} & \xrightarrow{g\times g'} & S^q\times S^q\\ \downarrow\phi_{p,p'} & & \downarrow\phi_{q,q'}\\ S^{p+q'} & \xrightarrow{g*g'} & S^{q+q'}\end{array}$$

where the maps $\phi_{p,p'}$ and $\phi_{q,q'}$ are defined in (2・8)″. Then $(g*g')*g''=g*(g'*g'')$ and $g*i_1=Eg$ for the identity $i_1: S^1\longrightarrow S^1$. By theorem 3.2. of [2], i_1*g represents $(-1)^{p+q}E\beta$ for the class $\beta\in\pi_p(S^q)$ of g. By (2・8)′, $(f_1)^{k-1}$ represents a suspension. Then there exists a map $g: (S^{(k-1)n-1}, e_0)\longrightarrow(S^{(k-1)m-1}, e_0)$ such that i_1*g is homotopic to $(f_1)^{k-1}$. Since $\rho(x, e_0)=\rho(f_1(x), e_0)$, we have that $\Omega(Ef_1)\circ\bar{i}=\bar{i}\circ\bar{f}_1$ in (2・6). Then from the definition of $\tilde{\phi}_1$, we see that the diagram

$$\begin{array}{ccc}S^n_{k-1}\times S^1 & \xrightarrow{\bar{f}_1\times i_1} & S^m_{k-1}\times S^1\\ \downarrow\tilde{\phi}_1 & & \downarrow\tilde{\phi}_1\\ S^{n+1} & \xrightarrow{Ef_1} & S^{m+1}\end{array}$$

is commutative. Since $\tilde{\phi}_{(k-1)r}(x, \phi_{1,(k-1)r-1}(y, z))=\phi_{n+1,(k-1)r-1}(\tilde{\phi}_1(x, y), z)$, the diagram

$$\begin{array}{ccc} S^n_{k-1}\times S^{(k-1)n} & \xrightarrow{\bar{f}_1\times(i_1*g)} & S^m_{k-1}\times S^{(k-1)m} \\ \downarrow \tilde{\phi}_{(k-1)n} & & \downarrow \tilde{\phi}_{(k-1)m} \\ S^{kn} & \xrightarrow{Ef_1*g} & S^{km} \end{array}$$

is commutative. We have easily that $Ef_1*g=(f_1*i_1)*g=f_1*(i_1*g)$ and $f_1*(f_1)^{k-1} =(f_1)^k$. Applying a homotopy between i_1*g and $(f_1)^{k-1}$, we have the homotopical commutativity of ③′. Then the homotopical commutativity of ③ is proved.

7. The group $\pi_i(\Omega(S^n_{p-1}), S^{n-1})$, n: even

In this § we suppose that n is even.

Let $w_r=\omega_{r,r}: S^{2r-1}\longrightarrow S^r$ be a map which represents the Whitehead product of the class ι_r of ψ_r. The Hopf invariant of the map w_r is ± 2 if r is even. It was proved in [12, Ch. IV, Prop. 5] (see also (4·6)) that

(7·1) *the correspondence* $(\alpha,\beta)\longrightarrow E\alpha+w_r(\beta)$ *defines a* $\mathcal{C}_p$*-isomorphism*: $\pi_{i-1}(S^{r-1}) +\pi_i(S^{2r-1})\longrightarrow\pi_i(S^r)$ *for a prime* $p>2$. $(\alpha\in\pi_{i-1}(S^{r-1}),\ \beta\in\pi_i(S^{2r-1}),\ r$*: even*$)$.

Define a map

$$(7\cdot 2)\qquad J_{n,k}:\Omega(S^{kn-1})\times\Omega^2(S^{2kn-1})\longrightarrow\Omega^2(S^{kn})$$

by the formula $J_{n,k}(x,y)=\tau(x)*\Omega^2w_{kn}(y)$ where τ is the map $\tau_{n,(k-1)n}$ as in (6·9) and $*$ indicates the product of loops in $\Omega^2(S^{kn})$ $(\Omega^2(S^{kn})=\Omega(\Omega(S^{kn})))$. Then

(7·2)′ *for an odd prime p, the map $J_{n,k}$ induces $\mathcal{C}_p$-isomorphisms of the homotopy and the cohomology groups* (n: even $\geqq 2$, $k>2$).

Proof. There is an isomorphism $\eta:\pi_{i-2}(\Omega(S^{kn-1}))+\pi_{i-2}(\Omega^2(S^{2kn-1}))\approx\pi_{i-2}(\Omega(S^{kn-1})\times\Omega^2(S^{2kn-1}))$. Then $J_{n,k_*}(\eta(\Omega(\alpha)+\Omega^2(\beta)))=\tau_*(\Omega(\alpha))+(\Omega^2w_{kn})_*(\Omega^2(\beta))=\Omega^2(E(\alpha)+w_{kn_*}(\beta))$ by (1·2) and (6·8)′. By (1·1), (6·1) and (7·1), we see that $J_{n,k_*}:\pi_{i-2}(\Omega(S^{kn-1})\times\Omega^2(S^{2kn-1}))\longrightarrow\pi_{i-2}(\Omega^2(S^{kn}))$ is a $\mathcal{C}_p$-isomorphism for all $i>2$. By (1·6), (7·2)′ is proved. q. e. d.

Let $P:\Omega(S^{kn-1})\times\Omega^2(S^{2kn-1})\longrightarrow\Omega(S^{kn-1})$ be the projection. Define a space

$$(7\cdot 3)\qquad Y=Y_{n,k}=\Omega(S^{kn-1})\cup\Omega(S^{kn-1})\times\Omega^2(S^{2kn-1})\times(0,1)\cup\Omega^2(S^{kn})$$

by identifying a space $\Omega(S^{kn-1})\cup\Omega(S^{kn-1})\times\Omega^2(S^{2kn-1})\times I\cup\Omega^2(S^{kn})$ with the relations $(x,y,0)\equiv P(x)$ and $(x,y,1)\equiv J_{n,k}(x,y)$. Then

(7·3)′ *the injection*: $\Omega(S^{kn-1})\subset Y$ *induces $\mathcal{C}_p$-isomorphisms of the homotopy and the cohomology groups for an odd prime p* (n: even $\geqq 2$, $k>2$).

Proof. Set $Y_+=\Omega(S^{kn-1})\times\Omega^2(S^{2kn-1})\times[\frac{1}{2},1)\cup\Omega^2(S^{kn})$ and $Y_-=\Omega(S^{kn-1})\cup\Omega(S^{kn-1})\times\Omega^2(S^{2kn-1})\times(0,\frac{1}{2}]$, then Y_+ is a mapping-cylinder of $J_{n,k}$ and the pairs (Y, Y_-) and $(Y,\Omega(S^{kn-1}))$ have the same homotopy type. Since $J_{n,k}$ induces $\mathcal{C}_p$-isomorphisms of the cohomology groups, $H^i(Y_+, Y_+\cap Y_-)\in\mathcal{C}_p$ for all i. Since $H^i(Y, \Omega(S^{kn-1}))\approx H^i(Y, Y_-)\approx H^i(Y_+, Y_+\cap Y_-)\in\mathcal{C}_p$ for all i, the injection homomorphism $i^*:H^i(Y)\longrightarrow H^i(\Omega(S^{kn-1}))$ is a $\mathcal{C}_p$-isomorphism for all i. Then (7·3)′ follows from (1·6). q. e. d.

Consider the map $\bar{h}$ of (6・5)'. By (6・6), $\bar{h}$ maps $S^{n-1}\subset\Omega(S^n)\subset\Omega(S^n_{k-2})$ to a single point e_0 of $\Omega^2(S^{kn})$, Then $\bar{h}$ defines a map

(7・4) $$h:(\Omega(S^n_{k-1}),\ S^{n-1})\longrightarrow(Y,e_0),\qquad k>2.$$

PROPOSITION (7・5). *The map h induces a* $\mathcal{C}_p$*-isomorphism* $h_*:\pi_i(\Omega(S^n_{k-1}),\ S^{n-1})\longrightarrow\pi_i(Y)$ *for all i and for a prime* $p\geqq k>2$ $(n:even)$.

