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Cohomology theory of a complex with a transformation of prime period and its applications

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Let W be a complex with a transformation t of prime period p, and denote by W_t the orbit space over W relative to t. This paper is concerned with a study of certain relations between the cohomology of W and of W_t. As its applications, the cohomology of the p-fold cyclic product of a complex and of the 3-fold symmetric product of a sphere will be considered. Such studies for the homology groups were first raised by M. Richardson and P. A. Smith [11] who introduced the notion of the special homology group. Recently S. D. Liao [5] studied the cohomology of the p-fold cyclic product of a sphere. Their, and also more extensive, results will be proved in the present paper by using of the systematic methods which are essentially due to R. Thom [18], W. T. Wu [19] and R. Bott [2]. The original papers of Thom and Wu are not easy reading, therefore we shall explain their theory in a complete form. Our exposition makes only use of the well-known simplicial cohomology theory.

In Chapter I, the theory is developed on a complex with a transformation of prime period. § 1 devoted to the exposition of the Smith-Richardson sequence and to its direct applications. In § 2, we define the basic homomorphism μ, ν and ϕ_0^*, and study their properties. We establish in § 3 certain relations of the basic homomorphisms to the well-known cohomology operations: the cup product, the squaring operation, the reduced power and the Bockstein homomorphism. § 4 and § 5 are devoted to the proof of certain theorems in § 3. In § 6 we define the notion of regularity and almost regularity, and prove the structure theorems.

Let K be a complex, and $\mathfrak{X}_{(p)}(K)$ the p-fold cartesian product of K. Denote by $\mathfrak{t}$ the transformation on $\mathfrak{X}_{(p)}(K)$ defined by the cyclic permutation of coordinates. Then, in Chapter II, the general theory in Chapter I is applied to the complex with the transformation $\mathfrak{t}$. The orbit space over $\mathfrak{X}_{(p)}(K)$ relative to $\mathfrak{t}$ is the p-fold cyclic product of K. After some preliminaries on the cohomology of the cartesian product given in § 7, we prove in § 8 that the pair $(\mathfrak{X}_{(p)}(K), \mathfrak{t})$ is almost regular in each dimension. In § 9 and § 10, we determine the structure of the kernel of the homomorphism induced by the projection of the cartesian product onto the cyclic product. Reduction formulas which stand deep relations with the reduced power of Steenrod are obtained in § 12. In the final theorem of this section, it is proved that the reduced power is characterized by the well-known properties. § 11 and § 13 are devoted to determine the cohomology of the p-fold cyclic product of a complex. The cohomology groups with coefficients in a field and the well-known cohomology operations are calculated. As for the integral cohomology groups of the cyclic product, we determine only those of certain special complexes.

In Chapter III we determine the cohomology of the 3-fold symmetric products of a sphere. The integral homology groups and the well-known cohomology operations will be given in explicite form.

Preliminary reports of our results have been published in [8, 9].

CHAPTER I. COHOMOLOGY OF ORBIT SPACES

1. Special cohomology group

Let W be a finite simplicial complex, and let $t: W \longrightarrow W$ be a periodic transformation with prime period p. Let us moreover suppose that t satisfies the conditions:

a) t is simplicial,

b) If a simplex is mapped onto itself by t, it remains point-wise fixed.

Then it is easily shown that the set $F=F(t)$ of fixed points under t form a subcomplex of W. Let W' be an arbitrary subcomplex of W invariant under t, and let G be an abelian group. Then t gives rise to a cochain map t^* in the group $C^r(W, W'; G)$ of r-cochains of the piar (W, W') with coefficient group G. Let σ, τ be cochain maps defined by

$$\sigma = \sum_{j=0}^{p-1} t^{*j}, \qquad \tau = 1 - t^*$$

respectively. We shall also denote these maps by ρ and $\bar{\rho}$ agreeing that ρ may stand for σ, $\bar{\rho}$ for τ or *vice versa,* but that the meaning of ρ and $\bar{\rho}$ shall remain fixed in any given discussion. Then we have $\rho\bar{\rho}=0$. Let ${}^{\rho}C^r(W, W'; G)$ and ${}^{\rho^{-1}}C^r(W, W'; G)$ denote respectively the image and the kernel of the map $\rho: C^r(W, W'; G) \longrightarrow C^r(W, W'; G)$. Then ${}^{\rho^{\varepsilon}}C^r(W, W'; G)$ ($\varepsilon=1$ or -1) for all r form a cochain complex under the coboundary δ in W, and hence we may define the cohomology group of ${}^{\rho^{\varepsilon}}C^r(W, W'; G)$, to be denoted by ${}^{\rho^{\varepsilon}}H^r(W, W'; G)$. This group is called *the special ρ^{ε}-cohomology group* of (W, W') with coefficients in G.

Since we have an exact sequence of cochain complexes

$$0 \longrightarrow {}^{\rho^{-1}}C^*(W, W'; G) \xrightarrow{i^*} C^*(W, W'; G) \xrightarrow{\rho} {}^{\rho}C^*(W, W'; G) \longrightarrow 0,$$

(i^*: inclusion homomorphism), we obtain by the well-known theorem [3, Chap. V] the following:

THEOREM (1·1). *The following sequence is exact:*

$$\cdots \longrightarrow {}^{\rho^{-1}}H^r(W, W'; G) \xrightarrow{\alpha_\rho} H^r(W, W'; G) \xrightarrow{\beta_\rho} {}^{\rho}H^r(W, W'; G)$$

$\xrightarrow{\gamma_\rho} {}^{\rho^{-1}}H^{r+1}(W, W'; G) \longrightarrow \cdots,$

where α_ρ *and* β_ρ *are respectively the homomorphisms induced by* i^* *and* ρ, *and* γ_ρ *is the homomorphism which sends a cohomology class containing* ρu *to a cohomology class containing* δu.

This is usualy called the *Smith-Richardson sequence* [4, 13, 18].

Let Π be the cyclic group of order p generated by t. Then Π operates freely on the r-dimensional integral chain group $C_r(W, W' \cup F)$. Let $\Omega_r = \Omega_r(W, W' \cup F)$ be a Π-free base for this group.

LEMMA (1·2). *We have*

(i) $\quad {}^{\tau^{-1}}C^r(W, W'; G) = {}^{\sigma}C^r(W, W'; G) + C^r(W' \cup F, W'; G),$ [1]

(ii) $\quad {}^{\sigma^{-1}}C^r(W, W'; G) = {}^{\tau}C^r(W, W'; G) + C^r(W' \cup F, W'; {}_pG).$ [2]

Proof. Let $u \in {}^{\rho^{-1}}C^r(W, W'; G)$, and $u = u_1 + u_2$ where $u_1 \in C^r(W, W' \cup F; G)$, $u_2 \in C^r(W' \cup F, W'; G)$.

Case 1: $\rho = \tau$. Denote by $t_\divideontimes$ the chain map induced by t. Since $\tau u = \tau u_1 = 0$ we have

$$u_1(x) = u_1(t_\divideontimes x) = \cdots = u_1(t_\divideontimes^{p-1} x) \quad \text{for} \quad x \in \Omega_r,$$

Define now $v \in C^r(W, W' \cup F; G)$ by

$$v(x) = u_1(x), \qquad v(t_\divideontimes^i x) = 0 \ (i \neq 0) \quad \text{for} \quad x \in \Omega_r.$$

Then $u_1 = \sigma v$ is obvious. Since $v \in C^r(W, W'; G)$; we have $u_1 \in {}^{\sigma}C^r(W, W'; G)$. Thus we see that ${}^{\tau^{-1}}C^r(W, W'; G) \subset {}^{\sigma}C^r(W, W'; G) + C^r(W' \cup F, W'; G)$. The inverse inclusion is obvious. This proves (i).

Case 2: $\rho = \sigma$. Since $\sigma u = 0$, we have

$$\textstyle\sum_{j=0}^{p-1} u_1(t_\divideontimes^j x) = 0 \quad \text{for} \quad x \in \Omega_r,$$
$$p u_2(x) = 0 \quad \text{for} \quad x \in C_r(W' \cup F, W').$$

Therefore $u_2 \in C^r(W' \cup F, W'; {}_pG)$. On the other hand, if we define $v \in C^r(W, W' \cup F; G)$ by

$$v(t_\divideontimes^i x) = \textstyle\sum_{j=i}^{p-1} u_1(t_\divideontimes^j x)$$

for any $x \in \Omega_r$ and $i = 0, 1, \ldots, p-1$, then it is obvious that $u_1 = \tau v$. Thus we have $u_1 \in {}^{\tau}C^r(W, W'; G)$, and hence ${}^{\sigma^{-1}}C^r(W, W' \cup F; G) \subset {}^{\tau}C^r(W, W'; G) + C^r(W' \cup F, W'; {}_pG)$. The inverse inclusion is obvious. This proves (ii).

THEOREM (1·3). (i). *If* $W' \supset F$, *then for any* G

$${}^{\bar\rho}H^r(W, W'; G) = {}^{\rho^{-1}}H^r(W, W'; G).$$

1) Let B and C be subgroups of an abelian group A, then we denote by $B+C$ a subgroup of A generated by B and C. If $B+C$ is the direct sum of B and C, we denote it by $B \oplus C$.

2) We write ${}_pG = \{g \in G \mid pg = 0\}$, $pG = \{pg \mid g \in G\}$ and $G_p = G/pG$.

(ii) *If G is a field of characteristic q, not a divisor of p, then for any W'*

$${}^{\bar\rho}H^r(W, W'; G) = {}^{\rho^{-1}}H^r(W, W'; G).$$

Proof. Under the assumption, it follows from (1·2) that ${}^{\rho^{-1}}C^r(W, W'; G) = {}^{\bar\rho}C^r(W, W'; G)$. In the case (ii), note that $C^r(W' \cup F, W'; G) = pC^r(W' \cup F, W'; G) = {}^{\sigma}C^r(W' \cup F, W'; G)$. From this, (1·3) is obvious.

The following is obvious.

LEMMA (1. 4). *Under the same assumption as in* (1. 3), *we have*

$$\alpha_{\bar\rho}\beta_\rho = \rho^*,$$

where $\rho^*: H^r(W, W'; G) \longrightarrow H^r(W, W'; G)$ is the homomorphism induced by ρ.

Let $Q_\rho : C^r(W, W'; G) \longrightarrow C^r(W, W'; G)$ be a cochain map defined by

$$(1\cdot5) \qquad \begin{aligned} &Q_\sigma = identity, \\ &Q_\tau = \textstyle\sum_{j=2}^{p}(-1)^j {}_pC_j\, \tau^{j-2}, \end{aligned}$$

where ${}_pC_j$ denotes the binomial coefficient. Then we have

LEMMA (1·6). $p\rho = Q_\rho\rho^2$.

Proof. Expand $\sigma^2 = (1+t^* + \cdots + t^{*p-1})^2$ and $t^{*p} = (1-\tau)^p$. Then we obtain (1·6) by easy calculations.

THEOREM (1·7). *Let a be any element of ${}^{\rho^{-1}}H^{r+1}(W, W'; G)$ which is contained in the image of Υ_ρ. Then we have $p(a) = 0$.*

Proof. For this purpose, it is sufficient to prove the following: Given a cochain $u \in C^r(W, W'; G)$ such that $\delta\rho u = 0$, there is a cochain $v \in C^r(W, W'; G)$ such that $p\delta u = \delta\bar\rho v$. In fact, we can take $v = Q_{\bar\rho}u$, as is proved in the following.

Case 1: $\rho = \tau$. Since $\tau\delta u = 0$, it follows from (1·2) that there exists $v \in C^r(W, W'; G)$ and $w \in C^r(W' \cup F, W'; G)$ such that $\delta u = \sigma v + w$. Then it follows from (1·6) and $\sigma w = pw$ that

$$\begin{aligned} p\delta u &= p\sigma v + pw = \sigma^2 v + pw \\ &= \sigma(\delta u - w) + pw = \sigma\delta u - \sigma w + pw \\ &= \delta\sigma Q_\sigma u. \end{aligned}$$

Case 2: $\rho = \sigma$. Since $\sigma\delta u = 0$, it follows from (1·2) that there exists $v \in C^r(W, W'; G)$ and $w \in C^r(W' \cup F, W'; G)$ such that $\delta u = \tau v + w$ and $pw = 0$. Then it follows from (1·6)

$$\begin{aligned} p\delta u &= p\tau v + pw = Q_\tau \tau^2 v = Q_\tau\tau(\delta u - w) \\ &= Q_\tau\tau\delta u = \delta\tau Q_\tau u. \end{aligned}$$

This completes the proof.

Let G be a field of characteristic q, not a divisor of p. Then it follows from (1·7) that $\Upsilon_\rho : {}^{\rho}H^{r-1}(W, W'; G) \longrightarrow {}^{\rho^{-1}}H^r(W, W'; G)$ is trivial for any r. Hence α_ρ is isomorphic into, β_ρ is onto, On the other hand, we have by (1·4) the commutative diagram

$$\begin{tikzcd} & H^r(W, W'; G) \arrow[dl, "\beta_{\bar\rho}"'] \arrow[dr, "\bar\rho^*"] & \\ {}^{\rho^{-1}}H^r(W, W'; G) \arrow[rr, "\alpha_\rho"] & & H^r(W, W'; G). \end{tikzcd}$$

Therefore we obtain

THEOREM (1·8). *Let G be a field of characteristic q, not a divisor of p. Then, for any r, the homomorphism α_ρ is isomorphic into, and its image is $\bar\rho^* H^r(W, W'; G)$. Moreover an element of ${}^{\rho}H^r(W, W'; G)$ is represented by $\beta_\rho(a)$ with $a \in H^r(W, W'; G)$.*

Denote by $W_t = O(W, t)$ the orbit space over W relative to t (*i.e.* the space obtained by identifying any two points x, x' of W into a single point whenever $x' = t^i(x)$ for some i), and let $\pi: W \longrightarrow W_t$ be the identification map. Then we can use as a simplicial decomposition of W_t the images of the simplexes of W in virtue of the assumptions a) and b). Thus W_t is a simplicial complex and π becomes a simplicial map[3]. Moreover $W_t' = O(W', t)$ and $F_t = O(F, t)$ are respectively the images by π of W' and F, and these form subcomplexes of W_t. It is obvious that

$$\pi: C^r(W_t, W_t'; G) \approx {}^{\tau^{-1}}C^r(W, W'; G),$$
$$\pi: C^r(F_t; G) \approx C^r(F; G).$$

Hence we have

THEOREM (1·9). $\mathrm{I}^*: H^r(W_t, W_t'; G) \approx {}^{\tau^{-1}}H^r(W, W'; G)$,
$\pi^*: H^r(F_t; G) \approx H^r(F; G)$,

where I^* is the homomorphism induced by π.

It is obvious that

(1·10) $$\alpha_\tau \mathrm{I}^* = \pi^*,$$

where $\pi^*: H^r(W_t\ W_t'; G) \longrightarrow H^r(W, W'; G)$ is the homomorphism induced by π. Thus (1·8) for $\rho = \tau$ yields the following:

THEOREM (1·11). *Under the same assumption as in* (1·8), *the homomorphism π^* is isomorphic into, and its image is $\sigma^* H^r(W, W'; G)$.*

2. Basic homomorphisms

Let $\eta: G \longrightarrow G_p$ be the natural projection[2], and consider the cochain map Q_ρ defined in (1·5). Then it follows from (1·6) and (1·2) that $\eta Q_\rho {}^{\rho}C^r(W, W' \cup F; G) \subset {}^{\bar\rho}C^r(W, W' \cup F; G_p)$. Thus ηQ_ρ induces a homomorphism Q'_ρ of ${}^{\rho}H^r(W, W' \cup F; G)$ to ${}^{\bar\rho}H^r(W, W' \cup F; G_p)$. Let $\varepsilon_\sigma = 1$ and $\varepsilon_\tau = -1$, and write $\psi_{\bar\rho}$ for $\varepsilon_\rho Q'_\rho$:

(2·1) $$\psi_{\bar\rho}: {}^{\rho}H^r(W, W' \cup F; G) \longrightarrow {}^{\bar\rho}H^r(W, W' \cup F; G_p).$$

Then we have

LEMMA (2·2). (i) *ψ_τ (or ψ_σ) sends an element of ${}^{\sigma}H^r(W, W' \cup F; G)$ (or ${}^{\tau}H^r(W, W' \cup F; G)$) containing σu (or τu) to an element of ${}^{\tau}H^r(W, W' \cup F; G_p)$ (or ${}^{\sigma}H^r(W, W' \cup F; G_p)$) containing σu.*

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3) Of course, W_t itself is not necessarily simplicial. In such a case, consider the first barycentric subdivision of W and W_t. Then W_t becomes a simplicial complex such that π is a simplicial map.