Proof. First we treat the case $n\geqq 4$. It is easily verified that the map h satisfies the conditions i) and ii) of (1・7). By (3・3)', $H^*(\Omega(S^n_{k-1}),\ Z_p)\approx P^*(a,\ b_j)\otimes Z_p$. Set $B=\{b_0,\ b_1,\ b_2,\cdots\}\otimes Z_p$ and $F=\{b_0,\ a\}\otimes Z_p$, then the conditions iii) and iv) are filfulled for the coefficient ring $R=Z_p$. By (7・3)', the injection homomorphism $i^*:H^*(Y)\longrightarrow H^*(\Omega(S^{kn-1}))$ is a $\mathcal{C}_p$-isomorphism. Take generators e_j of $H^{j(kn-2)}(\Omega(S^{kn-1}))$ such as in (2・4), i), then there are elements f'_j of $H^{j(kn-2)}(Y)$ such that $i^*(f'_j)=t'_je_j$ and $t'_j\not\equiv 0$ (mod. p). By (3・3), there exist integers u_j and s'_j such that $u_jh^*(f'_j)=s'_jb_j$ and $u_j\not\equiv 0$ (mod. p). Set $f_j=u_jf'_j$, $t_j=u_jt'_j$ and $s_j=u_js'_j$, then $i^*(f_j)=t_je_j$, $h^*(f_j)=s_jb_j$ and $t_j\not\equiv 0$ (mod. p). The homomorphism $h^*:\ H^{kn-2}(Y)\longrightarrow H^{kn-2}(\Omega(S^n_{k-1}))$ is divided into three homomorphisms $i_1^*:\ H^{kn-2}(Y)\longrightarrow H^{kn-2}(\Omega^2(S^{kn}))$, $\bar{h}^*:\ H^{kn-2}(\Omega^2(S^{kn}))\longrightarrow H^{kn-2}(\Omega(S^n_{k-1},\ S^n_{k-2}))$ and $i_2^*:\ H^{kn-2}(\Omega(S^n_{k-1},\ S^n_{k-2}))\longrightarrow H^{kn-2}(\Omega(S^n_{k-1}))$ where i_1^* and i_2^* are the injection homomorphisms. Obviously i_1^* is an isomorphism. By (6・7), $\bar{h}^*$ is an isomorphism. By (3・5), i_2^* is a $\mathcal{C}_p$-isomorphism. Therefore h^* is a $\mathcal{C}_p$-isomorphism and $s_1\not\equiv 0$ (mod. p). By (2・4), $i^*(t_jf_1^j-j\,!\,t_1^jf_j)=t_jt_1^j((e_1)^j-j\,!\,e_j)=0$. Sinec i^* is an $\mathcal{C}_p$-isomorphism, $t_jf_1^j-j\,!\,t_1^jf_j$ has a finite order. By (3・2), $h^*(t_jf_1^j-j\,!\,t_1^jf_j)=t_js_1^jb_1^j-j\,!\,t_1^js_jb_j=j\,!\,(t_js_1^j-t_1^js_j)b_j$. Since b_j has an infinite order, $j\,!\,(t_js_1^j-t_1^js_j)=0$. Then $s_j\equiv t_js_1^j/t_1^j\not\equiv 0$ (mod. p). Therefore $h^*:\ H^{j(kn-2)}(Y,Z_p)\longrightarrow H^{j(kn-2)}(\Omega(S^n_{k-1}),Z_p)$ is an isomorphism for all $j\geqq 0$. Then the condition v) of (1・7) is filfulled. By theorem (1・8), h^* $\pi_i(\Omega(S^n_{k-1}),\ S^{n-1})\longrightarrow\pi_i(Y)$ is a $\mathcal{C}_p$-isomorphism for all i.

Next consider the case $n=2$. The result $h^*:\ H^{j(kn-2)}(Y,\ Z_p)\approx H^{j(kn-2)}(\Omega(S^n_{k-1}),\ Z_p)$ is also true for the case $n=2$. By (4・3), there is a map $g_0:\ (S^{2k-2},e_0)\longrightarrow(\Omega(S^2_{k-1})e_0)$ such that $g_0^*:\ H^{2k-2}(\Omega(S^2_{k-1}),\ Z_p)\approx H^{2k-2}(S^{2k-2},\ Z_p)$. Define a map $g:S^{2k-1}\longrightarrow S^2_{k-1}$ by the formula $g(d_{2k-2}(x,\ t))=g_0(x)(t)$, then $g_0=\Omega g\,|\,S^{2k-2}$ for the induced map $\Omega g:\Omega(S^{2k-1})\longrightarrow\Omega(S^2_{k-1})$. Consider the homomorphism $\Omega g^*\circ h^*:\ H^{j(2k-2)}(Y)\longrightarrow H^{j(2k-2)}(\Omega(S^{2k-1}))$ and set $(\Omega g^*\circ h^*)(f_j)=t''_je_j$ for an integer t''_j. Obviously $t''_1\not\equiv 0$ (mod. p). Since $t_jf_1^j-j!\,t_1^jf_j$ has a finite order and since e_j is a free element, we have that $(\Omega g^*\circ h^*)(t_jf_1^j-j!\,t_1^jf_j)=(j!\,t_j(t''_1)^j-j!\,t_1^jt''_j)e_j$ and this implies that $j!(t_j(t''_1)^j-t_1^jt''_j)=0$. Therefore $t''_j\equiv t_j(t''_1)^j/t_1^j\not\equiv 0$ (mod. p), and then $\Omega g^*\circ h^*:H^*(Y,Z_p)\longrightarrow H^*(\Omega(S^{2k-1}),Z_p)$ is an isomorphism. By (1・6), we have a $\mathcal{C}_p$-isomorphism $h_*\circ\Omega g_*:\pi_i(\Omega(S^{2k-1}))\longrightarrow\pi_i(\Omega(S^2_{k-1}))\longrightarrow\pi_i(Y)$. By (4・4), (1・1) and (1・2), Ωg_* is a $\mathcal{C}_p$-isomorphism for $i>1$. Then h_* is a $\mathcal{C}_p$-isomorphism for $i>1$. Since $\pi_1(S^1)\approx\pi_1(\Omega(S^2_{k-1}))\approx Z$ and $\pi_i(S^1)=0$ for $i>1$, we have that $\pi_i(\Omega(S^2_{k-1}))\approx\pi_i(\Omega(S^2_{k-1}),S^1)$ for $i>1$ and that $h_*:\ \pi_1(\Omega(S^2_{k-1}),S^1)\longrightarrow\pi_i(Y)$ is a $\mathcal{C}_p$-isomorphism for $i>1$. q. e. d.

THEOREM (7·6) *The groups $\pi_i(\Omega(S^n_{k-1}), S^{n-1})$ and $\pi_{i+1}(S^{kn-1})$ are $\mathcal{C}_p$-isomorphic for a prim $p \geqq k \geqq 2$ and for $i > 1$ (n: even).*

Proof. If $k=2$, (2·10) implies (7·6). If $k > 2$, by (1·1), (7·3)′ and (7·5), we have (7·6). q. e. d.

Here we remark that a map $g: S^{kn-1} \longrightarrow S^n_{k-1}$ of (4·4), induces a $\mathcal{C}_p$-isomorphism

(7·7) $$\Omega g_* \circ \Omega : \pi_{i+1}(S^{kn-1}) \approx \pi_i(\Omega(S^{kn-1})) \longrightarrow \pi_i(\Omega(S^n_{k-1}), S^{n-1}).$$

For the p-primary components, we define *an isomorphism*

(7·8) $$\bar{H}_k : \pi_i(\Omega(S^n_{k-1}), S^{n-1}; p) \approx \pi_{i+1}(S^{kn-1}; p), p \geqq k \geqq 2,$$

by setting

i) $\bar{H}_k = H_2 = \Omega^{-1} \circ \tilde{i}_* \circ h_{2*} \circ \tilde{i}_*^{-1} : \pi_i(\Omega(S^n), S^{n-1}; p) \approx \pi_i(S^{n-1}_\infty, S^{n-1}; p) \approx \pi_i(S^{2n-2}_\infty; p) \approx \pi_i(\Omega(S^{2n-1}); p) \approx \pi_{i+1}(S^{2n-1}; p)$ *when* $k=2$,

ii) $\bar{H}_k = \Omega^{-1} \circ i_*^{-1} \circ h_* : \pi_i(\Omega(S^n_{k-1}), S^{n-1}; p) \approx \pi_i(Y; p) \approx \pi_i(\Omega(S^{kn-1}); p) \approx \pi_{i+1}(S^{kn-1}; p)$ *when* $k > 2$.

PROPOSITION (7·9). *Let $f: (S^n, e_0) \longrightarrow (S^m, e_0)$ be the suspension $Ef' = f$ of a map $f': (S^{n-1}, e_0) \longrightarrow (S^{m-1}, e_0)$, and let $\alpha \in \pi_n(S^m)$ be the class of f. Let $F': (S^{kn-1}, e_0) \longrightarrow (S^{km-1}, e_0)$ be a map which represents $E^{(k-1)m-1}(\alpha \circ E^{n-m}\alpha \circ \cdots \circ E^{(k-1)(n-m)}\alpha)$. Then the following diagram is commutative ($p \geqq k \geqq 2$; n, m: even):*

$$\begin{array}{ccc}
\pi_i(\Omega(S^n_{k-1}), S^{n-1}; p) & \xrightarrow[\approx]{\bar{H}_k} & \pi_{i+1}(S^{kn-1}; p) \\
\downarrow \Omega f_* & & \downarrow F'_* \\
\pi_i(\Omega(S^m_{k-1}), S^{m-1}; p) & \xrightarrow[\approx]{\bar{H}_k} & \pi_{i+1}(S^{km-1}; p)
\end{array}$$

Proof. If $k=2$, this follows from (2·9) and (2·8)′. Suppose that $k > 2$. Combining the formula (3·59) of [16] and theorem (2·4) of [2], we have the following formula. If $\alpha \in \pi_p(S^q)$ and $\alpha \in \pi_{p'}(S^{q'})$ are suspension elements, then $[\alpha, \alpha'] = [\iota_q, \iota_{q'}] \circ (-1)^{p(p'+q')} E^{q'-1}\alpha \circ E^{p-1}\alpha'$. Let $\alpha_{k-1} \in \pi_{(k-1)n}(S^{(k-1)m})$ be the class of $(f)^{k-1}$, then α_{k-1} is a suspension element by (2·8)′ and $(f \vee (f)^{k-1})_*[\iota_n, \iota_{(k-1)n}] = [\alpha, \alpha_{k-1}] = [\iota_m, \iota_{(k-1)m}] \circ E^{(k-1)m-1}\alpha \circ E^{n-1}\alpha_{k-1}$. The element $E^{(k-1)m-1}\alpha \circ E^{n-1}\alpha_{k-1}$ is represented by the map F', by (2·8)′. Therefore the first square in the following diagram is homotopically commutative:

$$\begin{array}{ccc}
\Omega(S^{kn-1}) & \xrightarrow{\Omega F'} & \Omega(S^{km-1}) \\
\downarrow \omega_{n,(k-1)n} & & \downarrow \omega_{m,(k-1)m} \\
\Omega(S^n \vee S^{(k-1)n}) & \xrightarrow{\Omega(f \vee (f)^{k-1})} & \Omega(S^m \vee S^{(k-1)m}) \\
\downarrow \bar{Q} & & \downarrow \bar{Q} \\
\Omega^2(S^n \times S^{(k-1)n}, S^n \vee S^{(k-1)n}) & \xrightarrow{\Omega^2(f \times (f)^{k-1})} & \Omega^2(S^m \times S^{(k-1)m}, S^m \vee S^{(k-1)m}) \\
\downarrow \phi_{n,(k-1)n} & & \downarrow \phi_{m,(k-1)m} \\
\Omega^2(S^{kn}) & \xrightarrow{\Omega^2(f)^k} & \Omega^2(S^{km})
\end{array}$$

The other two squares are exactly commutative. From the definition (6・8) of $\tau = \tau_{r,(k-1)r}(r=n \text{ or } m)$, we have the following homotopically commutative diagram:

$$\begin{array}{ccc} \Omega(S^{kn-1}) & \xrightarrow{\Omega F'} & \Omega(S^{km-1}) \\ \downarrow \tau & & \downarrow \tau \\ \Omega^2(S^{kn}) & \xrightarrow{\Omega^2(f)^k} & \Omega^2(S^{km}) \end{array}$$

Let $\alpha_k \in \pi_{kn}(S^{km})$ be the class of $(f)^k$, then $(f)^k_*[\iota_{kn}, \iota_{kn}]=[\alpha_k, \alpha_k]=[\iota_{km}, \iota_{km}]\circ E^{km-1}\alpha_k \circ E^{kn-1}\alpha_k$. Let $F^{(2)}: (S^{2kn-1}, e_0) \longrightarrow (S^{2km-1}, e_0)$ be a representative of $E^{km-1}\alpha_k \circ E^{kn-1}\alpha_k$. From the defiinition of $J_{r,k}$, we see that the diagram

$$\begin{array}{ccc} \Omega(S^{kn-1}) \times \Omega^2(S^{2kn-1}) & \xrightarrow{\Omega F' \times \Omega^2 F^{(2)}} & \Omega(S^{km-1}) \times \Omega^2(S^{2km-1}) \\ \downarrow J_{n,k} & & \downarrow J_{m,k} \\ \Omega^2(S^{kn}) & \xrightarrow{\Omega^2(f)^k} & \Omega^2(S^{km}) \end{array}$$

is homotopically commutative. Let $G_t: (\Omega(S^{kn-1}) \times \Omega^2(S^{2kn-1}), e_0) \longrightarrow (\Omega^2(S^{km}), e_0)$ be a homotopy between $G_0 = J_{m,k} \circ (\Omega F' \times \Omega^2 F^{(2)})$ and $G_1 = \Omega^2(f)^k \circ J_{n,k}$. Define a map

$$\tilde{F}: Y_{n,k} \longrightarrow Y_{m,k}$$

by setting $\tilde{F} \mid \Omega(S^{kn-1}) = \Omega F'$, $\tilde{F} \mid \Omega^2(S^{kn}) = \Omega^2(f)^k$ and

$$\tilde{F}(x, y, t) = \begin{cases} (\Omega F'(x),\ \Omega^2 F^{(2)}(y), 2t), & 0 \leqq t \leqq \frac{1}{2}, \\ G_{2t-1}(x, y), & \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

for $x \in \Omega(S^{kn-1})$, and $y \in \Omega^2(S^{2kn-1})$. Then the right square of the following diagram is commutative:

$$\begin{array}{ccccc} \pi_i(\Omega(S^n_{k-1}), S^{n-1}) & \xrightarrow{h_*} & \pi_i(Y_{n,k}) & \xleftarrow{i_*} & \pi_i(\Omega(S^{kn-1})) \\ \downarrow \Omega \bar{f}_* & & \downarrow \tilde{F}_* & & \downarrow \Omega F'_* \\ \pi_i(\Omega(S^m_{k-1}), S^{m-1}) & \xrightarrow{h_*} & \pi_i(Y_{m,k}) & \xleftarrow{i_*} & \pi_i(\Omega(S^{km-1})) \end{array}$$

The commntativity of the left square follows from the lemma (6・11). By (1・2) and (7・8), ii), the commutativity of the diagram (7・9) is proved for $k>2$. q. e. d.

8. Double suspension E^2 and the group $\pi_i(\Omega(S^{n+1}), S^{n-1})$, n: even.

In this § we suppose also n is even.

By (2・2)″ and (1・1), the suspension homomorphism $E: \pi_i(S^n) \longrightarrow \pi_{i+1}(S^{n+1})$ is equivalent to the injection homomorphism $i_*: \pi_i(S^n) \longrightarrow \pi_i(\Omega(S^{n+1}))$. Then the double suspension $E^2 = E \circ E: \pi_{i-1}(S^{n-1}) \longrightarrow \pi_{i+1}(S^{n+1})$ is equivalent to the injection homomorphism $i_*: \pi_{i-1}(S^{n-1}) \longrightarrow \pi_{i-1}(\Omega^2(S^{n+1}))$, i.e., we have a commutative diagram

$$(8\cdot1) \qquad \begin{array}{ccc} & & \pi_{i+1}(S^{n+1}) \\ & \nearrow^{E^2} & \wr\!\!\downarrow \Omega^2 \\ \pi_{i-1}(S^{n-1}) & \xrightarrow{i_*} & \pi_{i-1}(\Omega^2(S^{n+1})) \end{array}$$

for the injection $S^{n-1}\subset\Omega(S^n)\subset\Omega(\Omega(S^{n+1}))=\Omega^2(S^{n+1})$. From the exact homotopy sequence of the pair $(\Omega^2(S^{n+1}), S^{n-1})$, we have an exact sequence

(8・2) $\cdots\longrightarrow\pi_i(\Omega^2(S^{n+1}), S^{n-1})\xrightarrow{\partial}\pi_{i-1}(S^{n-1})\xrightarrow{E^2}\pi_{i+1}(S^{n+1})\xrightarrow{J}\pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1})\longrightarrow\cdots$

where $J=j_*\circ\Omega^2:\pi_{i+1}(S^{n+1})\approx\pi_{i-1}(\Omega^2(S^{n+1}))\longrightarrow\pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1})$. If $i\neq n$, then the groups $\pi_{i-1}(S^{n-1})$ and $\pi_{i+1}(S^{n-1})$ are finite by [10, Ch. V, Prop. 3]. If $i=n$, E^2 is an isomorphism. It follows from the exactness of the sequence (8・2) that the group $\pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1})$ is finite for each i and that the following sequence of the p-primary components is exact:

(8・2)′ $\cdots\longrightarrow\pi_{i-1}(S^{n-1};p)\xrightarrow{E^2}\pi_{i+1}(S^{n+1};p)\xrightarrow{J}\pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1};p)\longrightarrow\cdots$

where $\pi_i(X;p)$ and $\pi_i(X, A;p)$ indicate the p-primary components of $\pi_i(X)$ and $\pi_i(X, A)$ respectively.