(ii) $\psi_\sigma\psi_\tau=0$ $(p\geqq 3)$, *and* $=\eta_*$ $(p=2)$, *where* η_* *is the homomorphism induced by* η.

By the definition of ψ_ρ, this is a direct consequence of the following:

LEMMA (2·3). (i) $\varepsilon_\tau Q_\tau\tau\equiv\varepsilon_\sigma Q_\sigma\sigma=\sigma$ *mod* p.

(ii) $Q_\tau\sigma\equiv 0$ *mod* p $(p\geqq 3)$, *and* $=\sigma$ $(p=2)$.

Proof. It follows from (1·5) that

$$\begin{aligned}\varepsilon_\tau Q_\tau\tau &= \textstyle\sum_{j=2}^{p}(-1)^{j+1}\ {}_pC_j\tau^{j-1}\equiv(-1)^{p+1}\tau^{p-1}\\ &\equiv(t-1)^{p-1}=\textstyle\sum_{j=0}^{p-1}(-1)^{p-j-1}{}_{p-1}C_j\, t^j\\ &\equiv\textstyle\sum_{j=0}^{p-1}(-1)^{p-1}t^j=\sigma=\varepsilon_\sigma Q_\sigma\sigma \text{ mod } p.\end{aligned}$$

(Note that ${}_{p-1}C_j\equiv(-1)^j$ mod p, and that $Q_\sigma=1$.) This proves (i).

Since $\tau\sigma=0$, it follows from (1·5) that

$$\begin{aligned}Q_\tau\sigma &= \textstyle\sum_{j=2}^{p}(-1)^j{}_pC_j\tau^{j-2}\sigma={}_pC_2\sigma\\ &\equiv 0 \text{ mod } p\ (p\geqq 3), \text{ and}=\sigma\ (p=2).\end{aligned}$$

This proves (ii).

Furthermore we have

LEMMA (2·4). (i) $\psi_\rho\Upsilon_\rho=\Upsilon_{\bar\rho}\psi_{\bar\rho}$. (ii) $\psi_\sigma\beta_\tau=\eta_*\beta_\sigma$. (iii) $\alpha_\sigma\psi_\tau=\eta_*\alpha_\tau$.

Proof. (ii) and (iii) are obvious. We shall prove (i). Let $\rho u\in{}^\rho C^r(W, W'\cup F; G)$ be a cocycle representing $a\in{}^\rho H^r(W, W'\cup F; G)$. Then $\psi_{\bar\rho}(a)$ is represented by $\eta\varepsilon_\rho Q_\rho\rho u=\eta\varepsilon_{\bar\rho}Q_{\bar\rho}\bar\rho u$, in virtue of (i) of (2·3). Thus $\Upsilon_{\bar\rho}\psi_{\bar\rho}(a)$ is represented by $\delta\eta\varepsilon_{\bar\rho}Q_{\bar\rho}u=\eta\varepsilon_{\bar\rho}Q_{\bar\rho}\delta u$. On the other hand, $\Upsilon_\rho(a)$ is represented by δu, and hence $\psi_\rho\Upsilon_\rho(a)$ is represented by $\eta\varepsilon_{\bar\rho}Q_{\bar\rho}\delta u$. Thus we have (i). Q. E. D.

Define homomorphisms

$$(2\cdot 5)\qquad \begin{aligned}\mu &: H^r(W_t, W_t'\cup F_t; G)\longrightarrow H^{r+2}(W_t, W_t'\cup F_t; G),\\ \nu &: H^r(W_t, W_t'\cup F_t; G)\longrightarrow H^{r+1}(W_t, W_t'\cup F_t; G_p)\end{aligned}$$

as follows:[4)]

$$(2\cdot 6)\qquad \mu=\mathrm{I}^{*^{-1}}\Upsilon_\tau\Upsilon_\sigma\mathrm{I}^*,\qquad \nu=\mathrm{I}^{*^{-1}}\psi_\sigma\Upsilon_\sigma\mathrm{I}^*=\mathrm{I}^{*^{-1}}\Upsilon_\tau\psi_\tau\mathrm{I}^*.$$

Then we have

THEOREM (2·7). (i) $\nu^2=0$ *if* $p\geqq 3$, *and* $=\eta_*\mu$ *if* $p=2$. (ii) $\mu\nu=\nu\mu$.

Proof. It follows from (2·3) and (2·4) that

$$\begin{aligned}\nu^2=\mathrm{I}^{*^{-1}}\psi_\sigma\Upsilon_\sigma\psi_\sigma\Upsilon_\sigma\mathrm{I}^*=\mathrm{I}^{*^{-1}}\psi_\sigma\psi_\tau\Upsilon_\tau\Upsilon_\sigma\mathrm{I}^* &=0 \quad \text{if } p\geqq 3,\\ &=\eta_*\mu \quad \text{if } p\geqq 2.\end{aligned}$$

This proves (i). (ii) can be proved similarly. Q. E. D.

Let $a\in H^r(W, W'; G)$ be an element whose representative cocycle is u. Then $\rho u=\rho(u\mid W-F)$ mod pG, where $u\mid W-F$ denotes the restriction of u on $W-F$.

4) Note that the definition of ν is given without making use of local coefficients, different from the one given by R. Thom [18].

Therefore ρu is a cocycle of ${}^{\rho}C^{r}(W, W' \cup F; G_{p})$. Moreover, as is easily seen, the class of ${}^{\rho}H^{r}(W, W' \cup F; G_{p})$ containing ρu is independent of the choice of representatives of a. Thus the correspondence $u \longrightarrow \rho u$ provides a homomorphism

(2·8) $$\kappa_{\rho} : H^{r}(W, W'; G) \longrightarrow {}^{\rho}H^{r}(W, W' \cup F; G_{p}).$$

The following is obvious.

(2·9) $$\alpha_{\bar{\rho}}\kappa_{\rho} = \eta_{*}\rho_{0}^{*}, \qquad \psi_{\sigma}\kappa_{\tau} = \kappa_{\sigma},$$

where $\rho_{0}^{*} : H^{r}(W, W'; G) \longrightarrow H^{r}(W, W' \cup F; G)$ is the homomorphism induced by ρ.

Define a homomorphism

$$\phi : C^{r}(W, W'; G) \longrightarrow C^{r}(W_{t}, W_{t}'; G)$$

by

$$\phi u(\pi x) = \sigma u(x), \quad u \in C^{r}(W, W' G),$$

where x is any simplex of W. Then we have

LEMMA (2·10). $\delta\phi = \phi\delta, \qquad \pi\phi = \sigma.$

Proof. Let ∂ denote the boundary operator. Since

$$\delta\phi u(\pi x) = \phi u(\pi\partial x) = \sigma u(\partial x) = \delta\sigma u(x)$$
$$= \sigma\delta u(x) = \phi\delta u(\pi x),$$

we have $\delta\phi = \phi\delta$. $\pi\phi = \sigma$ is obvious. Q. E. D.

By (2·10), ϕ induces a homomorphism

(2·11) $$\phi^{*} : H^{r}(W, W'; G) \longrightarrow H^{r}(W_{t}, W_{t}'; G).$$

Furthermore, since it is obvious that $\eta\phi C^{r}(W, W'; G) \subset C^{r}(W_{t}, W_{t}' \cup F_{t}; G_{p})$, ϕ induces also a homomorphism

(2·12) $$\phi_{0}^{*} : H^{r}(W, W'; G) \longrightarrow H^{r}(W_{t}, W_{t}' \cup F_{t}; G_{p}).$$

The following is obvious.

LEMMA (2·13). $I^{*}\phi_{0}^{*} = \kappa_{\sigma}, \quad j^{*}\phi_{0}^{*} = \eta_{*}\phi^{*},$

where $j^{*} : H^{r}(W_{t}, W_{t}' \cup F_{t}; G_{p}) \longrightarrow H^{r}(W_{t}, W_{t}'; G_{p})$ *is the inclusion homomorphism.*

We shall prove

LEMMA (2·14). *If* $v \in C^{r}(F; G)$, *then* $\tau\delta v = 0$ *in* W.

Proof. Let c be an oriented $(r+1)$-simplex of W, and let

$$\partial c = \Sigma_{i}\alpha_{i}x_{i} + \Sigma_{j}\beta_{j}y_{j}, \qquad (\alpha_{i}, \beta_{j} : \text{integers}),$$

where x_{i}, y_{j} are oriented r-simplexes of $W-F$, F respectively. Then we have

$$\tau\delta v(c) = v(\tau\partial c) = v(\partial c) - v(\partial tc)$$
$$= \Sigma_{i}\alpha_{i}(v(x_{i}) - v(tx_{i})) + \Sigma_{j}\beta_{j}(v(y_{j}) - v(ty_{j})) = 0,$$

since $y_{j} = ty_{j}$ and $v(x_{i}) = v(tx_{i}) = 0$.

Let $b \in H^{r}(W' \cup F, W'; G)$ and let v be a representative of b. Then $\tau\delta v = 0$ by (2·14), and $\delta v \in Z^{r+1}(W, W' \cup F; G)$. Thus δv is a cocycle of ${}^{\tau^{-1}}C^{r+1}(W, W' \cup F; G) = {}^{\sigma}C^{r+1}(W, W' \cup F; G)$. Moreover, as is easily proved, the class of ${}^{\sigma}H^{r+1}(W, W' \cup F; G)$ containing δv is independent of the choice of representatives of b. Thus the correspondence $v \longrightarrow \delta v$ provides a homomorphism

$$\vartheta : H^{r}(W' \cup F, W'; G) \longrightarrow {}^{\sigma}H^{r+1}(W, W' \cup F; G)$$

Let us now define

$$(2\cdot 15) \qquad \vartheta_\rho : H^r(W'\cup F,\ W'; G) \longrightarrow {}^{\bar\rho}H^{r+1}(W,\ W'\cup F; G_p)$$

by

$$\vartheta_\tau = \eta_*\vartheta, \qquad \vartheta_\sigma = \psi_\tau\vartheta.$$

Then, by the definitions and (2·2), we have

LEMMA (2·16). (i) $\psi_\tau\vartheta_\tau = \vartheta_\sigma$.

(ii) $\psi_\sigma\vartheta_\sigma = 0$ $(p\geqq 3)$, *and* $=\vartheta_\tau$ $(p=2)$.

Consider the diagram[5]

$$\begin{array}{ccccc}
\cdots \longrightarrow {}^{\rho}H^r(W,\ W'\cup F; G_p) & \xrightarrow{\alpha_\rho} & H^r(W,\ W'\cup F; G_p) & \xrightarrow{\beta_\rho} & \\
\uparrow \vartheta_\rho & & \uparrow \eta_* & & \\
\cdots \longrightarrow H^{r-1}(W'\cup F,\ W'; G) & \xrightarrow{\delta^*} & H^r(W,\ W'\cup F; G) & \xrightarrow{j^*} &
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow {}^{\rho}H^r(W,\ W'\cup F; G_p) & \xrightarrow{\gamma_\rho} & {}^{\bar\rho}H^{r+1}(W,\ W'\cup F; G_p) & \longrightarrow \cdots \\
\uparrow \kappa_\rho & & \uparrow \vartheta_\rho & \\
\longrightarrow H^r(W,\ W';\ G) & \xrightarrow{i^*} & H^r(W'\cup F,\ W'; G) & \longrightarrow \cdots.
\end{array}$$

Then we have

LEMMA (2·17). (i) $\alpha_\rho\vartheta_\rho = \eta_*\delta^*$. (ii) $\beta_\rho\eta_* = \kappa_\rho j^*$. (iii) $\gamma_\rho\kappa_\rho = -\vartheta_\rho i^*$.

Proof. (i) and (ii) are obvious. (iii) is proved as follows: Let $a\in H^r(W,\ W';\ G)$ be a class whose representative is u, and let $u=u_1+u_2$, where $u_1\in C^r(W,\ W'\cup F;\ G)$ and $u_2\in C^r(W'\cup F,\ W'; G)$. Then we have $\eta\rho u=\eta\rho u_1$, and hence $\gamma_\rho\kappa_\rho(a)$ is represented by $\eta\delta u_1$. On the other hand, $\vartheta_\rho i^*(a)$ is represented by $\eta\delta u_2$. Thus $(\gamma_\rho\kappa_\rho+\vartheta_\rho i^*)(a)$ is represented by $\eta\delta(u_1+u_2)=\eta\delta u=0$. This proves (iii). Q. E. D.

Consider the diagram

$$\begin{array}{ccc}
H^r(W_t'\cup F_t,\ W_t'; G) & \xrightarrow{\eta_*\delta^*} & H^{r+1}(W_t,\ W_t'\cup F_t; G_p) \\
\wr\!\wr \downarrow \pi^* & & \wr\!\wr \downarrow \mathrm{I}^* \\
H^r(W'\cup F,\ W'; G) & \xrightarrow{\vartheta_\tau} & {}^{\sigma}H^{r+1}(W,\ W'\cup F; G_p).
\end{array}$$

We have then obviously

$$(2\cdot 18) \qquad \vartheta_\tau\pi^* = \mathrm{I}^*\eta_*\delta^*.$$

We shall prove

THEOREM (2·19). (i) $\mu\phi_0^* = -\nu\delta^*\pi^{*^{-1}}i^*$.

(ii) $\nu\phi_0^*=0$ $(p\geqq 3)$, *and* $=\eta_*\delta^*\pi^{*^{-1}}i^*$ $(p=2)$.

Proof. It follows from (2·6), (2·13), (2·16), (2·17) and (2·18) that

$$\begin{aligned}\mu\phi_0^* &= \mathrm{I}^{*^{-1}}\gamma_\tau\gamma_\sigma\kappa_\sigma = -\mathrm{I}^{*^{-1}}\gamma_\tau\vartheta_\sigma i^* = -\mathrm{I}^{*^{-1}}\gamma_\tau\psi_\tau\vartheta_\tau i^* \\ &= -\nu\mathrm{I}^{*^{-1}}\vartheta_\tau i^* = -\nu\delta^*\pi^{*^{-1}}i^*.\end{aligned}$$

It follows from (2·2), (2·6), (2·16), (2·17) and (2·18) that

$$\begin{aligned}\nu\phi_0^* &= \mathrm{I}^{*^{-1}}\psi_\sigma\gamma_\sigma\kappa_\sigma = -\mathrm{I}^{*^{-1}}\psi_\sigma\vartheta_\sigma i^* = -\mathrm{I}^{*^{-1}}\psi_\sigma\psi_\tau\vartheta_\tau i^* \\ &= 0 \quad (p\geqq 3), \text{ and } = \mathrm{I}^{*^{-1}}\vartheta_\tau i^* = \eta_*\delta^*\pi^{*^{-1}}i^* \quad (p=2).\end{aligned}$$

This completes the proof.

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5) The upper line is the Smith-Richardson sequence, and the lower line is the ordinary exact sequence.

Let

(2・20) $$\bar{\Gamma}_s^\rho \quad H^q(W' \cup F, W'; G) \longrightarrow {}^{\bar\rho}H^{q+s}(W, W' \cup F; G_p)$$

$(s>0)$ be a homomorphism defined as follows:

$$\bar{\Gamma}^\rho_{2\alpha+1} = (\varUpsilon_\rho \varUpsilon_{\bar\rho})^\alpha \vartheta_\rho, \qquad \bar{\Gamma}^\rho_{2\alpha+2} = (\varUpsilon_\rho \varUpsilon_{\bar\rho})^\alpha \varUpsilon_\rho \vartheta_{\bar\rho}.$$

Then we have

LEMMA (2・21). (i) $\varUpsilon_{\bar\rho} \bar{\Gamma}^\rho_s = \bar{\Gamma}^{\bar\rho}_{s+1}$,

(ii) *If* $p \geqq 3$, *then* $\psi_\sigma \bar{\Gamma}^\sigma_{2\alpha+1} = \psi_\tau \bar{\Gamma}^\tau_{2\alpha+2} = 0, \quad \psi_\sigma \bar{\Gamma}^\sigma_{2\alpha+2} = \bar{\Gamma}^\tau_{2\alpha+2}, \quad \psi_\tau \bar{\Gamma}^\tau_{2\alpha+1} = \bar{\Gamma}^\sigma_{2\alpha+1}$.

(iii) *If* $p=2$, $\psi_\rho \bar{\Gamma}^\rho_s = \bar{\Gamma}^{\bar\rho}_s$.