THEOREM (8・3). *We have an exact sequence*

$\cdots\longrightarrow\pi_{i+2}(S^{pn+1};p)\xrightarrow{\Delta}\pi_i(S^{pn-1};p)\longrightarrow\pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1};p)\longrightarrow\cdots\longrightarrow\pi_{pn-1}(\Omega^2(S^{n+1}), S^{n-1};p)\longrightarrow 0,$

where Δ is a homomorphism such that, if $p>2$, we have the formula

$$\Delta\circ E^2=f_{p*}:\pi_i(S^{pn-1};p)\longrightarrow\pi_i(S^{pn-1};p)$$

for a map $f_p:S^{pn-1}\longrightarrow S^{pn-1}$ of degree p. (n: even)

Proof. Consider the exact homotopy sequence of the triple $(\Omega^2(S^{n+1}), \Omega(S^n_{p-1}), S^{n-1})$: $\cdots\longrightarrow\pi_i(\Omega^2(S^{n+1}),\Omega(S^n_{p-1}))\longrightarrow\pi_{i-1}(\Omega(S^n_{p-1}),S^{n-1})\longrightarrow\pi_{i-1}(\Omega^2(S^{n+1}),S^{n-1})\longrightarrow\cdots$. By (7・6), the groups $\pi_{i-1}(\Omega(S^n_{p-1}), S^{n-1})$ and $\pi_i(S^{pn-1})$ are $\mathcal{C}_p$-isomorphic. Since $\pi_i(S^{pn-1})$ is finite for $i\neq pn-1$, $\pi_{i-1}(\Omega(S^n_{p-1}), S^{n-1})$ is finite for $i\neq pn-1$. By (1・1)′ and (2・11), the groups $\pi_i(\Omega^2(S^{n+1}), \Omega(S^n_{p-1}))$ and $\pi_{i+2}(S^{pn+1})$ are $\mathcal{C}_p$-isomorphic. Since $\pi_{i+2}(S^{pn+1})$ is finite for $i+2\neq pn+1$, $\pi_i(\Omega^2(S^{n+1}), \Omega(S^n_{p-1}))$ is finite for $i\neq pn-1$. Then the exactness of the above sequence implies that of the following sequence: $\cdots\longrightarrow\pi_i(\Omega^2(S^{n+1}), \Omega(S^n_{p-1});p)\longrightarrow\pi_{i-1}(\Omega(S^n_{p-1}), S^{n-1};p)\longrightarrow\pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1};p)\longrightarrow\cdots\longrightarrow\pi_{pn-1}(\Omega^2(S^{n+1}), S^{n-1};p)\longrightarrow\pi_{pn-1}(\Omega^2(S^{n+1}), \Omega(S^n_{p-1});p)=0$. By the isomorphisms H_p of (2・12) and $\bar{H}_p$ of (7・8), we have the exact sequence of (8・3). The homomorphism Δ is defined such that the diagram

(8・3)′
$$\begin{tikzcd}
\pi_i(\Omega^2(S^{n+1}), \Omega(S^n_{p-1});p) \arrow[r, "\partial"] & \pi_{i-1}(\Omega(S^n_{p-1}), S^{n-1});p) \arrow[dd, "\bar{H}_p", "\wr"'] \\
\pi_{i+1}(\Omega(S^{n+1}), S^n_{p-1};p) \arrow[u, "\Omega'", "\wr"'] \arrow[d, "H_p", "\wr"'] & \\
\pi_{i+2}(S^{pn+1};p) \arrow[r, "\Delta"] & \pi_i(S^{pn-1};p)
\end{tikzcd}$$

is commutative. Let $\mu:(I^{pn}, \dot{I}^{pn}, J^{pn-1})\longrightarrow(S^n_p, S^n_{p-1}, e_0)\subset(S^n_\infty, S^n_{p-1}, e_0)$ be a characteristic map of $e^{pn}=S^n_p-S^n_{p-1}$ such that $h_p'\circ\mu:(I^{pn}, \dot{I}^{pn})\longrightarrow(S^{pn}, e_0)$ is homotopic to ψ_{pn}. Define a map $\tilde{\mu}:(I^{pn-1}, \dot{I}^{pn-1})\longrightarrow(\Omega(S^n_\infty, S^n_{p-1}), e_0)$ by setting $\tilde{\mu}(t_1,\cdots,t_{pn-1})$ (t)

$=\mu(t_1,\cdots,t_{pn-1},t)$, then the composition $\Omega h_p\circ\tilde{\mu}:(I^{pn-1},\dot{I}^{pn-1})\longrightarrow(\Omega(S^n_\infty,S^n_{p-1}),e_0)\longrightarrow(\Omega(S^{pn}_\infty),e_0)$ is homotopic to a map $\tilde{\psi}$ which is defined by $\tilde{\psi}(t_1,\cdots,t_{pn-1})(t)=\psi_{pn}(t_1,\cdots,t_{pn-1},t)=d_{pn-1}(\psi_{pn-1}(t_1,\cdots,t_{pn-1}),t)$. Let $\tilde{\chi}:(S^{pn-1},e_0)\longrightarrow(\Omega(S^n_\infty,S^n_{p-1}),e_0)$ be a map such that $\tilde{\mu}=\tilde{\chi}\circ\psi_{pn-1}$, then the composition $\Omega h_p\circ\tilde{\chi}$ is homotopic to the canonical injection: $S^{pn-1}\subset\Omega(S^{pn})\subset\Omega(S^{pn}_\infty)$ of (2·2)$'$. Let $p:\Omega(S^n_\infty,S^n_{p-1})\longrightarrow S^n_{p-1}$ be a projection given by $p(f)=f(1)$, and set $\chi=p\circ\tilde{\chi}$, then $\mu\,|\,I^{pn-1}:(I^{pn-1},\dot{I}^{pn-1})\longrightarrow(S^n_{p-1},e_0)$ and $\chi\circ\psi_{pn-1}$ are homotopic to each other. Then the commutativity of the following diagram is verified without difficulties:

$$\begin{tikzcd}
 & & \pi_{i-1}(\Omega(S^n_{p-1}),S^{n-1}) \\
\pi_i(\Omega^2(S^{n-1}),\Omega(S^n_{p-1})) \arrow[urr, "\partial"] \arrow[rr, "\partial"] & & \pi_{i-1}(\Omega(S^n_{p-1})) \arrow[u, "j_*"'] \\
\pi_{i+1}(\Omega(S^{n+1}),S^n_{p-1}) \arrow[u, "\Omega'"] \arrow[rr, "\partial"] & & \pi_i(S^n_{p-1}) \arrow[u, "\Omega"'] \\
\pi_{i+1}(S^n_\infty,S^n_{p-1}) \arrow[u, "\tilde{i}_*"] \arrow[urr, "\partial"] \arrow[r, "\Omega"] \arrow[d, "h_{p*}"] & \pi_i(\Omega(S^n_\infty,S^n_{p-1})) \arrow[ur, "p_*"] \arrow[d, "\Omega h_{p*}"] & \\
\pi_{i+1}(S^{pn}_\infty) \arrow[r, "\Omega"] \arrow[d, "\tilde{i}_*"] & \pi_i(\Omega(S^{pn}_\infty)) \arrow[d, "\Omega\tilde{i}_*"] & \pi_i(S^{pn-1}) \arrow[l, "i_*"'] \arrow[ul, "\tilde{\chi}_*"'] \arrow[uu, bend right, "\chi_*"'] \arrow[ddl, "E^2"] \\
\pi_{i+1}(\Omega(S^{pn+1})) \arrow[r, "\Omega"] & \pi_i(\Omega^2(S^{pn+1})) & \\
 & \pi_{i+2}(S^{pn+1}) \arrow[ul, "\Omega"] \arrow[u, "\Omega^2"] &
\end{tikzcd}$$