Proof. (i) is obvious. It follows from (2・2), (2・4) and (2・16) that if $p \geqq 3$ then

$$\begin{aligned}\psi_\sigma \bar{\Gamma}^\sigma_{2\alpha+1} &= \psi_\sigma (\varUpsilon_\sigma \varUpsilon_\tau)^\alpha \vartheta_\sigma = (\varUpsilon_\tau \varUpsilon_\sigma)^\alpha \psi_\sigma \vartheta_\sigma \\ &= (\varUpsilon_\tau \varUpsilon_\sigma)^\alpha \psi_\sigma \psi_\tau \vartheta_\tau = 0.\end{aligned}$$

The proofs of the other formulas are similar. Q. E. D.

When $p=2$ and the coefficient group is G_2, there is a variation of the Smith-Richardson sequence, due to R. Bott [2]:

THEOREM (2・22). *Let* $p=2$, *then the sequence*

$$\cdots \longrightarrow H^{r-1}(W_t, W_t' \cup F_t; G_2) \xrightarrow{j^*\nu} H^r(W_t, W_t'; G_2) \xrightarrow{\pi^*}$$

$$\longrightarrow H^r(W, W'; G_2) \xrightarrow{\phi_0^*} H^r(W_t, W_t' \cup F_t; G_2) \longrightarrow \cdots,$$

is exact, and $\pi^* j^* \phi_0^* = \sigma^*$.

Proof. Since $H^r(W_t, W_t' \cup F_t; G_2) \approx {}^{\tau^{-1}}H^r(W, W' \cup F; G_2) = {}^{\tau}H^r(W, W'; G_2)$ and $H^r(W_t, W_t'; G_2) \approx {}^{\tau^{-1}}H^r(W, W'; G_2)$ by the isomorphisms induced by π, the exactness follows easily by the definitions of ν and ϕ_0^* from the Smith-Richardson sequence (1・1) for $\rho=\tau$ and $G=G_2$. $\pi^* j^* \phi_0^* = \sigma^*$ is obvious from (2・10).

Let $\bar W$ be a finite simplicial complex with a periodic map $\bar t$ satisfying the conditions a) and b) in § 1, and let $\bar W'$ be a $\bar t$-invariant subcomplex of $\bar W$. Given a simplicial map

$$f: (W, W') \longrightarrow (\bar W, \bar W')$$

which is t-equivariant (*i. e.* a map such that $\bar t f = f t$), it is obvious that f maps F in the set $\bar F$ of fixed points under $\bar t$, and that f induces a simplicial map $f: (W_t, W_t') \longrightarrow (\bar W_t, \bar W_t')$ such that $f\pi = \bar\pi f$. Thus f induces the homomorphisms $H^r(\bar W; \bar W'; G) \longrightarrow H^r(W, W'; G)$, ${}^\rho H^r(\bar W, \bar W' \cup \bar F; G) \longrightarrow {}^\rho H^r(W, W' \cup F; G)$, $H^r(\bar W_t, \bar W_t'; G) \longrightarrow H^r(W, W_t'; G)$ etc. Let us denote by f^* all these homomorphisms. Then it can be verified easily that f^* commutes with the various homomorphisms defined in § 1 and § 2.

3. Relations to the cohomology operations

In the preceeding section, we defined the homomorphisms μ, ν and ϕ_0^* which are basic in the discussions below. In this section we shall study some relationships between these homomorphisms and the well-known cohomology operations: the cup product, the Bockstein homomorphism, the reduced p-th power and the squaring operation. Whenever the argument are concerned with the cup product, we shall always suppose that coefficients are taken from a ring, and we shall denote by $\cup$ the cup product. Let (X, A) be a pair of simplicial complex X and its subcomplex A, and let Z be the group of integers. Then we shall denote the Bockstein homomorphism, the reduced p-th power, the squaring operation by

$$\Delta_p : H^r(X, A; Z_p) \longrightarrow H^{r+1}(X, A; Z_p),$$
$$\mathcal{P}^s : H^r(X, A; Z_p) \longrightarrow H^{r+2s(p-1)}(X, A; Z_p) \quad (p \geqq 3),$$
$$\mathrm{Sq}^s : H^r(X, A; Z_p) \longrightarrow H^{r+s}(X, A; Z_p) \quad (p=2)$$

respectively [14, 15, 16].

In virtue of the assumptions a) and b) in §1, we can define on W a locally simple ordering invariant under t. This ordering induces a locally simple ordering in W_t.[3] Using this ordering on W and W_t, we shall define as usual the cup product in W and W_t. Since the map π and $t^{\text{※}i}$ are order-preserving, we have the following [14].

LEMMA (3·1). *Let u, $v \in C^*(W, W'; G)$, then*

$$t^{\text{※}i}(u \cup v) = t^{\text{※}i}u \cup t^{\text{※}i}v, \qquad \pi^{\text{※}}(\phi u \cup \phi v) = \pi^{\text{※}}\phi u \cup \pi^{\text{※}}\phi v.$$

LEMMA (3·2). (i) $\sigma(u \cup \sigma v) = \sigma u \cup \sigma v$, (ii) $\tau(u \cup \sigma v) = \tau u \cup \sigma v$, (iii) $\phi(u \cup \sigma v) = \phi u \cup \phi v$.

Proof. By the definition of σ and (3·1), we have

$$\sigma(u \cup \sigma v) = \textstyle\sum_{i=0}^{p-1} t^{\text{※}i}(u \cup \sum_{j=0}^{p-1} t^{\text{※}j} v)$$
$$= \textstyle\sum_{i=0}^{p-1} (t^{\text{※}i}u \cup t^{\text{※}i}(\sum_{j=0}^{p-1} t^{\text{※}j} v)) = \sum_{i=0}^{p-1} t^{\text{※}i} u \cup \sum_{j=0}^{p-1} t^{\text{※}j} v$$
$$= \sigma u \cup \sigma v.$$

This proves (i). The proof of (ii) is similar.

From (2·10), (3·1) and above (i), we have

$$\pi^{\text{※}}\phi(u \cup \sigma v) = \sigma(u \cup \sigma v) = \sigma u \cup \sigma v = \pi^{\text{※}}\phi u \cup \pi^{\text{※}}\phi v = \pi^{\text{※}}(\phi u \cup \phi v).$$

Since $\pi^{\text{※}}$ is isomorphic into, we have (iii). Q. E. D.

As an immediate consequence of (3·2), we have

THEOREM (3·3). *Let a, $b \in H^*(W, W' : G)$, then*

$$\phi_0^*(a \cup \sigma^* b) = \phi_0^*(a) \cup \phi_0^*(b).$$

We shall prove

THEOREM (3·4). *For the above a and b, we have*

$$\nu(\phi_0^* a \cup \phi_0^* b) = 0, \qquad \mu(\phi_0^* a \cup \phi_0^* b) = 0.$$

Proof. By the definitions of μ and ν, it is sufficient to prove $\Upsilon_\sigma \mathrm{I}^*(\phi_0^* a \cup \phi_0^* b) = 0$. Let u and v be representative cocycles of a and b respectively. Then $\phi_0^* a \cup \phi_0^* b$ is represented by $\phi u \cup \phi v = \phi(u \cup \sigma v)$. (See (3·2).) Thus $\mathrm{I}^*(\phi_0^* a \cup \phi_0^* b)$ is represented by $\pi\phi(u \cup \sigma v) = \sigma(u \cup \sigma v)$. Since $\sigma v \in C^*(W, W' \cup F; G_p)$, we have $u \cup \sigma v \in C^*(W, W' \cup F; G_p)$. Thus $\Upsilon_\sigma \mathrm{I}^*(\phi_0^* a \cup \phi_0^* b)$ is represented by $\delta(u \cup \sigma v)$. However $u \cup \sigma v$ is a cocycle, and hence $\delta(u \cup \sigma v) = 0$. Namely we have $\Upsilon_\sigma \mathrm{I}^*(\phi_0^* a \cup \phi_0^* b) = 0$. Q. E. D.

Corollary (3·5). *Let $a_\alpha \in H^*(W, W'; G)$ for $\alpha = 1, 2, \cdots, k$, where $k \geqq 2$. Then we have*

$$\nu(\phi_0^* a_1 \cup \phi_0^* a_2 \cup \cdots \cup \phi_0^* a_k) = 0, \quad \mu(\phi_0^* a_1 \cup \phi_0^* a_2 \cup \cdots \cup \phi_0^* a_k) = 0.$$

Proof. Obvious from (3·3) and (3·4).

Theorem (3·6). *Let $a, b \in H^*(W_t, W_t' \cup F_t; G)$, and $\alpha, \beta \geqq 0$. Then we have*

(i) $\mu^\alpha(a) \cup \mu^\beta(b) = \mu^{\alpha+\beta}(a \cup b)$,

(ii) $\eta_* \mu^\alpha(a) \cup \nu(b) = (-1)^{\dim a} \mu^\alpha \nu(a \cup b)$,

(iii) $\nu(a) \cup \nu(b) = 0$ *if* $p \geqq 3$, *and* $= \eta_* \mu(a \cup b)$ *if* $p = 2$.

The proof is given in § 4.

Consider the diagram

$$\begin{tikzcd}
 & H^r(W, W'; Z_p) \arrow[ddl, "\phi_0^*"'] \arrow[d, "i^*"] \arrow[ddr, "\Delta_p"] & \\
 & H^r(W' \cup F, W'; Z_p) & \\
H^r(W_t, W_t' \cup F_t; Z_p) \arrow[ddr, "\Delta_p"'] & & H^{r+1}(W, W'; Z_p) \arrow[ddl, "\phi_0^*"] \\
 & H^r(W_t' \cup F_t, W_t'; Z_p) \arrow[uu, "\pi^*", "\wr"'] \arrow[d, "\delta^*"] & \\
 & H^{r+1}(W_t, W_t' \cup F_t; Z_p) &
\end{tikzcd}$$

Then we have

Theorem (3·7). $$\phi_0^* \Delta_p - \Delta_p \phi_0^* = \delta^* \pi^{*-1} i^*.$$

Proof. Let $a \in H^r(W, W'; Z_p)$ be any element, and let $u \in C^r(W, W'; Z)$ be a cocycle mod p which represents a. Then there is a cochain $v \in C^{r+1}(W, W'; Z)$ such that $\delta u = pv$. Let $u = u_1 + u_2$, where $u_1 \in C^r(W, W' \cup F; Z)$ and $u_2 \in C^r(W' \cup F, W'; Z)$. Then we have

$$\phi u = \phi u_1 + \phi u_2 = \phi u_1 + p u_2',$$

where $u_2' \in C^r(W_t' \cup F_t, W_t'; Z)$ is a cochain such that $\pi u_2' = u_2$. Make the coboundary of the both sides, then

$$\delta\phi u = \delta\phi u_1 + p\delta u_2'.$$

Since $\delta\phi u = \phi\delta u = p\phi v$, we have

$$\phi v = \frac{1}{p}\delta\phi u_1 + \delta u_2'$$

By the definitions, $\phi_0^* \varDelta_p(a)$ and $\delta^* \pi^{*^{-1}} i^*(a)$ are represented by ϕv and $\delta u_2'$ respectively. Since $\phi u \equiv \phi u_1$ mod p we see also that $\frac{1}{p} \delta \phi u_1$ represents $\varDelta_p \phi_0^*(a)$. Thus the above equation proves (3·7). Q.E.D.

THEOREM (3·8). (i) $\varDelta_p \nu + \nu \varDelta_p = \mu$, (ii) $\mu \varDelta_p = \varDelta_p \mu$.

Proof.[6] Let $a \in H^r(W_t, W_t' \cup F_t; Z_p)$ be an element whose representative cocycle is u mod p ($u \in C^r(W_t, W_t' \cup F_t; Z)$). Then there is a cochain $v \in C^{r+1}(W_t, W_t' \cup F_t; Z)$ such that $\delta u = pv$. Consider now $u_0 \in C^r(W, W' \cup F; Z)$ and $v_0 \in C^{r+1}(W, W' \cup F; Z)$ such that $\phi u_0 = u$ and $\phi v_0 = v$. Then it follows from (2·10) that

$$\sigma(\delta u_0 - p v_0) = \delta \sigma u_0 - p \sigma v_0 = \delta \pi \phi u_0 - p \pi \phi v_0$$
$$= \delta \pi u - p \pi v = \pi(\delta u - pv) = 0,$$

and

$$\sigma \delta v_0 = \delta \sigma v_0 = \delta \pi \phi v_0 = \delta \pi v = \pi \delta v = 0.$$

Therefore it follows from (1·2) that there exist cochains $u_1 \in C^{r+1}(W, W' \cup F; Z)$ and $v_1 \in C^{r+2}(W, W' \cup F; Z)$ satisfying

(A) $$\delta u_0 - p v_0 = \tau u_1,$$

(B) $$\delta v_0 = \tau v_1.$$

Then we have

$$\tau(\delta u_1 + p v_1) = \delta \tau u_1 + p \tau v_1 = \delta(\delta u_0 - p v_0) + p \tau v_1$$
$$= \delta \delta u_0 - p(\delta v_0 - \tau v_1) = 0,$$

and hence there is a cochain $y_1 \in C^{r+2}(W, W' \cup F; Z)$ such that

(C) $$\delta u_1 + p v_1 = \sigma y_1.$$

Applying ϕ to the both sides of equation, we have

(D) $$\frac{1}{p} \delta \phi u_1 = \phi y_1 - \phi v_1,$$

because of $\phi \sigma y_1 = p \phi y_1$.

Since $\pi u = \sigma u_0$, $\mathrm{I}^*(a)$ is represented by σu_0 mod p, and hence it follows from (A) that $\varUpsilon_\sigma \mathrm{I}^*(a)$ is represented by $\delta u_0 \equiv \tau u_1$ mod p. Thus it follows from (2·2) and (C) that $\psi_\sigma \varUpsilon_\sigma \mathrm{I}^*(a)$ and $\varUpsilon_\tau \varUpsilon_\sigma \mathrm{I}^*(a)$ are represented by σu_1 mod p and $\delta u_1 \equiv \sigma y_1$ mod p respectively. Therefore $\nu(a) = \mathrm{I}^{*^{-1}} \psi_\sigma \varUpsilon_\sigma \mathrm{I}^*(a)$ and $\mu(a) = \mathrm{I}^{*^{-1}} \varUpsilon_\tau \varUpsilon_\sigma \mathrm{I}^*(a)$ are represented by $\phi(u_1)$ mod p and $\phi(y_1)$ mod p. On the other hand, $\varDelta_p(a)$ is represented by v mod p, and hence $\mathrm{I}^* \varDelta_p(a)$ by σv_0 mod p. Thus it follows from (B) that $\varUpsilon_\sigma \mathrm{I}^* \varDelta_p(a)$ is represented by $\delta v_0 = \tau v_1$. From this, we see that $\nu \varDelta_p(a) = \mathrm{I}^{*^{-1}} \psi_\sigma \varUpsilon_\sigma \mathrm{I}^* \varDelta_p(a)$ is represented by $\phi(v_1)$ mod p. Now (i) is clear from (D).

It follows from above (i) and the well-known property: $\varDelta_p^2 = 0$ that

6) See the Remark (1) at the end of §4.

$$\mu\Delta_p=(\Delta_p\nu+\nu\Delta_p)\Delta_p=\Delta_p\nu\Delta_p$$
$$=(\mu-\nu\Delta_p)\Delta_p=\mu\Delta_p.$$

This completes the proof.

THEOREM (3·9). (i) *Let* $p\geqq 3$, *then*

$$\wp^s\mu-\mu\wp^s=\mu^p\wp^{s-1}, \qquad \wp^s\nu=\nu\wp^s.$$

(ii) *Let* $p=2$, *then*

$$\mathrm{Sq}^s\nu-\nu\mathrm{Sq}^s=\nu^2\mathrm{Sq}^{s-1}.$$ [7]

The proof will be given in § 4 by no making use of the original definitions of $\wp^s$ and Sq^s. We shall use only the following properties [16]:

(I) *If* $f:(X, A)\longrightarrow(X', A')$ *is a simplicial map,*
$f^*\wp^s=\wp^s f^*$, $\qquad f^*\mathrm{Sq}^s=\mathrm{Sq}^s f^*$.

(II) *For the coboundary operator* δ^*, *we have*
$\delta^*\wp^s=\wp^s\delta^*$, $\qquad \delta^*\mathrm{Sq}^s=\mathrm{Sq}^s\delta^*$.

(III) $\wp^s(a\cup b)=\sum_{j+k=s}\wp^j(a)\cup\wp^k(b)$,
$\mathrm{Sq}^s(a\cup b)=\sum_{j+k=s}\mathrm{Sq}^j(a)\cup\mathrm{Sq}^k(b)$.

(IV) $\wp^0$ *and* Sq^0 *are the identity.*

(V) $\wp^s(a)=a\cup a\cup\cdots\cup a$ (*p-fold cup product*) *if dim a is even and* $s=\frac{1}{2}$ *dim a, and* $=0$ *if* $s>\frac{1}{2}$ *dim a or* <0. $\mathrm{Sq}^s(a)=a\cup a$ *if dim* $a=s$, *and* $=0$ *if* $s>$ *dim a or* <0.

By iterations of (3·9), we have

COROLLARY (3·10). (i) $\mu\wp^s=\sum_{k=0}^s(-1)^k\mu^{k(p-1)}\wp^{s-k}\mu \qquad (p\geqq 3)$,

(ii) $\nu\mathrm{Sq}^s=\sum_{k=0}^s\nu^k\mathrm{Sq}^{s-k}\nu \qquad (p=2)$.

By (2·19), (3·9) and the property (II) of $\wp^s$ and Sq^s, we have

COROLLARY (3·11). (i) $\mu\phi_0^*\wp^s=\wp^s\mu\phi_0^* \qquad (p\geqq 3)$,

(ii) $\nu\phi_0^*\mathrm{Sq}^s=\mathrm{Sq}^s\nu\phi_0^* \qquad (p=2)$.

THEOREM (3·12). (i) $\phi_0^*\wp^s-\wp^s\phi_0^*=\mu^{p-1}\wp^{s-1}\phi_0^* \qquad (p\geqq 3)$,

(ii) $\phi_0^*\mathrm{Sq}^s-\mathrm{Sq}^s\phi_0^*=\nu\mathrm{Sq}^{s-1}\phi_0^* \qquad (p=2)$. [7]

This will be proved in § 5 by making use of the original definitions of $\wp^s$ and Sq^s, due to N. E. Steenrod.

COROLLARY (3·13). *If* $p=2$, *then*

$$\phi_0^*\mathrm{Sq}^s-\mathrm{Sq}^s\phi_0^*=\mu\mathrm{Sq}^{s-2}\phi_0^*+\mathrm{Sq}^{s-1}\nu\phi_0^*.$$

---

7) For the formula for $p=2$, see R. Bott [2].

Proof. Since $\nu^2=\mu$ for $p=2$ by (2·17), it follows from (ii) of (3·9) that

$$\mu\mathrm{Sq}^{s-2}\phi_0^* = \mathrm{Sq}^{s-1}\nu\phi_0^* - \nu\mathrm{Sq}^s\phi_0^*.$$

From this and (ii) of (3·12), we have (3·13).

By iterations of (3·12), we have

COROLLARY (3·14). (i) $\mathcal{P}^s\phi_0^* = \sum_{k=0}^{s}(-1)^k\mu^{k(p-1)}\phi_0^*\mathcal{P}^{s-k}$ $(p\geqq 3)$,

(ii) $\mathrm{Sq}^s\phi_0^* = \sum_{k=0}^{s}\nu^k\phi_0^*\mathrm{Sq}^{s-k}$ $(p=2)$.

4. Proof of (3·6) and (3·9)

We shall first state an important property of μ and ν for a special case in which t operates on W without fixed points.

Let $U_2=U_2(W)\in H^2(W_t;Z)$ and $V_1=V_1(W)\in H^1(W_t;Z_p)$ be elements defined by $\mu(\mathbf{1})$ and $\nu(\mathbf{1})$ respectively[8], where $\mu: H^0(W_t;Z)\longrightarrow H^2(W_t;Z)$ and $\nu: H^0(W_t;Z)\longrightarrow H^1(W_t;Z_p)$. Then we have the following:

THEOREM (4·1).[9] *Suppose that the transformation $t: W\longrightarrow W$ has no fixed point, and let $a\in H^r(W_t, W_t';G)$. Then, for the homomorphism $\mu: H^r(W_t, W_t';G)\longrightarrow H^{r+2}(W_t, W_t';G)$ and $\nu: H^r(W_t, W_t';G)\longrightarrow H^{r+1}(W_t, W_t';G_p)$, it holds that*

$$\mu(a)=U_2(W)\cup a, \qquad \nu(a)=V_1(W)\cup a,$$

where the cup products are taken with respect to the natural pairing $G\otimes Z\longrightarrow G$ and $G\otimes Z_p\longrightarrow G_p$ respectively.

Proof. Let $u\in Z^r(W_t, W_t';G)$ be a cocycle which represents a, and $1\in Z^0(W_t;Z)$ the unit cocycle. Let $u_0\in C^r(W, W';G)$ be an element such that $\phi u_0=u$, and let $s_i\in C^i(W;Z)$ $(i=0,1,2)$ be elements such that

(A) $$\phi s_0=1, \qquad \delta s_0=\tau s_1, \qquad \delta s_1=\sigma s_2.$$

It is clear that such s_i exists. Then it holds by the definitions of μ and ν that ϕs_2 and ϕs_1 mod p are representative cocycles of U and V respectively. Moreover it follows easily from (A) by making use of (3·2) and (2·10) that

$$\phi(s_0\cup\sigma u_0)=u, \qquad \delta(s_0\cup\sigma u_0)=\tau(s_1\cup\sigma u_0),$$
$$\delta(s_1\cup\sigma u_0)=\sigma(s_2\cup\sigma u_0).$$

Thus $\phi(s_2\cup\sigma u_0)$ and $\phi(s_1\cup\sigma u_0)$ mod pG represent $\mu(a)$ and $\nu(a)$ respectively. On the other hand, it follows from (3·2) that $\phi(s_2\cup\sigma u_0)=\phi(s_2)\cup u$ and $\phi(s_1\cup\sigma u_0)=\phi(s_1)\cup u$. Therefore $\phi(s_2\cup\sigma u_0)$ and $\phi(s_1\cup\sigma u_0)$ mod pG represent $U_2\cup a$ and

8) Let Y be a complex, then we denote by $\mathbf{1}$ the cohomology class containing the fundamental zero-cocycle 1.

9) See W. T. Wu [19].

$V_1 \cup a$ respectively. This proves (4·1).

THEOREM (4·2). (i) $\Delta_p V_1(W) = U_2(W)$ *mod* p.

(ii) $V_1(W) \cup V_1(W) = 0$ *if* $p \geqq 3$, *and* $= U_2(W)$ *mod* p *if* $p = 2$.

Proof. Using the notations in the above proof, V_1 and U_2 are represented by ϕs_1 mod p and ϕs_2 respectively. Moreover it follows from (2·10) and (A) that

$$\delta\phi s_1 = \phi\delta s_1 = \phi\sigma s_2 = p\phi s_2.$$

This proves (i). It follows from (4·1) that

$$V_1 \cup V_1 = \nu(V_1) = \nu^2(1).$$

Therefore (ii) is obvious from (2·7). Q. E. D.

We shall again consider the general case in which t may have fixed points. We shall retain the notations in above sections. Let W^* and $(W' \cup F)^*$ be the *second* barycentric subdivisions of W and $W' \cup F$ respectively, and let N be the regular neighborhood of $(W' \cup F)^*$ in W^*. (*i. e.* $N = \bigcup_A St(A)$, where the union is extended over all the vertices A of $(W' \cup F)^*$ and St denotes the open star in W^*.) Let $\bar{N}$ be the closure of N in W^*, and denote

$$M = W^* - N, \qquad E = \bar{N} - N.$$

Then $\bar{N}$, E and M are subcomplexes of W^*. It is obvious that the map t is simplicial with respect to W^* and satisfies the conditions a) and b) in § 1. Moreover it is easy to verify that $\bar{N}$, E and M are t-invariant subcomplexes of W^*.

Let W_t^* be the second subdivision of W_t, then the map π: $W^* \longrightarrow W_t^*$ is simplicial. Let $\bar{N}_t$, E_t and M_t be the images by π of $\bar{N}$, E and M respectively. Then it can be easily proved that those are subcomplexes of W_t^* and that N_t is the regular neighborhood of $(W_t' \cup F_t)^*$ which is the second subdivision of $W_t' \cup F_t$. Thus, by the THEOREM 9·9 of Chapter II in [3], we see that $W_t' \cup F_t$ is a strong deformation retract of $\bar{N}_t$. Therefore we have

$$k_1^* : H^r(W_t, \bar{N}_t; G) \approx H^r(W_t, W_t' \cup F_t; G)$$

for any r and G, where $k_1 : (W_t, W_t' \cup F_t) \longrightarrow (W_t, \bar{N}_t)$ is the inclusion map. On the other hand, it follows from the excision property that

$$k_2^* : H^r(W_t, \bar{N}_t; G) \approx H^r(M_t, E_t; G)$$

for any r and G, where k_2: $(M_t, E_t) \longrightarrow (W_t, \bar{N}_t)$ is the inclusion map. Thus we have proved

LEMMA (4·3). $k_2^* k_1^{*-1} : H^r(W_t, W_t' \cup F_t; G) \approx H^r(M_t, E_t; G)$ *for any* r *and* G.

Consider now the complex M, then t operates on M with the properties a) and b) in § 1. Moreover the transformation $t: M \longrightarrow M$ has no fixed point, and E is the t-invariant subcomplex of M. Therefore we may apply (4·1) with $W = M$ and $W' = E$. Namely we have

LEMMA (4·4). *For the homomorphism* $\mu: H^r(M_t, E_t; G) \longrightarrow H^{r+2}(M_t, E_t; G)$ *and* $\nu: H^r(M_t, E_t; G) \longrightarrow H^{r+1}(M_t, E_t; G_p)$, *it holds that*

$$\mu(a) = U_2(M) \cup a, \qquad \nu(a) = V_1(M) \cup a,$$

where $a \in H^r(M_t, E_t; G)$.

We shall now give a proof of (3·6) and (3·9).[10] In virtue of (4·3) and the naturality of μ, ν, $\mathcal{P}^s$ and $\cup$, it is sufficient to prove them for the special case: $(W, W') = (M, E)$.

Write U_2 and V_1 for $U_2(M)$ and $V_1(M)$ respectively, and let $U_2 \cup U_2 \cup \cdots \cup U_2$ denotes the p-fold cup product of U_2. Then, from (4·4) and the properties of $\mathcal{P}^s$ and Sq^s described in § 3, we have for an element $a \in H^r(M_t, E_t; Z_p)$ the following:

$$\text{(i)} \qquad \begin{aligned} \mathcal{P}^s\mu(a) &= \mathcal{P}^s(U_2 \cup a) = \textstyle\sum_{j+k=s} \mathcal{P}^j(U_2) \cup \mathcal{P}^k(a) \\ &= (U_2 \cup \mathcal{P}^s(a)) + (U_2 \cup U_2 \cup \cdots \cup U_2 \cup \mathcal{P}^{s-1}(a)) \\ &= \mu\mathcal{P}^s(a) + \mu^p \mathcal{P}^{s-1}(a), \\ \mathcal{P}^s\nu(a) &= \mathcal{P}^s(V_1 \cup a) = V_1 \cup \mathcal{P}^s(a) = \nu\mathcal{P}^s(a), \end{aligned}$$

$$\text{(ii)} \qquad \begin{aligned} \mathrm{Sq}^s\nu(a) &= \mathrm{Sq}^s(V_1 \cup a) = (V_1 \cup \mathrm{Sq}^s(a)) + (V_1 \cup V_1 \cup \mathrm{Sq}^{s-1}(a)) \\ &= \nu\mathrm{Sq}^s(a) + \nu^2 \mathrm{Sq}^{s-1}(a). \end{aligned}$$

This proves (3·9).
As for (3·6), (i) and (ii) are obvious from the anti-commutativity of the cup product. In addition to this, if we use (4·2) we have (iii). Thus we obtain (3·6).

REMARK (1) We proved (i) of (3·8) directly by the definitions of μ, ν and Δ_p. However, since it follows from (4·2) and a well-known property of Δ_p that

$$\begin{aligned} \Delta_p\nu(a) &= \Delta_p(V_1 \cup a) = (\Delta_p V_1 \cup a) - (V_1 \cup \Delta_p a) \\ &= (U_2 \cup a) - (V_1 \cup \Delta_p a) = \mu a - \nu\Delta_p a \end{aligned}$$

for $a \in H^r(M_t, E_t; Z_p)$, we can prove (i) of (3·8) by the same way as the above.

(2) It is not difficult to give a direct proof of (ii) of (3·9) without making use of (M_t, E_t). Such a proof is seen in R. Bott [2].

5. Proof of (3·12)

We shall prove only the formula for $p \geqq 3$. The proof of the one for $p=2$ is similar.[7]

Let X be an arbitrary simplicial complex, and A its subcomplex. Denote by $\mathfrak{X}_{(p)}(X)$ the p-fold cartesian product of X. Then N. E. Steenrod [15, 16] defined

$$\mathcal{P}^s: H^q(X, A; Z_p) \longrightarrow H^{q+2s(p-1)}(X, A; Z_p) \qquad \text{for} \quad p \geqq 3,$$

by making use of the homomorphism

$$D_i: C^r(\mathfrak{X}_{(p)}(X); Z_p) \longrightarrow C^{r-i}(X; Z_p)$$

as follows:

$$\text{(5·1)} \qquad \mathcal{P}^s(a) = (-1)^{ts+tq(q-1)/2}(t!)^{2s-q}\{D_{(q-2s)(p-1)}(u \times u \times \cdots \times u)\},$$

10) Compare with the original proof in R. Thom [18] where the Cartan-Leray cohomology theory is used.

where u is a representative cocycle of $a\in H^q(X, A; Z_p)$, $\{v\}$ denotes the cohomology class containing v, and $t=(p-1)/2$.

We shall recall some properties of D_i.

Let

$$T: C^r(\mathfrak{X}_{(p)}(X); Z_p) \longrightarrow C^r(\mathfrak{X}_{(p)}(X); Z_p)$$

be an automorphism defined by

$$T(u_1\times u_2\times\cdots\times u_p)=(-1)^{q_1(q_2+\cdots+q_p)}(u_2\times u_3\times\cdots\times u_1)$$

$(u_i\in C^{q_i}(X; Z_p))$, and denote by σ_T and τ_T the homomorphism $\sum_{j=0}^{p-1}T^j$ and $1-T$ respectively. Then the following holds [15].

LEMMA (5·2). $\delta D_i(u_1\times u_2\times\cdots\times u_p)$

$$=(-1)^i D_i\delta(u_1\times u_2\times\cdots\times u_p)+(-1)^{i+1}D_{i-1}\alpha_i(u_1\times u_2\times\cdots\times u_p),$$

where $\alpha_i=\sigma_T$ *for even* i, *and* $=-\tau_T$ *for odd* i.

The following property of ρ_T is verified easily.

LEMMA (5·3). *If* $\dim u_1=\dim u_2=\cdots=\dim u_p$, *then we have*

$$\sigma_T(u_1\times u_2\times\cdots\times u_p)=\tau_T(\textstyle\sum_{j=1}^{p} jT^{j-1}(u_1\times u_2\times\cdots\times u_p)).$$

D_i are not determined uniquely for a given X. However if a locally simple ordering is introduced on X, we can construct uniquely D_i by making use of this ordering, as is shown by S. Araki [1]. Construct D_i by such a method, then the following is immediate from the definition due to Araki.

LEMMA (5·4). *Let X and X' be simplicial complexes on which locally simple orderings are given, and let $f: X\longrightarrow X'$ be an order-preserving simplical map, then the naturality*:

$$D_i(fu_1\times fu_2\times\cdots\times fu_p)=fD_i(u_1\times u_2\times\cdots\times u_p)$$

holds.

Take now as X especially W considered in above sections. Then we can consider on $C^r(\mathfrak{X}_{(p)}(W); Z_p)$ an another automorphism t than above-mentioned T. The definition of t is as follows:

$$t(u_1\times u_2\times\cdots\times u_p)=tu_1\times tu_2\times\cdots\times tu_p,$$

where $u_i\in C^{q_i}(W; Z_p)$. Therefore $C^r(\mathfrak{X}_{(p)}(W); Z_p)$ becomes a group with two operators T and t. Denote $\sigma=\sum_{j=0}^{p-1} t^j$ and $\tau=1-t$. Then the following is obvious.

LEMMA (5·5). (i) $tT=Tt$.