From the definition of the homomorphism H_p, we have the commutativity of ① of the following diagram

$$(8\cdot3)''\quad\begin{tikzcd}
\pi_i(\Omega^2(S^{n+1}),\Omega(S^n_{p-1});p) \arrow[rr, "\partial"] & & \pi_{i-1}(\Omega(S^n_{p-1}),S^{n-1};p) \arrow[dd, "\bar{H}_p"] \\
\pi_{i+1}(\Omega(S^{n+1}),S^n_{p-1};p) \arrow[u, "\Omega'"] \arrow[d, "H_p"'] \arrow[ur, phantom, "(1)"] & \pi_i(S^{pn-1};p) \arrow[ur, "\nu"] \arrow[dl, "E^2"'] \arrow[dr, "f_{p*}"] \arrow[r, phantom, "(2)"] \arrow[d, phantom, "(3)"] & {} \\
\pi_{i+2}(S^{pn+1};p) \arrow[rr, "\Delta"] & {} & \pi_i(S^{pn-1};p),
\end{tikzcd}$$

where $\nu=j_*\circ\Omega\circ\chi_*$. To prove the commutativity of the triangle ②, we consider a diagram $(p>2)$:

$$\begin{tikzcd}
 & & \pi_{i-1}(\Omega(S^n_{p-1}),S^{n-1}) \arrow[d, "\bar{h}_*"] \arrow[dr, "h_*"] & \\
 & \pi_{i-1}(\Omega(S^n_{p-1})) \arrow[ur, "j_*"] \arrow[r, "h_*"] & \pi_{i-1}(\Omega^2(S^{pn})) \arrow[r, "i'_*"] & \pi_{i-1}(Y) \\
\pi_i(S^n_{p-1}) \arrow[ur, "\Omega"] & \pi_{i-1}(\Omega(S^{pn-1})) \arrow[u, "\Omega\chi_*"] \arrow[r, "\Omega f_{p*}"] & \pi_{i-1}(\Omega(S^{pn-1})) \arrow[u, "\tau_*"] \arrow[ur, "i_*"'] & \\
\pi_i(S^{pn-1}) \arrow[u, "\chi_*"] \arrow[ur, "\Omega"] \arrow[rr, "f_{p*}"] & & \pi_i(S^{pn-1}). \arrow[u, "\Omega"] &
\end{tikzcd}$$

Since χ is an attaching map of e^{pn}, we have the commutativity of the central square from the lemma (6·9). The commutativity of the other two squares follows from (1·2). The commutativity of three triangles follows from the definition of the space Y. Then from the definition of the isomorphism $\bar{H}_p$, we have the commutativity of the triangle ② of (8·3)″. Since the homomorphism $\varDelta$ is defined by the commutativity of (8·3)′, we have the commutativity of the triangle ③ of (8·3)″. Therefore we have that $f_{p*}=\varDelta\circ E^2$. for an odd prime p. q. e. d.

Let $f':(S^{n-1},e_0)\longrightarrow(S^{m-1},e_0)$ be a map. Define a suspension $f=Ef':(S^n,e_0)\longrightarrow(S^m,e_0)$ of f' by the formula $Ef'(d_{n-1}(x,t))=d_{m-1}(f(x),t)$, then the induced map $\Omega f:\Omega(S^n)\longrightarrow\Omega(S^m)$ maps S^{n-1} into S^{m-1} and coincides with the map f' on S^{n-1}. Define a double suspension E^2f' of f' by $E^2f'=Ef=E(Ef')$, then we have a map

$$\Omega^2(E^2f'):(\Omega^2(S^{n+1}),S^{n-1})\longrightarrow(\Omega^2(S^{m+1}),S^{m-1}).$$

THEOREM (8·4). *Let $\alpha'\in\pi_{n-1}(S^{m-1})$ be represented by f'. Let $F':S^{pn-1}\longrightarrow S^{pm-1}$ be a representative of an element $E^{(p-1)m}(\alpha'\circ E^{n-m}\alpha'\circ\cdots\circ E^{(p-1)(n-m)}\alpha')\in\pi_{pn-1}(S^{pm-1})$. Then in the diagram*

$$\begin{array}{ccccccc}\cdots\longrightarrow\pi_{i+2}(S^{pn+1};p) & \xrightarrow{\varDelta} & \pi_i(S^{pn-1};p) & \longrightarrow & \pi_{i-1}(\Omega(S^{n+1}),S^{n-1};p) & \longrightarrow\cdots\\ \downarrow E^2F'_* & & \downarrow F'_* & & \downarrow \Omega^2(E^2f')_* & \\ \cdots\longrightarrow\pi_{i+2}(S^{pm+1};p) & \xrightarrow{\varDelta} & \pi_i(S^{pm-1};p) & \longrightarrow & \pi_{i-1}(\Omega^2(S^{m+1}),S^{m-1};p) & \longrightarrow\cdots\end{array}$$

the commutativity holds, where the sequences are defined in (8·3) *and $i>pn-1$.*

Proof. Remark that the inclusion $\Omega^2(E^2f')(\Omega(S^n_{p-1}))\subset\Omega(S^m_{p-1})$ is not true in general. Let $\bar{f}:S^n_\infty\longrightarrow S^m_\infty$ be the combinatorial extension of $f=Ef'$. Consider the homotopy f_θ in the proof of (2·6). If $x\in S^n\subset S^n_\infty$, then $f_\theta(x)=f(x)$. Thus $\Omega f_\theta:\Omega(S^n_\infty)\longrightarrow\Omega(S^m_\infty)$ is a homotopy such that $\Omega f_\theta\mid\Omega(S^n)=\Omega f$. In particular $f'=\Omega f_\theta\mid S^{n-1}$. Therefore, by (2·6), in the diagram

$$\begin{array}{ccc}\pi_i(\Omega(S^n_\infty),S^{n-1}) & \underset{\approx}{\xrightarrow{\Omega i_*}} & \pi_i(\Omega^2(S^{n+1}),S^{n-1})\\ \downarrow\Omega\bar{f}_* & & \downarrow\Omega^2(E^2f')_*\\ \pi_i(\Omega(S^m_\infty),S^{m-1}) & \underset{\approx}{\xrightarrow{\Omega i_*}} & \pi_i(\Omega^2(S^{m+1}),S^{m-1})\end{array}$$

the commutativity holds. From the commutativity of the diagram

$$\begin{array}{ccccccc}\cdots\longrightarrow\pi_i(\Omega(S^n_\infty),\Omega(S^n_{p-1})) & \longrightarrow & \pi_{i-1}(\Omega(S^n_{p-1}),S^{n-1}) & \longrightarrow & \pi_{i-1}(\Omega(S^n_\infty),S^{n-1}) & \longrightarrow\cdots,\\ \downarrow\Omega\bar{f}_* & & \downarrow\Omega\bar{f}_* & & \downarrow\Omega\bar{f}_* & \\ \cdots\longrightarrow\pi_i(\Omega(S^m_\infty),\Omega(S^m_{p-1})) & \longrightarrow & \pi_{i-1}(\Omega(S^m_{p-1}),S^{m-1}) & \longrightarrow & \pi_{i-1}(\Omega(S^m_\infty),S^{m-1}) & \longrightarrow\cdots,\end{array}$$

we see that it is sufficient to prove the commutativity of the following two diagrams

$$(8\cdot4)'\quad\begin{array}{ccccc}\pi_i(\Omega(S^n_\infty),\Omega(S^n_{p-1});p) & \xleftarrow{\Omega'} & \pi_{i+1}(S^n_\infty,S^n_{p-1};p) & \xrightarrow{H'_p} & \pi_{i+2}(S^{pn+1};p)\\ \downarrow\Omega\bar{f}_* & & \downarrow\bar{f}_* & & \downarrow E^2F'_*\\ \pi_i(\Omega(S^m_\infty),\Omega(S^m_{p-1});p) & \xleftarrow{\Omega'} & \pi_{i+1}(S^m_\infty,S^m_{p-1};p) & \xrightarrow{H'_p} & \pi_{i+2}(S^{pm+1};p)\end{array}$$

and (8·4)″

$$\begin{array}{ccc} \pi_{i-1}(\Omega(S^n_{p-1}),\, S^{n-1};\, p) & \xrightarrow{\bar{H}_p} & \pi_i(S^{pn-1};\, p) \\ \downarrow{\scriptstyle \Omega\bar{f}_*} & & \downarrow{\scriptstyle F'_*} \\ \pi_{i-1}(\Omega(S^m_{p-1}),\, S^{m-1};\, p) & \xrightarrow{\bar{H}_p} & \pi_i(S^{pm-1};\, p). \end{array}$$

By (2·8)′, the maps E^2F' and $E(f)^k$ are homotopic to each other. Then the commutativity of (8·4)′ follows from (1·2)′ and (2·9). The commutativity of (8·4)″ follows from (7·9). q. e. d.