(ii) $\sigma(u_1\times u_2\times\cdots\times u_p)=\tau(\sum_{j=1}^{p} jt^{j-1}(u_1\times u_2\times\cdots\times u_p)).$

Let us define $D_i: C^r(\mathfrak{X}_{(p)}(W); Z_p)\longrightarrow C^{r-i}(W; Z_p)$ by making use of a locally simple ordering invariant under t on W, and define $D_i: C^r(\mathfrak{X}_{(p)}(W_t); Z_p)\longrightarrow C^{r-i}(W_t; Z_p)$ by making use of the locally simple ordering on W_t induced from that on W. Then t^j $(j=1, 2,\cdots, p-1)$ and π are order-preserving simplicial maps. Therefore we have by (5·3) and the definition of t the following

LEMMA (5·6). (i) $D_i(\pi\phi u_1\times\pi\phi u_2\times\cdots\times\pi\phi u_p)=\pi D_i(\phi u_1\times\phi u_2\times\cdots\times\phi u_p).$

(ii) $D_i t^j(u_1\times u_2\times\cdots\times u_p)=t^j D_i(u_1\times u_2\times\cdots\times u_p)$,
and hence $D_i\rho=\rho D_i$.

Let $(m_0, m_1,\cdots, m_{p-1})$ be a sequence of integers mod p such that $m_0+m_1+\cdots+m_{p-1}\equiv 0$ mod p. Such two sequences $(m_0, m_1,\cdots, m_{p-1})$ and $(m_0', m_1',\cdots, m_{p-1}')$ are called to be equivalent if there is an integer β mod p such that $m_j\equiv m'_{\beta+j}$ for $j=0, 1,\cdots, p-1$. The equivalence class containing a sequence $(m_0, m_1,\cdots, m_{p-1})$ will be denoted by $\mathfrak{m}=[m_0, m_1,\cdots, m_{p-1}]$. Given an equivalence class $\mathfrak{m}$ and a cochain $u\in C^q(W, W'; Z_p)$, let us denote by $\Omega(u;\mathfrak{m})$ a cochain

$$\sum_{(i_1, i_2,\cdots, i_p)} t^{i_1}u\times t^{i_2}u\times\cdots\times t^{i_p}u,$$

where the summation is extended over all sequences $(i_1, i_2,\cdots, i_p)$ of integers mod p such that $(i_2-i_1, i_3-i_2,\cdots, i_1-i_p)\in\mathfrak{m}$. Then we have obviously

LEMMA (5·7). $\sigma u\times\sigma u\times\cdots\times\sigma u=\sum_{\mathfrak{m}}\Omega(u;\mathfrak{m})$,
where $\mathfrak{m}$ *runs over every possible equivalence class.*

Assume that $(m_0, m_1,\cdots, m_{p-1})=(m_\beta, m_{\beta+1},\cdots, m_{\beta+p-1})$ for some $\beta\not\equiv 0$ mod p. Then we have $m_j=m_{\beta+j}$ for any j, and hence $m_0=m_\beta=\cdots=m_{(p-1)\beta}$. Since p is prime and $\beta\not\equiv 0$, the set $\{0,\beta,2\beta,\cdots,(p-1)\beta\}$ and $\{0, 1, 2,\cdots, p-1\}$ are same. Therefore we have $m_0=m_1=\cdots=m_{p-1}$. Thus it holds that if $m_0, m_1,\cdots,m_{p-1}$ are not all same then the equivalence class $[m_0, m_1,\cdots,m_{p-1}]$ consists of p different sequences. On the other hand, it is obvious that the class $[m, m,\cdots, m]$ consists of only one sequence. These considerations deduce readily the following:

LEMMA (5·8). *Let* $(m_0, m_1,\cdots, m_{p-1})$ *be a sequence, and let* $\mathfrak{m}$ *be the class containing it. Put* $\omega(u;m_0, m_1,\cdots, m_{p-1})=u\times t^{m_0}u\times t^{m_0+m_1}u\times\cdots\times t^{m_0+m_1+\cdots+m_{p-2}}u$. *Then if* $m_0, m_1,\cdots, m_{p-1}$ *are not all same, then*

$$\Omega(u;\mathfrak{m})=\sigma\sigma_T\omega(u;m_0, m_1,\cdots, m_{p-1});$$

if $m_0=m_1=\cdots=m_{p-1}$, *then*

$$\Omega(u;\mathfrak{m})=\sigma\omega(u;m_1, m_2,\cdots, m_{p-1}).$$

If $m_0=m_1=\cdots=m_{p-1}(=m)$, we shall abbreviate $\omega(u;m_0, m_1,\cdots m_{p-1})$ as $\omega(u;m)$. Throughout the remainder of this section, we shall assume that u denotes a *cocycle* (*i.e.* $u\in Z^q(W, W';Z_p)$) and that i denotes an *even* number.

LEMMA (5·9) $D_i(\sigma u\times\sigma u\times\cdots\times\sigma u)$ *and* $\sigma\sum_{m=0}^{p-1}D_i\omega(u;m)$ *represent the same cohomology class of* ${}^\sigma H^{pq}(W, W'\cup F; Z_p)$.

Proof. It follows from (5·7), (5·8), (5·6) and (5·2) that

$$\begin{aligned}
&D_i(\sigma u\times\sigma u\times\cdots\times\sigma u)\\
&=D_i\textstyle\sum_{m=0}^{p-1}\sigma\omega(u;m)+D_i\sum'_{\mathfrak{m}}\sigma\sigma_T(u;m_0, m_1,\cdots, m_{p-1})\\
&=\sigma\textstyle\sum_{m=0}^{p-1}D_i\omega(u;m)+\sigma\sum'_{\mathfrak{m}}D_i\sigma_T\omega(u;m_0, m_1,\cdots, m_{p-1})\\
&=\sigma\textstyle\sum_{m=0}^{p-1}D_i\omega(u;m)+\sigma\sum'_{\mathfrak{m}}D_i\tau_T(\sum_{j=1}^{p}jT^{j-1}\omega(u;m_0, m_1,\cdots, m_{p-1}),
\end{aligned}$$

where $\sum'_{\mathfrak{m}}$ denotes the summation extended over every possible equivalence class

$\mathfrak{m}=[m_0, m_1,\cdots, m_{p-1}]$ such that $m_0, m_1,\cdots, m_{p-1}$ are not all same. However it follows from (5·1) that

$$D_i\tau_T(\textstyle\sum_{j=1}^{p} jT^{j-1}\omega(u\,;m_0, m_1,\cdots, m_{p-1}))$$
$$= -\delta D_{i+1}(\textstyle\sum_{j=1}^{p} jT^{j-1}\omega(u\,;m_0, m_1,\cdots, m_{p-1})).$$

This proves (5·9).

LEMMA (5·10). *$\delta D_i\omega(u\,;m)=0$ if $m\equiv 0$, and*
$$= -\tau D_{i-1}(\textstyle\sum_{j=1}^{p} jt^{j-1}\omega(u\,;m)) \text{ if } m\not\equiv 0.$$

Proof. It follows from (5·1) that

$$\delta D_i\omega(u\,;m) = -D_{i-1}\sigma_T\omega(u\,;m)$$
$$= -D_{i-1}(\textstyle\sum_{j=1}^{p} t^{jm}(u\times t^m u\times \cdots \times t^{(p-1)}u))$$
$$= -D_{i-1}\sigma\omega(u\,;m) \text{ if } m\not\equiv 0, \text{ and } =0 \text{ if } m\equiv 0.$$

Since it follows from (5·5) and (5·6) that

$$D_{i-1}\sigma\omega(u\,;m)=\tau D_{i-1}\textstyle\sum_{j=1}^{p} jt^{j-1}\omega(u\,;m)),$$

we have (5·10).

Denote by $\{\rho v\}_\rho$ the element of ${}^\rho H^j(W, W'\cup F; Z_p)$ containing a cocycle $\rho v\in {}^\rho C^j(W, W'\cup F; Z_p)$. Then we have

LEMMA (5·11). *Let k be an integer such that $0\leqq k\leqq p-2$. Then we have*

$$\Upsilon_\sigma\{\sigma D_i(\textstyle\sum_{m=0}^{p-1} m^k\omega(u\,;m))\}_\sigma$$
$$= -\{\tau D_{i-1}(\textstyle\sum_{m=1}^{p-1}\sum_{j=1}^{p} m^k jt^{j-1}\omega(u\,;m))\}_\tau.$$

Proof. It is an elementary fact that $\sum_{m=0}^{p-1} m^k\equiv 0$ if $0\leqq k\leqq p-2$. Therefore, if we recall that the dual chain map of D_i is carried by the diagonal carrier, it follows easily that $D_i(\sum_{m=0}^{p-1} m^k\,\omega(u\,;m))$ is an element of $C^{pq-i}(W, W'\cup F\,; Z_p)$. By the definition of Υ_σ, it follows that $\Upsilon_\sigma\{\sigma D_i\sum_{m=0}^{p-1} m^k\,\omega(u\,;m)\}_\sigma$ is represented by $\delta D_i(\sum_{m=0}^{p} m^k\omega(u\,;m))$. Thus (5·11) is clear from (5·10).

LEMMA (5·12). $\Upsilon_\tau\{\tau D_{i-1}\sum_{j=1}^{p} jt^{j-1}\omega(u\,;m))\}_\tau = -m\{\sigma D_{i-2}\omega(u\,;m)\}_\sigma.$

Proof. Since $\sum_{j=1}^{p} j=0$, $D_{i-1}(\sum_{j=1}^{p} jt^{-1}\omega(u\,;m))$ is an element of $C^{pq-i+1}(W, W'\cup F; Z_p)$. Therefore $\Upsilon_\tau\{\tau D_{i-1}(\sum_{j=1}^{p} jt^{j-1}\omega(u\,;m))\}_\tau$ is represented by $\delta D_{i-1}(\sum_{j=1}^{p} jt^{j-1}\omega(u\,;m))$. However we have by (5·1) and (5·6) that

$$-\delta D_{i-1}\textstyle\sum_{j=1}^{p} jt^{j-1}\omega(u\,;m)$$
$$=D_{i-2}(\textstyle\sum_{j=1}^{p}\tau_T jt^{j-1}\omega(u\,;m))$$
$$=D_{i-2}(\textstyle\sum_{j=1}^{p}(jt^{j-1}-jt^{j-1+m})\omega(u\,;m))$$
$$=D_{i-2}(\textstyle\sum_{j=1}^{p}(j-(j-m))t^{j-1}\omega(u\,;m))$$
$$=mD_{i-2}(\textstyle\sum_{j=1}^{p} t^{j-1}\omega(u\,;m))$$
$$=m\sigma D_{i-2}\omega(u\,;m)$$

This proves (5·12).

From (5·11) and (5·12), we have

LEMMA (5·13). *Let* $0 \leqq k \leqq p-2$, *then*

$$\Upsilon_\tau \Upsilon_\sigma \{\sigma D_i (\textstyle\sum_{m=0}^{p-1} m^k \omega(u;m))\}_\sigma = \{\sigma D_{i-2} \textstyle\sum_{m=1}^{p-1} m^{k+1} \omega(u;m))\}_\sigma.$$

Since $m^{p-1} \equiv 1$ for an integer $m \not\equiv 0 \bmod p$, we have by the iterations of (5·13) the following

LEMMA (5·14). $(\Upsilon_\tau \Upsilon_\sigma)^{p-1} \{\sigma D_i (\sum_{m=0}^{p-1} \omega(u;m)\}_\sigma = \{\sigma D_{i-2(p-1)} (\sum_{m=1}^{p-1} \omega(u;m))\}_\sigma$.

THEOREM (5·15). $(\Upsilon_\tau \Upsilon_\sigma)^{p-1} \{D_i(\sigma u \times \sigma u \times \cdots \times \sigma u)\}_\sigma$

$$= \{D_{i-2(p-1)}(\sigma u \times \sigma u \times \cdots \times \sigma u\}_\sigma - \{\sigma D_{i-2(p-1)}(u \times u \times \cdots \times u)\}_\sigma.$$

Proof. It follows from (5·9) and (5·14)

$$\begin{aligned}
&(\Upsilon_\tau \Upsilon_\sigma)^{p-1} \{D_i(\sigma u \times \sigma u \times \cdots \times \sigma u)\}_\sigma \\
&= (\Upsilon_\tau \Upsilon_\sigma)^{p-1} \{\sigma D_i (\textstyle\sum_{m=0}^{p-1} \omega(u;m))\}_\sigma \\
&= \{\sigma D_{i-2(p-1)} (\textstyle\sum_{m=1}^{p-1} \omega(u;m))\}_\sigma \\
&= \{\sigma D_{i-2(p-1)} (\textstyle\sum_{m=0}^{p-1} \omega(u;m)\}_\sigma - \{\sigma D_{i-2(p-1)} \omega(u;0)\}_\sigma \\
&= \{D_{i-2(p-1)}(\sigma u \times \sigma u \times \cdots \times \sigma u)\}_\sigma - \{\sigma D_{i-2(p-1)}(u \times u \times \cdots \times u)\}_\sigma
\end{aligned}$$

This proves (5·15).

We shall now give a proof of (3·11).

Let $a \in H^q(W, W'; Z_p)$ be an element whose representative cocycle is $u \in Z^q(W, W'; Z_p)$. Let $p \geqq 3$, then it follows from (2·10), (5·6) and (5·15) by the definitions of $\wp^s$ and μ that

$$\begin{aligned}
&\mu^{p-1} \wp^s \phi_0^*(a) = \mathrm{I}^{*^{-1}} (\Upsilon_\tau \Upsilon_\sigma)^{p-1}\ \mathrm{I}^* \wp^s \phi_0^*(a) \\
&= (-1)^l (t!)^{2s-q} \mathrm{I}^{*^{-1}} (\Upsilon_\tau \Upsilon_\sigma)^{p-1} \{D_{(q-2s)(p-1)}(\sigma u \times \sigma u \times \cdots \times \sigma u)\}_\sigma \\
&= (-1)^l (t!)^{2s-q} \mathrm{I}^{*^{-1}} (\{D_{(q-2s-2)(p-1)}(\sigma u \times \sigma u \times \cdots \times \sigma u)\}_\sigma \\
&\qquad\qquad - \{\sigma D_{(q-2s-2)(p-1)}(u \times u \times \cdots \times u)\}_\sigma) \\
&= (-1)^l (t!)^{2s-q} (\{D_{(q-2s-2)(p-1)}(\phi u \times \phi u \times \cdots \times \phi u)\}_\sigma \\
&\qquad\qquad - \{\phi D_{(q-2s-2)(p-1)}(u \times u \times \cdots \times u)\}_\sigma) \\
&= (-1)^t (t!)^2 (\wp^{s+1} \phi_0^*(a) - \phi_0^* \wp^{s+1}(a)),
\end{aligned}$$

where we put $l = ts + tq(q-1)/2$. However we know that $(t!)^2 \equiv -1$ if t is even, and $\equiv 1$ if t is odd. Hence we have

$$\mu^{p-1} \wp^s \phi_0^* = \phi_0^* \wp^{s+1} - \wp^{s+1} \phi_0^*$$

which is (i) of (3·11).

6. Regularity and almost regularity

Consider the sequence

$$(\mathrm{R}_1) \qquad H^r(W, W'; Z_p) \xrightarrow{\rho_0^*} H^r(W, W' \cup F; Z_p) \xrightarrow{\beta_{\bar\rho}} {}^{\bar\rho}H^r(W, W' \cup F; Z_p).$$

Then $\beta_{\bar\rho} \rho_0^* = 0$ is obvious from $\bar\rho \rho = 0$, but the sequence is not necessarily exact. When the sequence (R_1) is exact for $\rho = \sigma$ and τ, we shall call that (W, W', t) is *almost* *regular* in dimension r. Consider also the sequence

$$(\mathrm{R}_2)\qquad H^r(W, W'; Z_p) \xrightarrow{\rho^*} H^r(W, W'; Z_p) \xrightarrow{\bar\rho^*} H^r(W, W'; Z_p).$$

Then $\bar\rho^*\rho^*=0$, but (R_2) is not necessarily exact. When (R_2) is exact for $\rho=\sigma$ and τ, and in addition the inclusion homomorphism $j^*: H^r(W, W'\cup F; Z_p) \longrightarrow H^r(W, W'; Z_p)$ is isomorphic into, we shall call that (W, W', t) is *regular* in dimension r.

THEOREM (6·1). *If (W, W', t) is regular in dimension r, then it is almost regular in dimension r.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
H^r(W, W'; Z_p) \arrow[r, "\rho_0^*"] \arrow[dr, "\rho^*"'] & H^r(W, W'\cup F; Z_p) \arrow[r, "\beta_{\bar\rho}"] \arrow[d, "j^*"] & {}^{\rho}H^r(W, W'\cup F; Z_p) \arrow[d, "\alpha_\rho"] \\
 & H^r(W, W'; Z_p) \arrow[r, "\rho_0^*"] \arrow[dr, "\bar\rho^*"'] & H^r(W, W'\cup F; Z_p) \arrow[d, "j^*"] \\
 & & H^r(W, W' Z_p).
\end{tikzcd}$$

Assume that $\beta_{\bar\rho}(a)=0$ for $a\in H^r(W, W'\cup F; Z_p)$. Then, since $\bar\rho^*j^*(a)=j^*\alpha_\rho\beta_{\bar\rho}(a)=0$, it follows by the assumption of the theorem that there is an element $b\in H^r(W, W'; Z_p)$ such that $\rho^*(b)=j^*(a)$. Since $\rho^*=j^*\rho_0^*$, we have $j^*(a-\rho_0^*(b))=0$. Since j^* is isomorphic into by the assumption of the theorem, we have $a=\rho_0^*(b)$. This proves (6·1).