For a map $f: (X, x_0) \longrightarrow (Y, y_0)$ we define a mapping-cylinder

$$Y_f = X\times[0, 1) \cup Y$$

by identifying a space $X\times I \cup Y$ by the relations $(x, 1) \equiv f(x)$, $x\in X$ and $(x_0, t) \equiv y_0$, $t\in I$. Here we state several elementary properties of the group $\pi_i(X_f, X)$.

(8·5), i) *If f is an infection*: $X \subset Y$, *then* $\pi_i(Y_f, X) \approx \pi_i(Y, X)$.

ii) *If* $f\simeq g: (X, x_0) \longrightarrow (Y, y_0)$, *then* $\pi_i(Y_f, X) \approx \pi_i(Y_g, X)$.

iii) $\pi_{i+1}(Y_f, X) \approx \pi_i(\Omega(Y)_{\Omega f}, \Omega(X))$.

iv) *For maps* $f: X \longrightarrow Y$ *and* $g: Y \longrightarrow Z$, *we have an exact sequence*

$$\cdots \longrightarrow \pi_i(Y_f, X) \longrightarrow \pi_i(Z_{g\circ f}, X) \longrightarrow \pi_i(Z_g, Y) \longrightarrow \pi_{i-1}(Y_f, X) \longrightarrow \cdots.$$

Proof. i) Define a map $F: Y_f \longrightarrow Y$ be setting $F(y) = y$, $y\in Y$ and $F(x, t) = x$, $x\in X$, $t\in I$. Consider the following diagram

$$\begin{array}{ccccccc} \cdots\longrightarrow \pi_i(X) & \longrightarrow & \pi_i(Y_f) & \longrightarrow & \pi_i(Y_f, X) & \longrightarrow\cdots \\ \uparrow{\scriptstyle (F\mid X)_*} & & \uparrow{\scriptstyle F_*} & & \uparrow{\scriptstyle F_{**}} & \\ \cdots\longrightarrow \pi_i(X) & \longrightarrow & \pi_i(Y) & \longrightarrow & \pi_i(Y, X) & \longrightarrow\cdots. \end{array}$$

$(F\mid X)_*$ is an isomorphism since $F\mid X$ is the identity. F_* is an isomorphism since F is a (deformation) retraction. Applying the five lemma to the above diagram, we have that F_{**} is an isomorphism.

ii). It is not so difficult to prove that the pairs (Y_f, X) and (Y_g, X) have the same homotopy type. Then $\pi_i(Y_f, X) \approx \pi_i(Y_g, X)$.

iii). Define a map $F: \Omega(Y)_{\Omega f} \longrightarrow \Omega(Y_f)$ by the formulas $F(x, t)(u) = (x(u), t)$, $x\in\Omega(X)$, $t, u\in I$ and $F(y)(u) = y(u)$, $y\in\Omega(Y)$, $u\in I$. Since Y is a deformation retract of Y_f, $\Omega(Y)$ is a deformation retract of $\Omega(Y_f)$. Also $\Omega(Y)$ is a deformation retract of $\Omega(Y)_{\Omega f}$. Since $F\mid\Omega(Y)$ is the identity, we have the following commutative diagram

$$\begin{array}{ccc} \pi_i(\Omega(Y)_{\Omega f}) & \xrightarrow{F_*} & \pi_i(\Omega(Y_f)) \\ \nwarrow & & \nearrow \\ & \pi_i(\Omega(Y)). & \end{array}$$

F_* is an isomorphism since the other (injection) homomorphisms are isomorphisms. $(F\mid\Omega(X))_*$ is an isomorphism since $F\mid\Omega(X)$ is the identity. Then similar methods to i) shows that $F_{**}: \pi_i(\Omega(Y)_{\Omega f}, \Omega(X)) \approx \pi_i(\Omega(Y_f), \Omega(X))$. By (1·1)′,

$\pi_i(\Omega(Y_f), \Omega(X)) \approx \pi_{i+1}(Y_f, X)$, and then we have the isomorphism of iii).

iv) Consider a mapping-cylinder $(Z_g)_f$ of the map $f: X \longrightarrow Y \subset Z_g$. Since Z_g is a deformation retract of $(Z_g)_f$, we have an isomorphism $\pi_i(Z_g, Y) \approx \pi_i((Z_g)_f, Y_f)$. As is easily seen, the pairs $((Y_g)_f, X)$ and $(Z_{g\circ f}, X)$ have the same homotopy type, and $\pi_i((Z_g)_f, X) \approx \pi_i(Z_{g\circ f}, X)$. Then the sequence of iv) is equivalent to the homotopy exact sequence of the triple $((Z_g)_f, Y_f, X)$. q. e. d.

As a corollary of (8・5) we have the following lemma.

(8・6) *For three maps $f: X \longrightarrow Y$, $g: Y \longrightarrow Z$ and $h: X \longrightarrow Z$ suppose that $h \simeq g\circ f$. Let $\mathcal{C}$ be a class of abelian groups.*

i) *If f induces $\mathcal{C}$-isomorphisms of the homotopy groups, then the homotopy groups of the pairs (Z_g, Y) and (Z_h, X) are $\mathcal{C}$-isomorphic for each dimension.*

ii) *If g induces $\mathcal{C}$-isomorphisms of the homotopy groups, then the homotopy groups of the pairs (Y_f, X) and (Z_h, X) are $\mathcal{C}$-isomorphic for each dimension.*

iii) *If h induces $\mathcal{C}$-isomorphisms of the homotopy groups, then the homotopy groups of the pairs (Y_f, X) and (Z_g, Y) are $\mathcal{C}$-isomorphic for each dimension.*

Proof. By, (8・5), ii) we may suppose that $h = g\circ f$. If f induces $\mathcal{C}$-isomorphisms of the homotopy groups, then $\pi_i(Y_f, X) \in \mathcal{C}$ for all i. It follows from the exactness of the sequence (8・5), iv) that the homomorphism $\pi_i(Z_h, X) \longrightarrow \pi_i(Z_g, Y)$ is a $\mathcal{C}$-isomorphism for all i. The proof of ii) and) iii) is similar. q. e. d.

THEOREM (8・7). *Let n be even and let p be an odd prime. Let $f_p: S^{pn-1} \longrightarrow S^{pn-1}$ be a map of degree p and let $S^{pn-1}_{f_p}$ be the mapping cylinder of f_p. Then there is an exact sequence*

$$\cdots \longrightarrow \pi_i(\Omega^2(S^{pn+1}), S^{pn-1}: p) \longrightarrow \pi_i(S^{pn-1}_{f_p}, S^{pn-1}) \longrightarrow$$
$$\pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1}; p) \longrightarrow \pi_{i-1}(\Omega^2(S^{pn+1}), S^{pn-1}; p) \longrightarrow \cdots.$$

COROLLARY (8・7)′.

$$\pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1}; p) \begin{cases} =0 \ \textit{for} \ i < pn-1, \\ \approx \pi_i(S^{pn-1}_{f_p}, S^{pn-1}) \ \textit{for} \ i < p^2n-2. \end{cases}$$

Proof. Let E be a space of singular 2-cubes given by

$E = \{f: I^2 \longrightarrow \Omega(S^n_\infty) \mid f(I\times(0)) \subset \Omega(S^n_{p-1}), f(0,0) \in S^{n-1}$ and $f(I\times(1) \cup (1)\times I) = e_0\}$.

Then we have two fiberings

$p_1: E \longrightarrow \Omega(\Omega(S^n_{p-1}), S^{n-1})$ with the fibre $\Omega(\Omega^2(S^n_\infty, S^n_\infty))$,

$p_2: E \longrightarrow \Omega(\Omega(S^n_\infty), S^{n-1})$ with the fibre $\Omega^2(\Omega(S^n_\infty), \Omega(S^n_{p-1}))$

which are given by setting $p_1(f)(t) = f(t, 0)$ and $p_2\ (f)(t) = f(0, t)$.

According to the proof of (8・3), we take an attaching map $\chi: S^{pn-1} \longrightarrow S^n_{p-1}$ of e^{pn} and a map $\tilde{\chi}: S^{pn-1} \longrightarrow \Omega(S^n_\infty, S^n_{p-1})$ such that the diagram

$$\begin{tikzcd} \Omega(S^n_\infty, S^n_{p-1}) \arrow[r, "p"] \arrow[d, "\Omega h_p"] & S^n_{p-1} \\ \Omega(S^{pn}_\infty) & S^{pn-1} \arrow[l, "i"] \arrow[u, "\chi"'] \arrow[ul, "\tilde{\chi}"'] \end{tikzcd}$$

is homotopically commutative. Let $\eta : \Omega^2(\Omega(S^n_\infty, S^n_{p-1})) \longrightarrow \Omega^2(\Omega(S^n_\infty), \Omega(S^n_{p-1}))$ be a homeomorphism given by $\eta f(t_1, t_2)(u) = f(t_1, u)(t_2)$, $f \in \Omega^2(\Omega(S^n_\infty, S^n_{p-1}))$, $(t_1, t_2) \in I^2$, $u \in I$. Applying (8·5), iv) to the maps $\Omega^2\tilde{\chi} : \Omega^2(S^{pn-1}) \longrightarrow \Omega^2(\Omega(S^n_\infty, S^n_{p-1}))$ and $i_1 \circ \eta$: $\Omega^2(\Omega(S^n_\infty, S^n_{p-1})) \longrightarrow \Omega^2(\Omega(S^n_\infty), \Omega(S^n_{p-1})) \subset E$, we have an exact sequence

$$\cdots \longrightarrow \pi_{i-2}(\Omega^2(\Omega(S^n_\infty, S^n_{p-1}))_{\Omega^2\tilde{\chi}}, \Omega^2(S^{pn-1})) \longrightarrow \pi_{i-2}(E_{i_1\circ\eta\circ\Omega^2\tilde{\chi}}, \Omega^2(S^{pn-1})) \longrightarrow \pi_{i-2}(E_{i_1\circ\eta}, \Omega^2(\Omega(S^n_\infty, S^n_{p-1}))) \longrightarrow \cdots.$$

In the followings we shall prove that

(8·8), i) *$\pi_{i-2}(\Omega^2(\Omega(S^n_\infty, S^n_{p-1}))_{\Omega^2\tilde{\chi}}, \Omega^2(S^{pn-1}))$ and $\pi_i(\Omega^2(S^{pn+1}), S^{pn-1})$ are $\mathcal{C}_p$-isomorphic,*

ii) *$\pi_{i-2}(E_{i_1\circ\eta\circ\Omega^2\tilde{\chi}}, \Omega^2(S^{pn-1}))$ and $\pi_i(S^{pn-1}_{f_p}, S^{pn-1})$ are $\mathcal{C}_p$-isomorphic,*

iii) *$\pi_{i-2}(E_{i_1\circ\eta}, \Omega^2(\Omega(S^n_\infty, S^n_{p-1})))$ and $\pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1})$ are isomorphic.*

Then these groups are finite and the exactness of the above sequence implies that of the sequence of p-primary components. The $\mathcal{C}_p$-isomorphisms of (8·8) induce isomorphisms of p-primary components. Therefore we have the exact sequence of (8·7) from (8·8).

Proof of (8·8), i). We have a commutative diagram

$$\begin{tikzcd} & \Omega^2(\Omega(S^n_\infty, S^n_{p-1})) & \\ \Omega^3(S^{pn}_\infty) = \Omega^2(\Omega(S^{pn}_\infty)) \arrow[ur, "\Omega^3 h_p"] & & \Omega^2(S^{pn-1}). \arrow[ul, "\Omega^2\tilde{\chi}"'] \arrow[ll, "\Omega^2 i"'] \end{tikzcd}$$

The homomorphism $\Omega^3 h_{p*}$: $\pi_i(\Omega^2(\Omega(S^n_\infty, S^n_{p-1}))) \longrightarrow \pi_i(\Omega^3(S^{pn}))$ is equivalent to the homomorphism $h_{p*} : \pi_{i+3}(S^n_\infty, S^n_{p-1}) \longrightarrow \pi_{i+3}(S^{pn}_\infty)$ which is a $\mathcal{C}_p$-isomorphism by the (2·11). Then, by (8·6), ii), we have that the groups $\pi_{i-2}(\Omega^2(\Omega(S^n_\infty, S^n_{p-1}))_{\Omega^2\tilde{\chi}}, \Omega^2(S^{pn-1}))$ and $\pi_{i-2}(\Omega^3(S^{pn}), \Omega^2(S^{pn-1}))$ are $\mathcal{C}_p$-isomorphic. By (1·2)′ and (2·3)′ we have that $\pi_{i-2}(\Omega^3(S^{pn}_\infty), \Omega^2(S^{pn-1})) \approx \pi_i(\Omega^2(S^{pn+1}), S^{pn-1})$. Then (8·8), i) is proved.

Proof of (8·8), ii). Consider the diagram

$$\begin{tikzcd} \Omega^2(\Omega(S^n_\infty, S^n_{p-1})) \arrow[r, "i_1"] & E \arrow[d, "p_1"] \\ \Omega^2(S^{pn-1}) \arrow[u, "\Omega^2\tilde{\chi}"] \arrow[r, "\Omega^2\chi"] \arrow[d, "\Omega^2 f_p"] & \Omega(\Omega(S^n_{p-1}), S^{n-1}) \arrow[d, "\Omega h"] \\ \Omega^2(S^{pn-1}) \arrow[r, "\Omega i"] & \Omega(Y). \end{tikzcd}$$

The commutativity of the upper square is verified from the definition of mappings. The homotopical commutativity of the lower square is verified from (6·9) and the definition of Y. Since the fibre $\Omega(\Omega^2(S^n_\infty, S^n_\infty))$ is contractible, the fibering p_1 induces isomorphisms of the homotopy groups. Then $\pi_{i-1}(E_{i_1\circ\eta\circ\Omega^2\tilde{\chi}}, \Omega^2(S^{pn-1})) \approx \pi_{i-2}(\Omega(\Omega(S^n_{p-1}), S^{n-1})_{\Omega^2\chi}, \Omega^2(S^{pn-1}))$ by ii) of (8·6). Since Ωh induces $\mathcal{C}_p$-isomorphisms of the homotopy groups by (7·5) and (1·2), the groups $\pi_{i-2}(\Omega(\Omega(S^n_{p-1}), S^{n-1})_{\Omega^2\chi}, \Omega^2$

$(S^{pn-1}))$ and $\pi_{i-2}(\Omega(Y)_{\Omega h\circ\Omega^2\chi}, \Omega^2(S^{pn-1}))$ are $\mathcal{C}_p$-isomorphic by ii) of (8·6). By ii) of (8·5), $\pi_{i-2}(\Omega(Y)_{\Omega h\circ\Omega^2\chi}, \Omega^2(S^{pn-1})) \approx \pi_{i-2}(\Omega(Y)_{\Omega i\circ\Omega^2 f_p}, \Omega^2(S^{pn-1}))$. Since Ωi induces $\mathcal{C}_p$-isomorphisms of homotopy groups by (7·3)′ and (1·2), the groups $\pi_{i-2}(\Omega(Y)_{\Omega i\circ\Omega^2 f_p}, \Omega^2(S^{pn-1}))$ and $\pi_{i-2}(\Omega^2(S^{pn-1})_{\Omega^2 f_p}, \Omega^2(S^{pn-1}))$ are $\mathcal{C}_p$-isomorphic by ii) of (8·6). By iii) of (8·5), $\pi_{i-2}(\Omega^2(S^{pn-1})_{\Omega^2 f_p}, \Omega^2(S^{pn-1})) \approx \pi_i(S^{pn-1}_{f_p}, S^{pn-1})$. Consequently (8·8), ii) is proved.

Proof of (8·8), iii). Since η is a homeomorphism, η induces isomorphisms of the homotopy groups. Then $\pi_{i-2}(E_{i_1\circ\eta}, \Omega^2(\Omega(S^n_\infty, S^n_{p-1}))) \approx \pi_{i-2}(E_{i_1}, \Omega^2(\Omega(S^n_\infty), \Omega(S^n_{p-1}))) \approx \pi_{i-2}(E, \Omega^2(\Omega(S^n_\infty), \Omega(S^n_{p-1})))$ by (8·6), i) and (8·5), i). By the fibering $p_2 : E \longrightarrow \Omega(\Omega(S^n_\infty), S^{n-1})$, we have an isomorphism $p_{2*} : \pi_{i-2}(E, \Omega^2(\Omega(S^n_\infty), \Omega(S^n_{p-1}))) \approx \pi_{i-2}(\Omega(\Omega(S^n_\infty), S^{n-1}))$. By (1·1) and (2·3), $\pi_{i-2}(\Omega(\Omega(S^n_\infty), S^{n-1})) \approx \pi_{i-1}(\Omega(S^n_\infty), S^{n-1}) \approx \pi_{i-1}(\Omega^2(S^{n+2}), S^{n-1})$. Then (8·8), iii) is proved. q. e. d.