Let us denote by ${}^{\rho}N^r(W, W'\cup F; Z_p)$ the kernel of the homomorphism $\alpha_{\bar\rho}: {}^{\rho}H^r(W, W'\cup F; Z_p) \longrightarrow H^r(W, W'\cup F; Z_p)$. Then we have

THEOREM (6·2)[11]. *If (W, W', t) is almost regular in dimension r, it holds that*

$${}^{\bar\rho}N^{r+1}(W, W'\cup F; Z_p)=\varUpsilon_\rho{}^{\rho}N^r(W, W'\cup F; Z_p)+\vartheta_\rho i^*H^r(W, W'; Z_p)^{1)}.$$

Proof. Consider the following diagram

$$\begin{tikzcd}
H^r(W, W'\cup F; Z_p) \arrow[d, "\beta_{\bar\rho}"] & {}^{\rho}H^r(W, W'\cup F; Z_p) \arrow[l, "\alpha_{\bar\rho}"'] \arrow[r, "\varUpsilon_\rho"] & {}^{\bar\rho}H^{r+1}(W, W'\cup F; Z_p) \\
{}^{\bar\rho}H^r(W, W'\cup F; Z_p) & H^r(W, W'; Z_p) \arrow[ul, "\rho_0^*"'] \arrow[u, "\kappa_\rho"'] \arrow[r, "i^*"] & H^r(W'\cup F, W'; Z_p) \arrow[u, "\vartheta_\rho"']
\end{tikzcd}$$

Let $a=\varUpsilon_\rho(b)\in{}^{\bar\rho}N^{r+1}(W, W'\cup F; Z_p)$, where $b\in{}^{\rho}H^r(W, W'\cup F; Z_p)$. Then, since $\beta_{\bar\rho}(\alpha_{\bar\rho}b)=0$, it follows by the assumption that there is an element $c\in H^r(W, W'; Z_p)$ such that $\alpha_{\bar\rho}(b)=\rho_0^*(c)$. Put $d=b-\kappa_\rho(c)$, then it follows from (2·9) that

$$\alpha_{\bar\rho}(d)=\alpha_{\bar\rho}(b)-\alpha_{\bar\rho}\kappa_\rho(c)=\alpha_{\bar\rho}(b)-\rho_0^*(c)=0,$$

so that $d\in{}^{\rho}N^r(W, W'\cup F; Z_p)$. On the other hand, it follows from (2·17) that

$$\begin{aligned} a&=\varUpsilon_\rho(b)=\varUpsilon_\rho(d+\kappa_\rho(c))=\varUpsilon_\rho(d)+\varUpsilon_\rho\kappa_\rho(c)\\ &=\varUpsilon_\rho(d)-\vartheta_\rho i^*(c).\end{aligned}$$

This proves ${}^{\bar\rho}N^{r+1}(W, W'\cup F; Z_p)\subset\varUpsilon_\rho{}^{\rho}N^r(W, W'\cup F; Z_p)+\vartheta_\rho i^*H^r(W, W'; Z_p)$.

11) See Theorem 1 in R. Thom [18.]

The inverse inclusion is obvious. This completes the proof of (6・2).

THEOREM (6・3)[12]. *If* (W, W', t) *is regular in dimension* r, *then it holds that*

$$^{\bar{\rho}}N^{r+1}(W, W'\cup F; Z_p) = \gamma_{\rho}{}^{\rho}N^{r}(W, W'\cup F; Z_p) \oplus \vartheta_{\rho} i^{*} H^{r}(W, W'; Z_p)^{1)}.$$

In addition $\gamma_{\rho}: {}^{\rho}N^{r}(W, W'\cup F; Z_p) \longrightarrow {}^{\bar{\rho}}N^{r+1}(W, W'\cup F; Z_p)$ *is isomorphic into.*

Proof. In virtue of (6・1) and (6・2), it is sufficient to prove that $\gamma_{\rho}(a)+\vartheta_{\rho} i^{*}(b)=0$ implies $a=0$, where $a\in {}^{\rho}N^{r}(W, W'\cup F; Z_p)$ and $b\in H^{r}(W, W'; Z_p)$.

It follows from (2・17) that $\gamma_{\rho}(a-\kappa_{\rho}(b))=0$. Hence there is an element $c\in H^{r}(W, W'\cup F; Z_p)$ such that $a-\kappa_{\rho}(b)=\beta_{\rho}(c)$. Since $\alpha_{\bar{\rho}}(a)=0$, it follows from (1・4) and (2・9) that $-\rho_0^{*}(b)=\rho^{*}(c)$. Thus we have

$$\begin{aligned}\rho^{*}(b+j^{*}(c)) &= j^{*}\rho_0^{*}(b)+\rho^{*}j^{*}(c)\\ &= j^{*}(\rho_0^{*}(b)+\rho_0^{*}(c))=0.\end{aligned}$$

Therefore it follows from the assumption that there is an element $d\in H^{r}(W, W'; Z_p)$ such that $b+j^{*}(c)=\bar{\rho}^{*}(d)$. Then $a-\kappa_{\rho}(\bar{\rho}^{*}(d)-j^{*}(c))=\beta_{\rho}(c)$. Since $\kappa_{\rho}\bar{\rho}^{*}=0$ and $\kappa_{\rho}j^{*}=\beta_{\rho}$, we obtain $a=0$, which is our purpose. This completes the proof of (6・3).

THEOREM (6・4). *If* (W, W', t) *is almost regular in dimension* r, *then we have*

$$^{\rho}H^{r}(W, W'\cup F; Z_p) = {}^{\rho}N^{r}(W, W'\cup F; Z_p) + \kappa_{\rho}H^{r}(W, W'; Z_p).$$

Proof. Consider the following commutative diagram

$$\begin{tikzcd} {}^{\rho}H^{r}(W, W'\cup F; Z_p) \arrow[r, "\alpha_{\bar{\rho}}"] & H^{r}(W, W'\cup F; Z_p) \arrow[r, "\beta_{\bar{\rho}}"] & {}^{\bar{\rho}}H^{r}(W, W'\cup F; Z_p) \\ & H^{r}(W, W'; Z_p) \arrow[ul, "\kappa_{\rho}"] \arrow[u, "\rho_0^{*}"'] & \end{tikzcd}$$

Let $a\in {}^{\rho}H^{r}(W, W'\cup F; Z_p)$. Then, since $\beta_{\bar{\rho}}\alpha_{\bar{\rho}}(a)=0$, it follows from the assumption that there is an element $b\in H^{r}(W, W'; Z_p)$ such that $\alpha_{\bar{\rho}}(a)=\rho_0^{*}(b)$. Put $c=a-\kappa_{\rho}(b)$, then we have

$$\alpha_{\bar{\rho}}(c)=\alpha_{\bar{\rho}}(a)-\alpha_{\bar{\rho}}\kappa_{\rho}(b)=\alpha_{\bar{\rho}}(a)-\rho_0^{*}(b)=0.$$

Therefore $c\in {}^{\rho}N^{r}(W, W'\cup F; Z_p)$. Since $a=c+\kappa_{\rho}(b)$, we have (6・4).

CHAPTER II. COHOMOLOGY OF CYCLIC PRODUCTS

7. Cartesian products and cyclic products

Let K be a finite simplicial complex, and denote by $\mathfrak{X}_{(p)}(K)$ the p-fold cartesian product $K\times K\times\cdots\times K$ of K. Suppose that a locally simple ordering is given in K. Then, as is well known [3], $\mathfrak{X}_{(p)}(K)$ is simplicially decomposed as follows: the vertices of $\mathfrak{X}_{(p)}(K)$ are all the points $a=(a_1, a_2, \cdots, a_p)$, where a_j are vertices of K; Different $(n+1)$ vertices $a^{i}=(a_1^{i}, a_2^{i}, \cdots, a_p^{i})$ $(i=0, 1, 2, \cdots, n)$ of $\mathfrak{X}_{(p)}(K)$ form an n-dimensional simplex if and only if $a_k^0, a_k^1, \cdots, a_k^n$ are contained in a simplex of K and the relations $a_k^0\leqq a_k^1\leqq\cdots\leqq a_k^n$ $(k=1, 2, \cdots, p)$ hold with respect to the order $<$ given in K.

12) See Theorem 2 in R. Thom [18].

Define now an order $<$ among such vertices a^i $(i=0, 1,\cdots, n)$ by $a^0<a^1<\cdots<a^n$. Then a locally simple ordering is introduced in $\mathfrak{X}_{(p)}(K)$. In the following, $\mathfrak{X}_{(p)}(K)$ will be always understood as a simplicial complex with such an ordering.

Let $\mathfrak{t}:\mathfrak{X}_{(p)}(K)\longrightarrow\mathfrak{X}_{(p)}(K)$ be a map defined by

$$\mathfrak{t}(x_1, x_2,\cdots, x_p)=(x_2, x_3,\cdots, x_1), \qquad x_i\in K. \tag{7·1}$$

Then it is easily seen that $\mathfrak{t}$ is a periodic transformation of period p, and satisfies the conditions a) and b) in § 1. Moreover $\mathfrak{t}$ is order-preserving. Thus we may apply the results in Chapter I for $W=\mathfrak{X}_{(p)}(K)$, W'=empty set, $t=\mathfrak{t}$. Obviously the fixed points set under $\mathfrak{t}$ is the diagonal $\mathfrak{D}_{(p)}(K)=\{(x, x,\cdots,x)\mid x\in K\}$. The orbit space $\mathrm{O}(\mathfrak{X}_{(p)}(K), \mathfrak{t})$ is called usually the *p-fold cyclic product* of K [5, 11]. This corresponds to W_t in the theory in Chapter I, and hence we see that it is a simplicial complex. In the following, this complex will be denoted by $\mathfrak{Z}_{(p)}(K)$. Denote also by $\mathfrak{d}_{(p)}(K)$ the image of $\mathfrak{D}_{(p)}(K)$ by the projection $\pi:\mathfrak{X}_{(p)}(K)\longrightarrow\mathfrak{Z}_{(p)}(K)$. $\pi:\mathfrak{D}_{(p)}(K)\longrightarrow\mathfrak{d}_{(p)}(K)$ is a homeomorphism.

Applying the theory of Chapter I to the complex $\mathfrak{X}_{(p)}(K)$ with the transformation $\mathfrak{t}$ defined by (7·1), we shall in the present chapter study the cohomology of $\mathfrak{Z}_{(p)}(K)$. Here p is an arbitrary prime number, but p shall remain fixed throughout the discussion of this chapter. Therefore we write briefly $\mathfrak{X}(K)$, $\mathfrak{Z}(K)$, $\cdots$ for $\mathfrak{X}_{(p)}(K)$, $\mathfrak{Z}_{(p)}(K)$,$\cdots$.

Let $d:K\longrightarrow\mathfrak{X}(K)$ be the diagonal map (*i. e.* a map defined by $d(x)=(x, x,\cdots, x)$ for any $x\in K$), and let $d^*: H^r(\mathfrak{X}(K);G)\longrightarrow H^r(K;G)$ be the homomorphism induced by d. Then, for any element $a\in H^r(K;G)$ and the unit class $\mathbf{1}\in H^0(K;Z)$, we have by the definition of cup product

$$d^*(a\times\mathbf{1}\times\cdots\times\mathbf{1})=a\cup\mathbf{1}\cup\cdots\cup\mathbf{1}=a, \tag{7·2}$$

where the cross and cup products are taken with respect to the natural multiplication $G\otimes Z\otimes\cdots\otimes Z\longrightarrow G$. Especially we see that d^* is onto. Let $d_0:K\longrightarrow\mathfrak{D}(K)$ be the map defined by d, then d_0 is a homeomorphism. Hence $d_0^*:H^r(\mathfrak{D}(K);G)\longrightarrow H^r(K;G)$ is isomorphic onto. Since the commutativity holds in the diagram

$$\begin{tikzcd} H^r(\mathfrak{X}(K);G) \arrow[rr, "i^*"] \arrow[dr, "d^*"'] & & H^r(\mathfrak{D}(K);G) \arrow[dl, "d_0^*", "\approx"'] \\ & H^r(K;G) & \end{tikzcd}$$

the above consideration implies that i^* is onto. Thus, by the exact sequence for $(\mathfrak{X}(K),\mathfrak{D}(K))$, we have the following:

THEOREM (7·3). *The sequence*

$$0\longrightarrow H^r(\mathfrak{X}(K),\mathfrak{D}(K);G)\xrightarrow{j^*}H^r(\mathfrak{X}(K);G)\xrightarrow{i^*}H^r(\mathfrak{D}(K);G)\longrightarrow 0$$

is exact; moreover it holds that $d_0^*i^*=d^*$ and $d_0^*:H^r(\mathfrak{D}(K);G)\approx H^r(K;G)$.

Let G be a field, and let $\Omega^*(K;G)$ be a (homogeneous) base for the vector space $H^*(K;G)$. Then the cross product $b_1\times b_2\times\cdots\times b_p$ $(b_j\in\Omega^*(K;G))$ is an element of $H^*(\mathfrak{X}(K);G)$. It is well-known as the Künneth formula that we can take as a base for the vector space $H^r(\mathfrak{X}(K);G)$ the following set:

$$B^r(\Omega^*(K;G))=\{b_1\times b_2\times\cdots\times b_p \mid b_j\in\Omega^*(K;G), \textstyle\sum_{j=1}^p q_j=r\},$$

where $\dim b_j=q_j$. Denote by $B''^r(\Omega^*(K;G))$ a subset of $B^r(\Omega^*(K;G))$ consisting of all the 'diagonal' elements:

$$B''(\Omega^*(K;G))=\{b\times b\times\cdots\times b \mid b\in\Omega^*(K;G), pq=r\},$$

where $\dim b=q$, and let

$$B'^r(\Omega^*(K;G))=B^r(\Omega^*(K;G))-B''^r(\Omega^*(K;G)).$$

Furthermore we shall denote by $V'^r(\Omega^*(K;G))$ and $V''^r(\Omega^*(K;G))$ the vector subspaces spanned by $B'^r(\Omega^*(K;G))$ and $B''^r(\Omega^*(K;G))$ respectively. Then, since

$$\mathfrak{t}^*(b_1\times b_2\times\cdots\times b_p)=(-1)^{q_1(q_2+\cdots+q_p)}(b_2\times b_3\times\cdots\times b_1)$$

for the homomorphism $\mathfrak{t}^*: H^r(\mathfrak{X}(K);G)\longrightarrow H^r(\mathfrak{X}(K);G)$ induced by $\mathfrak{t}$, it is obvious that $B'^r(\Omega^*(K;G))$ and $B''^r(\Omega^*(K;G))$ are $\mathfrak{t}^*$-invariant subspaces. This, together with the fact p is prime, proves that there is a set $B_t'^r(\Omega^*(K;G))$ such that

(i) any element of $B_t'^r(\Omega^*(K;G))$ is written εw, where $\varepsilon=1$ or -1 and $w\in B'^r(\Omega^*(K;G))$,

(ii) the set $\{\mathfrak{t}^{*j}(\bar{w}) \mid \bar{w}\in B_t'^r(\Omega^*(K;G)),\ 0\leqq j\leqq p-1\}$ is a base for $V'^r(\Omega^*(K;G))$.

Thus we can now prove by the same arguments as in the proof of $(1\cdot 3)$ the following:

THEOREM $(7\cdot 4)$. *Let G be a field, and let $a\in H^r(\mathfrak{X}(K);G)$ be an element such that $\bar{\rho}^*a=0$. Then there are two elements $x, y\in H^r(\mathfrak{X}(K);G)$ such that $a=\rho^*x+y$ and y is a linear combination of diagonal elements for a base of $H^*(K;G)$.*

If r is not divisible by p, there is no diagonal element. Therefore we have

COROLLARY $(7\cdot 5)$. *Let G be a field, and assume that r is not divisible by p. Then the sequence*

$$H^r(\mathfrak{X}(K);G)\xrightarrow{\rho^*}H^r(\mathfrak{X}(K);G)\xrightarrow{\bar{\rho}^*}H^r(\mathfrak{X}(X);G)$$

is exact.

8. Proof of almost regularity

We shall in this section prove that $(\mathfrak{X}(K), \emptyset, \mathfrak{t})$ is almost regular in every dimension, where $\emptyset$ denotes the empty set. We abbreviate $(\mathfrak{X}(K), \emptyset, \mathfrak{t})$ as $(\mathfrak{X}(K), \mathfrak{t})$.

From $(7\cdot 3)$ and $(7\cdot 5)$, we have immediately

THEOREM $(8\cdot 1)$. *$(\mathfrak{X}(K), \mathfrak{t})$ is regular in dimension r which is not divisible by p.*

LEMMA $(8\cdot 2)$. *Let $a\in H^{pq}(\mathfrak{X}(K), \mathfrak{D}(K);Z_p)$ be an element such that $\beta_{\bar{\rho}}a=0$ and j^*a is a linear combination of diagonal elements for a base of $H^*(K;Z_p)$. Then we have $a=0$.*

Proof. Let $\dim k=n$.

Case $1: q=n$. Consider the following commutative diagram

$$\begin{tikzcd}
 & {}^{\rho}H^{pn}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p) \arrow[dr, "\alpha_{\bar\rho}"] & & \\
H^{pn}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p) \arrow[ur, "\beta_\rho"] \arrow[rr, "\rho^*"] \arrow[d, "j^*"'] & & H^{pn}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p) \arrow[d, "j^*"] \arrow[dr, "\beta_{\bar\rho}"] & \\
H^{pn}(\mathfrak{X}(K);Z_p) \arrow[rr, "\rho^*"'] & & H^{pn}(\mathfrak{X}(K);Z_p) & {}^{\bar\rho}H^{pn}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)
\end{tikzcd}$$

Since $\beta_{\bar\rho}(a)=0$, there is an element $b\in{}^{\rho}H^{pn}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)$ such that $\alpha_{\bar\rho}(b)=a$. β_ρ is onto, so that there is an element $c\in H^{pn}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)$ such that $\beta_\rho(c)=b$. Thus we have $a=\alpha_{\bar\rho}(b)=\alpha_{\bar\rho}\beta_\rho(c)=\rho^*(c)$. Let $\Omega^*(K;Z_p)$ be the base of $H^*(K;Z_p)$ stated in the assumption of (8・2), and use the notations in §7. Let $j^*(c)=c'+c''$, where $c'\in V'(\Omega^*(K;Z_p))$ and $c''\in V''(\Omega^*(K;Z_p))$. Then it follows that

$$\begin{aligned} j^*(a) &= j^*\rho^*(c)=\rho^*j^*(c)\\ &=\rho^*(c')+\rho^*(c'')=\rho^*(c'), \end{aligned}$$

so that $j^*(a)\in V'(\Omega^*(K;Z_p))$. However $j^*(a)\in V''(\Omega^*(K;Z_p))$ by the assumption. Since $V'(\Omega^*(K;Z_p))\cap V''(\Omega^*(K;Z_p))=0$, we have $j^*(a)=0$. Since j^* is isomorpic into by (7・3), we concude $a=0$.

Case $2: q<n$. Denote by K^q the q-skelton of K. Let $g:K^q\longrightarrow K$ be the inclusion, and let $G:\mathfrak{X}(K^q)\longrightarrow\mathfrak{X}(K)$ be the map given by $g\times g\times\cdots\times g$. Let $\Omega^*(K;Z_p)$ be the base of $H^*(K;Z_p)$ stated in the assumption of (8・2). Since $g^*:H^q(K;Z_p)\longrightarrow H^q(K^q;Z_p)$ is isomorphic into, there is a base $\Omega^*(K^q;Z_p)$ of $H^*(K^q;Z_p)$ which contains all the elements $g^*(b)$ such that $b\in\Omega^*(K;Z_p)$ and dim $b=q$. It is obvious that $G^*(V''^{pq}(\Omega^*(K;Z_p))\subset V'^{pq}(\Omega^*(K^q;Z_p))$. We shall first prove that $G^*:V''^{pq}(\Omega^*(K;Z_p))\longrightarrow V''^{pq}(\Omega^*(K^q;Z_p))$ is isomorphic into. Let $\otimes_pH^q(K;Z_p)$ denotes the p-fold tensor product of $H^q(K;Z_p)$, then we have the natural into-isomorphism ξ: $V''^{pq}(\Omega^*(K;Z_p))\longrightarrow\otimes_pH^q(K;Z_p)$. We have also the similar into-isomorphism ξ for K^q. Consider the following diagram

$$\begin{tikzcd}
V''^{pq}(\Omega^*(K;Z_p)) \arrow[r, "\xi"] \arrow[d, "G^*"] & \otimes_pH^q(K;Z_p) \arrow[d, "\otimes_pg^*"] \\
V''^{pq}(\Omega^*(K^q;Z_p)) \arrow[r, "\xi"] & \otimes_pH^q(K^q;Z_p),
\end{tikzcd}$$

where $\otimes_p g^*=g^*\otimes g^*\otimes\cdots\otimes g^*$. Then it is obvious that the commutativity holds in this diagram. Moreover, since $g^*:H^q(K;Z_p)\longrightarrow H^q(K^q;Z_p)$ is isomorphic into, it follows that $\otimes_p g^*$ is also isomorphic into. This shows that G^* is isomorphic into.

Consider next the commutative diagram

$$\begin{tikzcd}
H^{pq}(\mathfrak{X}(K);Z_p) \arrow[d, "G^*"] & H^{pq}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p) \arrow[l, "j^*"'] \arrow[r, "\beta_{\bar\rho}"] \arrow[d, "G^*"] & {}^{\bar\rho}H^{pq}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p) \arrow[d, "G^*"] \\
H^{pq}(\mathfrak{X}(K^q);Z_p) & H^{pq}(\mathfrak{X}(K^q),\mathfrak{D}(K^q);Z_p) \arrow[l, "j^*"'] \arrow[r, "\beta_{\bar\rho}"] & {}^{\bar\rho}H^{pq}(\mathfrak{X}(K^q),\mathfrak{D}(K^q);Z_p),
\end{tikzcd}$$

then the assumption $\beta_{\bar\rho}(a)=0$ implies that $\beta_{\bar\rho}G^*(a)=G^*\beta_{\bar\rho}(a)=0$. Since $j^*(a)\in V''^{pq}(\Omega^*(K;Z_p))$ by the assumption, it holds $j^*G^*(a)=G^*j^*(a)\in V''^{pq}(\Omega^*(K^q;Z_p))$. Therefore we have $G^*(a)=0$ by the fact proved in case 1. Thus we have also $G^*j^*(a)=j^*G^*(a)=0$. Since $j^*(a)\in V''^{pq}(\Omega^*(K;Z_p))$, we obtain $j^*a=0$ by the above-mentioned fact. This means $a=0$ by (7・3). Q. E. D.

THEOREM (8・3)[13]. *$(\mathfrak{X}(K),t)$ is almost regular in dimension $r=pq$ divisible by p.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
H^{pq}(\mathfrak{X}(K);Z_p) \arrow[rr, "\rho_0^*"] \arrow[d, "\rho^*"] & & H^{pq}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p) \arrow[r, "\beta_{\bar\rho}"] \arrow[dll, "j^*"] \arrow[dr, "\bar\rho^*"] & {}^{\bar\rho}H^{pq}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p) \arrow[d, "\alpha_\rho"] \\
H^{pq}(\mathfrak{X}(K);Z_p) \arrow[dr, "\bar\rho^*"] & & & H^{pq}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p) \arrow[dll, "j^*"] \\
& H^{pq}(\mathfrak{X}(K);Z_p) & &
\end{tikzcd}$$

Let $a\in H^{pq}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)$ be an element such that $\beta_{\bar\rho}(a)=0$. Then $\bar\rho^*j^*(a)=j^*\bar\rho^*(a)=j^*\alpha_\rho\beta_{\bar\rho}(a)$, so that there is an element $x,y\in H^{pq}(\mathfrak{X}(K);Z_p)$ such that $j^*(a)=\rho^*(x)+y$ and y is a linear combination of diagonal elements for a base of $H^*(K;Z_p)$, in virtue of (7・4). Therefore $j^*(a)=j^*\rho_0^*(x)+y$, and hence $j^*(a-\rho_0^*(x))=y$. Since $\beta_{\bar\rho}(a-\rho_0^*(x))=\beta_{\bar\rho}(a)-\beta_{\bar\rho}\rho_0^*(x)=0$, it follows from (8・2) that $a-\rho_0^*(x)=0$. Therefore $a=\rho_0^*(x)$. Q. E. D.

Summarizing (8・1) and (8・3), we have by (6・1) the following:

THEOREM (8・4). *$(\mathfrak{X}(K),t)$ is almost regular in every dimension.*

9. The homomorphism Γ_s^ρ

Let

$$\Gamma_s^\rho : H^q(K;Z_p) \longrightarrow {}^{\bar\rho}H^{q+s}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)$$

be the homomorphism defined by $\bar\Gamma_s^\rho d_0^{*-1}$. (See (2・20) as for the definition of $\bar\Gamma_s^\rho$.) The purpose of this section is to prove

THEOREM (9・1)[14]. *The homomorphism $\Gamma_s^\rho : H^q(K;Z_p) \longrightarrow {}^{\bar\rho}H^{q+s}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)$ is isomorphic into for $1\leqq s\leqq (p-1)q$.*

As is proved in § 8, $(\mathfrak{X}(K),t)$ is almost regular in every dimension. Therefore the following is obvious from (7・2).

LEMMA (9・2). *For any r, we have*

$${}^{\bar\rho}N^{r+1}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)=\gamma_\rho{}^\rho N^r(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)+\Gamma_1^\rho H^r(K;Z_p).$$

Let S^n be an n-dimensional sphere. Then $H^q(S^n;Z_p)=0$ if $q\neq 0,n$. Since $\kappa_\rho(\mathbf{1})$

13) See Theorem 3 in [18].
14) See Theorem 4 in [18].

$=0$ by the definition of κ_ρ, it follows from (2·17) that $\vartheta_\rho i^*(1)=-\gamma_\rho\kappa_\rho(1)=0$. Therefore $\Gamma^\rho_s H^0(S^n;Z_p)=0$. Thus, by iterations of (9·2), we obtain

LEMMA (9·3). ${}^\rho N^{np}(\mathfrak{X}(S^n),\mathfrak{D}(S^n);Z_p)=\Gamma^{\bar\rho}_{n(p-1)}H^n(S^n;Z_p)$.

Consider the commutative diagram

$$\begin{tikzcd}
{}^\rho H^{np}(\mathfrak{X}(S^n),\mathfrak{D}(S^n);Z_p) \arrow[r, "\alpha_{\bar\rho}"] & H^{np}(\mathfrak{X}(S^n),\mathfrak{D}(S^n);Z_p) \arrow[r, "\beta_{\bar\rho}"] \arrow[dr, "j^*"] & {}^{\bar\rho}H^{np}(\mathfrak{X}(S^n),\mathfrak{D}(S^n);Z_p) \\
H^{np}(\mathfrak{X}(S^n),\mathfrak{D}(S^n);Z_p) \arrow[u, "\beta_\rho"] \arrow[r, "j^*"'] & H^{np}(\mathfrak{X}(S^n);Z_p) \arrow[ul, "\kappa_\rho"] \arrow[u, "\rho_0^*"'] \arrow[r, "\rho^*"'] & H^{np}(\mathfrak{X}(S^n);Z_p)
\end{tikzcd}$$

Since np is the maximal dimension, β_ρ and $\beta_{\bar\rho}$ are onto. Let $e_n\in H^n(S^n;Z_p)$ be a generator, then $H^{np}(\mathfrak{X}(S^n);Z_p)\approx Z_p$ is generated by $e_n\times e_n\times\cdots\times e_n$. Since $t^*(e_n\times e_n\times\cdots\times e_n)=e_n\times e_n\times\cdots\times e_n$, we have $\rho^*(e_n\times e_n\times\cdots\times e_n)=0$. Thus $j^*\rho_0^*=\rho^*$ is trivial, so that it follows from (7·3) that $\alpha_{\bar\rho}\kappa_\rho=\rho_0^*$ is trivial. However, since $\beta_\rho=\kappa_\rho j^*$ is onto, κ_ρ is onto. Therefore $\alpha_{\bar\rho}$ is trivial, and hence $\beta_{\bar\rho}$ is isomorphic into. Since $\beta_{\bar\rho}$ is onto, $\beta_{\bar\rho}$ is an isomorphism. Thus we have

LEMMA (9·4). ${}^\rho H^{np}(\mathfrak{X}(S^n),\mathfrak{D}(S^n);Z_p)={}^\rho N^{np}(\mathfrak{X}(S^n),\mathfrak{D}(S^n);Z_p)=\kappa_\rho H^{np}(\mathfrak{X}(S^n);Z_p)\approx Z_p$, *where κ_ρ is an isomorphism.*

From (9·3) and (9·4), we have

THEOREM (9·5). *There is an integer mod p $\chi_{\rho,n}\not\equiv 0$ such that*

$$\chi_\rho(e_n\times e_n\times\cdots\times e_n)=\chi_{\rho,n}\Gamma^{\bar\rho}_{n(p-1)}(e_n).$$

REMARK. We can assert easily that $\chi_{\rho,n}$ mod p is independent of the choice of generators e_n of $H^n(S^n;Z_p)$. For some n, we can determine $\chi_{\rho,n}$ explicitely. (See (12·7).)

Proof of (9·1). For this purpose, it is sufficient to prove that $\Gamma^{\bar\rho}_{q(p-1)}:H^q(K;Z_p)\longrightarrow {}^\rho H^{pq}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)$ is isomorphic into.

Let $a\in H^q(K;Z_p)$ be an element such that $\Gamma^{\bar\rho}_{q(p-1)}(a)=0$. Our purpose is to prove $a=0$.

Case $1:q=n$, where $n=\dim K$. In this case there is a map $f:K\longrightarrow S^n$ such that $f^*(e_n)=a$. Let $F:\mathfrak{X}(K)\longrightarrow\mathfrak{X}(S^n)$ be the map given by $F=f\times f\times\cdots\times f$. Since $f^*d_0^*=d_0^*F$,* it follows from (9·5) by the naturality of γ_ρ, ϑ_ρ and κ_ρ that

$$\begin{aligned}
&\Gamma^{\bar\rho}_{n(p-1)}(a)=\Gamma^{\bar\rho}_{n(p-1)}(f^*e_n)=F^*\Gamma^{\bar\rho}_{n(p-1)}(e_n)\\
&=\chi^{-1}_{\rho,n}F^*\kappa_\rho(e_n\times e_n\times\cdots\times e_n)\\
&=\chi^{-1}_{\rho,n}\kappa_\rho F^*(e_n\times e_n\times\cdots\times e_n)\\
&=\chi^{-1}_{\rho,n}\kappa_\rho(a\times a\times\cdots\times a).
\end{aligned}$$

Therefore we have $\kappa_\rho(a\times a\times\cdots\times a)=0$ by the assumption. Consider the commutative diagram

$$\begin{array}{ccccc} H^{pn}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p) & \xrightarrow{j^*} & H^{pn}(\mathfrak{X}(K);Z_p) & \xrightarrow{i^*} & H^{pn}(\mathfrak{D}(K);Z_p) \\ \downarrow \beta_\rho & \swarrow \kappa_\rho & & & \\ {}^\rho H^{pn}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p) & & & & \end{array}$$

Since $H^{pn}(\mathfrak{X}(K);Z_p)\approx H^{pn}(K;Z_p)=0$, we see that j^* is onto. Thus there is an element $b\in H^{pn}(\mathfrak{X}(K),\ \mathfrak{D}(K);Z_p)$ such that $a\times a\times\cdots\times a=j^*b$. Then we have $\beta_\rho(b)=\kappa_\rho j^*(b)=\kappa_\rho(a\times a\times\cdots\times a)=0$. Since there is a base of $H^*(K;Z_p)$ containing a, $j^*(b)$ is a diagonal element for such a base of $H^*(K;Z_p)$. Thus it follows from (8·2) that $b=0$, so that $a\times a\times\cdots\times a=0$. This means $a=0$.

Case $2:q<n$. Let $g:K^q\longrightarrow K$ be the inclusion, and let $G:\mathfrak{X}(K^q)\longrightarrow\mathfrak{X}(K)$ be the map given by $g\times g\times\cdots\times g$. Then we have by the assumption

$$\Gamma^{\bar\rho}_{q(p-1)}(g^*a)=G^*\Gamma^{\bar\rho}_{q(p-1)}(a)=0,$$

and hence it follows from the fact proved in Case 1 that $g^*(a)=0$. Since $g^*: H^q(K;Z_p)\longrightarrow H^q(K^q;Z_p)$ is isomorphic into, we have $a=0$. This completes the proof of (9·1).