Consider the exact sequence of the homotopy groups of the pair $(S^{pn-1}_{f_p}, S^{pn-1})$: $\cdots \longrightarrow \pi_i(S^{pn-1}) \xrightarrow{i_*} \pi_i(S^{pn-1}_{f_p}) \longrightarrow \pi_i(S^{pn-1}_{f_p}, S^{pn-1}) \longrightarrow \cdots$. The injection homomorphism i_* is equivalent to the homomorphism $f_{p*} : \pi_i(S^{pn-1}) \longrightarrow \pi_i(S^{pn-1})$ induced by f_p.

(8·9) *Let m be even and let q be an integer. Let $f_q : S^{m+1} \longrightarrow S^{m+1}$ be a map of degree q. If p is an odd prime and if $\alpha \in \pi_i(S^{m+1}; p)$, then $f_{q*}(\alpha) = q\alpha$.*

Proof. By (7·1). the suspension homomorphism E maps $\pi_i(S^{m+1}; p)$ isomorphically into $\pi_{i+1}(S^{m+2})$. Then the fact $E(f_{q*}(\alpha)) = E(q\alpha)$ implies that $f_{q*}(\alpha) = q\alpha$. q. e. d.

We see that the kernel and the cokernel of the homomorphism f_{p*} consist of the elements of order p. Therefore

$$p^2(\pi_i(S^{pn-1}_{f_p}, S^{pn-1})) = 0$$

and then

$$p^2(\pi_{i-2}(\Omega^2(S^{n+1}), S^{n-1}; p)) = 0 \qquad \textit{for } i < p^2n-2.$$

From the exactness of the sequence (8·2)′, we have that

$$E^2(\pi_{i-1}(S^{n-1}; p)) \supset p^2(\pi_{i+1}(S^{n+1}; p)) \quad \textit{for } i < p^2n-2.$$

More generally we have that

THEOREM (8·10). *$E^2(\pi_{i-1}(S^{n-1}; p)) \supset p^2(\pi_{i+1}(S^{n+1}; p))$ for all i (n: even, p: odd prime).*

Since $\pi_i(S^1; p) = 0$,

COROLLAY (8·11) $p^n(\pi_i(S^{n+1}; p)) = 0$ *for all i.*

Proof of (8·10). From the exactness of the sequence (8·2), it is sufficient to prove that $p^2(J(\alpha)) = 0$ for arbitrary $\alpha \in \pi_{i+1}(S^{n+1}; p)$. We may suppose that $i > pn-1$ since $\pi_{pn-2}(\Omega^2(S^{n+1}), S^{n-1}; p) \approx Z_p$ and $\pi_i(\Omega^2(S^{n+1}), S^{n-1}; p) = 0$ for $i < pn-1$ by (8·7)′. Let $f' : (S^{n-1}, e_0) \longrightarrow (S^{n-1}, e_0)$ be a map of degree p. From the commutativity of the diagram

$$\begin{array}{ccc}
\pi_{i+1}(S^{n+1};\ p) & \xrightarrow{J} & \pi_{i-1}(\Omega^2(S^{n+1}),\ S^{n-1};\ p) \\
\big\downarrow E^2f'_* & & \big\downarrow \Omega^2(E^2f')_* \\
\pi_{i+1}(S^{n+1};\ p) & \xrightarrow{J} & \pi_{i-1}(\Omega^2(S^{n+1}),\ S^{n-1};\ p)
\end{array}$$

we have that $\Omega^2(E^2f')_*(J(\alpha))=J(E^2f'_*(\alpha))=J(p\alpha)=pJ(\alpha)$ by (8·9). In the theorem (8·4), the map F': $S^{pn-1}\longrightarrow S^{pn-1}$ is a map of degree p^p. From the commutativity (8·4) of the diagram

$$\begin{array}{ccc}
\pi_{i-1}(\Omega^2(S^{n+1}),\ S^{n-1};\ p) & \xrightarrow{I} & \pi_{i+1}(S^{pn+1};\ p) \\
\big\downarrow \Omega^2(E^2f')_* & & \big\downarrow E^2F'_* \\
\pi_{i-1}(\Omega^2(S^{n+1}),\ S^{n-1};\ p) & \xrightarrow{I} & \pi_{i+1}(S^{pn+1};\ p)
\end{array}$$

we have that $I((p^p-p)J(\alpha))=p^pI(J(\alpha))-I(pJ(\alpha))=E^2F'_*(I(J(\alpha)))-I(\Omega^2(E^2f')_*(J(\alpha)))=0$ $(E^2F'_*\gamma=p^p\gamma$ by (8·9)). From the exactness (8·3) of the sequence

$$\pi_i(S^{pn-1};\ p)\xrightarrow{I'}\pi_{i-1}(\pi^2(S^{n+1}),\ S^{n-1};\ p)\xrightarrow{I}\pi_{i+1}(S^{pn+1};\ p),$$

there exists an element β of $\pi_i(S^{pn-1};\ p)$ such that $I'(\beta)=(p^p-p)J(\alpha)$. From the commutativity (8·4) of the diagram

$$\begin{array}{ccc}
\pi_i(S^{pn-1};\ p) & \xrightarrow{I'} & \pi_{i-1}(\Omega^2(S^{n+1}),\ S^{n-1};\ p) \\
\big\downarrow F'_* & & \big\downarrow \Omega^2(E^2f')_* \\
\pi_i(S^{pn-1};\ p) & \xrightarrow{I'} & \pi_{i-1}(\Omega^2(S^{n+1}),\ S^{n-1};\ p)
\end{array}$$

we have that $(p^p-p)^2J(\alpha)=(p^p-p)I'(\beta)=I'(p^p\beta)-pI'(\beta)=I'(F'_*(\beta))-p(p^p-p)J(\alpha)=\Omega^2(E^2f')_*(I'(\beta))-J(p(p^p-p)(\alpha))=\Omega^2(E^2f')_*(J((p^p-p)(\alpha)))-J(p(p^p-p)(\alpha))=p(J((p^p-p)(\alpha)))-J(p(p^p-p)(\alpha))=0$ by (8·9). Since $p^{p-1}-1\not\equiv 0$ (mod. p), $(p^p-p)^2J(\alpha)=(p^{p-1}-1)^2J(p^2\alpha)=0$ implies that $J(p^2\alpha)=0$. Then the theorem (8·10) is proved. q. e. d.

Appendix

Here we list the following values of the group $\pi_i(S^{2m+1};\ p)$ for *an odd prime p*.

i) $1\leqq k\leqq p-1$,

$$\pi_{2m+2k(p-1)-1}(S^{2m+1};\ p)=\begin{cases} Z_p, & 1\leqq m\leqq k-1, \\ 0, & k\leqq m, \end{cases}$$

$$\pi_{2m+2k(p-1)}(S^{2m+1},\ p)=Z_p,\qquad 1\leqq m,$$

ii) $(k=p)$

$$\pi_{2m+2p(p-1)-1}(S^{2m+1};\ p)=\begin{cases} Z_{pm}, & 1\leqq m\leqq p-1, \\ Z_{pp-1}, & p-1\leqq m, \end{cases}$$

$$\pi_{2m+2p(p-1)}(S^{2m+1};\ p)=\begin{cases} Z_{pm}, & 1\leqq m\leqq p, \\ Z_{pp}, & p\leqq m. \end{cases}$$

iii) $(k=p+1)$

$$\pi_{2m+2(p+1)(p-1)-2}(S^{2m+1};\ p)=Z_p,\qquad 1\leqq m,$$

$$\pi_{2m+2(p+1)(p-1)-1}(S^{2m+1};\ p)=\begin{cases} Z_p, & 1\leqq m\leqq p, \\ 0, & p+1\leqq m, \end{cases}$$

$\pi_{2m+2(p+1)(p-1)}(S^{2m+1};\ p) \quad = \quad Z_p, \qquad\qquad 1 \leqq m,$

iv) $\pi_i(S^{2m+1};\ p)=0$ otherwise for $i<2m+2(p+2)(p-1)-3$.

These resulats are caluculated, by making use of $(8\cdot 2)'$ and $(8\cdot 3)$, from the results of H. Cartan for the stable case. His proofs are cohomological and not yet published, however, the author note that the proofs are made from the results for $H^*(\Pi,\ n,\ Z_p)$[5] and the relations of Adem [1], [6] without difficulties and rather automatically. (*cf.* [11] for $p=2$).

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