We shall now define a homomorphism

$$E_s: H^q(K;Z_p)\longrightarrow H^{q+s}(\mathfrak{Z}(K),\mathfrak{d}(K);Z_p) \qquad (s>0)$$

as follows:

(9·6) $$E_{2\alpha+1}=\mu^\alpha\delta^*\pi^{*^{-1}}d_0^{*^{-1}},\qquad E_{2\alpha+2}=\mu^\alpha\nu\delta^*\pi^{*^{-1}}d_0^{*^{-1}}\qquad(\alpha\geqq 0).$$

Then it is immediate that

(9·7) $$E^s=\mathrm{I}^{*^{-1}}\Gamma^\tau_s$$

Obviously we have

(9·8) $$E_{2\alpha+2}=\nu E_{2\alpha+1}\quad(\alpha\geqq 0),\qquad E_{s+2\alpha}=\mu^\alpha E_s\qquad(s>0).$$

The following is a translation of (9·1) for $\rho=\sigma$.

THEOREM (9·9). *The homomorphism* $E_s: H^q(K;Z_p)\longrightarrow H^{q+s}(\mathfrak{Z}(K),\mathfrak{d}(K);Z_p)$ *is isomorphic into for* $1\leqq s\leqq(p-1)q$.

10. Kernel of π^*

LEMMA (10·1). *Let r be not divisible by p. Then*

$${}^{\bar\rho}N^{r+1}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)=\gamma_\rho{}^\rho N^r(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)\oplus\Gamma^\rho_1 H^r(K;Z_p).$$

Moreover γ_ρ: ${}^\rho N^r(\mathfrak{X}(K),\ \mathfrak{D}(K);Z_p)\longrightarrow{}^{\bar\rho}N^{r+1}(\mathfrak{X}(K),\ \mathfrak{D}(K);Z_p)$ *and* Γ^ρ_1 *are both isomorphic into.*

Proof. This is obvious from (8·1), (6·3) and (9·1).

LEMMA (10·2). *Let* $x\in H^q(K;Z_p)$, *then*

$$\kappa_\rho(x\times x\times\cdots\times x)\in{}^\rho N^{pq}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p).$$

Proof. It follows that

$$j^*\alpha_{\bar\rho}\kappa_\rho(x\times x\times\cdots\times x)=j^*\rho_0^*(x\times x\times\cdots\times x)$$
$$=\rho^*(x\times x\times\cdots\times x)=0.$$

Since j^* is isomorphic into, $\alpha_{\bar{p}}\kappa_\rho(x\times x\times\cdots\times x)=0$. This proves (19·2).

LEMMA (10·3). *Assume that*

$$ {}^{\rho}N^{pq}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)=\oplus\sum_{s=q}^{pq-1}\Gamma_{pq-s}^{\bar{p}}H^s(K;Z_p)^{15)} $$

holds for some q. Then the component in $\Gamma_{(p-1)q}^{\bar{p}}H^q(K;Z_p)$ *of* $\kappa_\rho(x\times x\times\cdots\times x)$ *for the above direct decomposition is* $\chi_{\rho,q}\Gamma_{(p-1)q}^{\bar{p}}(x)$, *where* $\chi_{\rho,q}$ *is the integer mod p in* (9·5).

Proof. Let $g:K^q\longrightarrow K$ and $G:\mathfrak{X}(K^q)\longrightarrow\mathfrak{X}(K)$ be the maps in the proof of (9·1). Then, as is shown there, it follows that

$$ \begin{aligned} G^*\kappa_\rho(x\times x\times\cdots\times x)&=\kappa_\rho(g^*x\times g^*x\times\cdots\times g^*x)\\ &=\chi_{\rho,q}\Gamma_{(p-1)q}^{\bar{p}}(g^*x). \end{aligned} $$

On the other hand, if we put $\kappa_\rho(x\times x\times\cdots\times x)=\sum_{s=q}^{pq-1}\Gamma_{pq-s}^{\bar{p}}\,y_{\rho,s}$ with $y_{\rho,s}\in H^s(K;Z_p)$, then we have

$$ \begin{aligned} G^*\kappa_\rho(x\times x\times\cdots\times x)&=G^*\sum_{s=q}^{pq-1}\Gamma_{pq-1}^{\bar{p}}(y_{\rho,s})\\ &=\sum_{s=q}^{pq-1}\Gamma_{pq-s}^{\bar{p}}(g^*y_{\rho,s})=\Gamma_{q(p-1)}^{\bar{p}}g^*(y_{\rho,s}), \end{aligned} $$

since $H^s(K^q;Z_p)=0$ if $s>q$. Therefore we have $\chi_{\rho,q}\Gamma_{q(p-1)}^{\bar{p}}g^*(x)=\Gamma_{q(p-1)}^{\bar{p}}\,g^*(y_{\rho,s})$. Since $\Gamma_{q(p-1)}^{\bar{p}}$ and g^* is isomorphic into, we conclude $y_{\rho,s}=\chi_{\rho,s}x$.

We are now in a position to prove

THEOREM (10·4)[16)]

$$ {}^{\bar{p}}N^r(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)=\oplus\sum_{s=[(r+p-1)/p]}^{r-1}\Gamma_{r-s}^{\rho}H^s(K;Z_p)^{17)}. $$

Note that each Γ_{r-s}^{ρ} is isomorphic into by (9·1). Such a direct decomposition of ${}^{\bar{p}}N^r(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)$ will be called *canonical.*

Proof. Mathematical induction on r will be used. For $r<0$, the both sides are obviously zero, and hence (10·4) holds for $r<0$. We shall now assume that (10·4) is valid for $r\leqq r_0$, and prove (10·4) for $r=r_0+1$.

Case 1 : r_0 is not divisible by p.

By (10·1) and the hypothesis of induction, we have

$$ \begin{aligned} {}^{\bar{p}}N^{r_0+1}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)&=\gamma_\rho{}^{\bar{p}}N^{r_0}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)\oplus\Gamma_1^{\rho}H^{r_0}(K;Z_p)\\ &=\gamma_\rho((\oplus\sum_{s=[(r_0+p-1)/p]}^{r_0-1}\Gamma_{r_0-s}^{\bar{p}}H^s(K;Z_p)\oplus\Gamma_1^{\rho}H^{r_0}(K;Z_p)\\ &=\oplus\sum_{s=[(r_0+p-1)/p]}^{r_0-1}\Gamma_{r_0-s+1}^{\rho}H^s(K;Z_p)\oplus\Gamma_1^{\rho}H^{r_0}(K;Z_p)\\ &=\oplus\sum_{s=[(r_0+p)/p]}^{r_0}\Gamma_{r_0-s+1}^{\rho}H^s(K;Z_p), \end{aligned} $$

since $[(r_0+p)/p]=[(r_0+p-1)/p]$ if r_0 is not divisible by p. This proves (10·4) for $r=r_0+1$.

---

15) Let A_j $(j=1,2,\cdots,r)$ be subgroups of an abelian group. Then we write $A_1+A_2+\cdots+A_r$ as $\sum_{j=1}^r A_j$, and $A_1\oplus A_2\oplus\cdots\oplus A_r$ as $\oplus\sum_{j=1}^r A_j$

16) See Theorem 5 in [18].

19) Let r be a number, then we denote by $[r]$ the greatest integer $\leqq r$.

Case $2: r_0$ is divisible by p.

Let $r_0=pq$. We must prove

I) ${}^{\bar{p}}N^{pq+1}(\mathfrak{X}(K), \mathfrak{D}(K); Z_p)=\sum_{s=q+1}^{pq} \Gamma^{\rho}_{pq-s+1}H^s(K; Z_p)$.

II) *The above decomposition is direct.*

Proof of I). It follows from (9•2) and the hypothesis of induction that

$$
\begin{aligned}
{}^{\bar{p}}N^{pq+1}(\mathfrak{X}(K), \mathfrak{D}(K); Z_p) &= \gamma_{\rho}{}^{\rho}N^{pq}(\mathfrak{X}(K), \mathfrak{D}(K); Z_p)+\Gamma^{\rho}_1 H^{pq}(K; Z_p) \\
&= \gamma_{\rho}(\textstyle\sum_{s=q}^{pq-1}\Gamma^{\bar{\rho}}_{pq-s}H^s(K; Z_p))+\Gamma^{\rho}_1 H^{pq}(K; Z_p)=\sum_{s=q}^{pq}\Gamma^{\rho}_{pq-s+1}H^s(K; Z_p).
\end{aligned}
$$

Thus the proof of I) completes if we prove $\Gamma^{\rho}_{(p-1)q+1}H^q(K; Z_p)\subset\sum_{s=q+1}^{pq}\Gamma^{\rho}_{pq-s+1} H^s(K; Z_p)$. For this purpose, we shall consider $\kappa_{\rho}(x\times x\times\cdots\times x)$ for any $x\in H^q(K; Z_p)$. Then we have $\kappa_{\rho}(x\times x\times\cdots\times x)\in{}^{\rho}N^{pq}(\mathfrak{X}(K), \mathfrak{D}(K); Z_p)$ by (10•2). Let $y_{\rho,s}\in H^s(K; Z_p)$ $(s=q, q+1, \cdots, pq-1)$ be elements such that $\kappa_{\rho}(x\times x\times\cdots\times x)=\sum_{s=q}^{pq-1}\Gamma^{\bar{\rho}}_{pq-s}(y_{\rho,s})$. Then we have $\Gamma^{\bar{\rho}}_{(p-1)q}(y_{\rho,q})=\chi_{\rho,q}\Gamma^{\bar{\rho}}_{(p-1)q}(x)$ by (10•3). Therefore $\chi_{\rho,q}\Gamma^{\bar{\rho}}_{(p-1)q}(x)=\kappa_{\rho}(x\times x\times\cdots\times x)-\sum_{s=q+1}^{pq-1}\Gamma^{\bar{\rho}}_{pq-s}(y_{\rho,s})$. Applying γ_{ρ} to the both sides of this equation, we have

$$\chi_{\rho,q}\Gamma^{\rho}_{(p-1)q+1}(x)=\gamma_{\rho}\kappa_{\rho}(x\times x\times\cdots\times x)-\sum_{s=q+1}^{pq-1}\Gamma^{\rho}_{pq-s+1}(y_{\rho,s}).$$

Since $\gamma_{\rho}\kappa_{\rho}(x\times x\times\cdots\times x)=-\vartheta_{\rho}i^*(x\times x\times\cdots\times x)=-\vartheta_{\rho}d_0^{*-1}d^*(x\times x\times\cdots\times x)=-\vartheta_{\rho} d_0^{*-1}(x\cup x\cup\cdots\cup x)$ by (2•17), (7•3) and the definition of cup product [15], we have

(A) $$\chi_{\rho,q}\Gamma^{\rho}_{(p-1)q+1}(x)=-\sum_{s=q+1}^{pq}\Gamma^{\rho}_{pq-s+1}(y_{\rho,s}),$$

where we put $y_{\rho,pq}=x\cup x\cup\cdots\cup x$. This proves $\Gamma^{\rho}_{(p-1)q+1}H^q(K; Z_p)\subset\sum_{s=q+1}^{pq}H^s(K; Z_p)$, and completes the proof of I).

Proof of II). Assume that $\sum_{s=q+1}^{pq}\Gamma^{\rho}_{pq-s+1}(a_s)=0$ with $a_s\in H^s(K; Z_p)$. Then our purpose is to prove $\Gamma^{\rho}_{pq-s+1}(a_s)=0$ for $q+1\leqq s\leqq pq$. Since $\Gamma^{\rho}_1(a_{pq})=\vartheta_{\rho}d_0^{*-1}(a_{pq})=\vartheta_{\rho}i^*(a_{pq}\times 1\times\cdots\times 1)=-\gamma_{\rho}\kappa_{\rho}(a_{pq}\times 1\times\cdots\times 1)$, it follows from the assumption that

$$\gamma_{\rho}\{\kappa_{\rho}(a_{pq}\times 1\times\cdots\times 1)-\sum_{s=q+1}^{pq-1}\Gamma^{\bar{\rho}}_{pq-s}(a_s)\}=0.$$

Therefore there is an element $b_{pq}\in H^{pq}(\mathfrak{X}(K), \mathfrak{D}(K); Z_p)$ such that

(B) $$\kappa_{\rho}(a_{pq}\times 1\times\cdots\times 1)=\beta_{\rho}(b_{pq})+\sum_{s=q+1}^{pq-1}\Gamma^{\bar{\rho}}_{pq-s}(a_s).$$

Applying $j^*\alpha_{\bar{\rho}}$ to the both sides of (B), we have

$$\rho^*(a_{pq}\times 1\times\cdots\times 1)=\rho^*j^*(b_{pq}),$$

since $j^*\alpha_{\bar{\rho}}\kappa_{\rho}=\rho^*$, $j^*\alpha_{\bar{\rho}}\beta_{\bar{\rho}}=j^*\rho^*=\rho^*j^*$, $\alpha_{\bar{\rho}}\gamma_{\bar{\rho}}=0$ and $\alpha_{\bar{\rho}}\vartheta_{\rho}i^*=-\alpha_{\bar{\rho}}\gamma_{\bar{\rho}}\kappa_{\bar{\rho}}=0$. Thus it follows from (7•4) that there exist elements $x, y\in H^{pq}(\mathfrak{X}(K); Z_p)$ such that

$$(a_{pq}\times 1\times\cdots\times 1)-j^*(b_{pq})=\bar{\rho}^*(x)+y,$$

and such that y is a linear combination of diagonal elements for a base of $H^*(K; Z_p)$. We have

(C) $$\begin{aligned}\kappa_{\rho}(y)&=\kappa_{\rho}(a_{pq}\times 1\times\cdots\times 1)-\kappa_{\rho}j^*(b_{pq})-\kappa_{\rho}\bar{\rho}^*(x)\\ &=\kappa_{\rho}(a_{pq}\times 1\times\cdots\times 1)-\beta_{\rho}(b_{pq}).\end{aligned}$$

Hence it follows from (B) and (C) that

(D) $$\kappa_{\rho}(y)=\sum_{s=q+1}^{pq-1}\Gamma^{\bar{\rho}}_{pq-s}(a_s).$$

Since y is a linear combination of diagonal elements, it follows from (10•3) by the hypothesis of induction that the component of $\kappa_{\rho}(y)$ in $\Gamma^{\bar{\rho}}_{(p-1)q}H^q(K; Z_p)$ is $\chi_{\rho,q}\Gamma^{\bar{\rho}}_{(p-1)q}(y)$. Thus we see $y=0$ from (D). This, together with (C), gives

$$\kappa_\rho(a_{pq}\times 1\times\cdots\times 1)=\beta_\rho(b_{pq}).$$

Thus it follows from (B) that

$$\sum_{s=q+1}^{pq-1}\Gamma^{\bar\rho}_{pq-s}(a_s)=0.$$

By the hypothesis of induction, this implies that $\Gamma^{\bar\rho}_{pq-s}(a_s)=0$ for $q+1\leqq s\leqq pq-1$. Thus we have $\Gamma^{\rho}_{pq-s+1}(a_s)=0$ for $q+1\leqq s\leqq pq-1$, and hence also $\Gamma^{\rho}_1(a_{pq})=0$. This proves II). Thus the proof of (10•4) is complete.

By (10•2), (10•3) and (10•4), we have

THEOREM (10•5). *Let $x\in H^q(K;Z_p)$, then there exists a unique system of $(p-1)q$ elements $\{y_{\rho,s}(x)\}$ $(s=q,\ q+1,\cdots pq-1)$ with $y_{\rho,s}(x)\in H^s(K;Z_p)$ such that*

$$\kappa_\rho(x\times x\times\cdots\times x)=\sum_{s=q}^{pq-1}\Gamma^{\bar\rho}_{pq-s}(y_{\rho,s}(x))\ ;$$

it holds that $y_{\rho,q}(x)=\chi_{\rho,q}x$ with $\chi_{\rho,q}$ in (9•5).

We have also

THEOREM (10•6). *Given $x\in H^q(K;Z_p)$, there exists a unique system of $(p-1)q$ elements $\{y_{\rho,s}(x)\}$ $(s=q+1,q+2,\cdots,pq)$ with $y_{\rho,s}(x)\in H^s(K;Z_p)$ satisfying an equation*

$$\sum_{s=q}^{pq}\Gamma^{\rho}_{pq-s+1}(y_{\rho,s}(x))=0,$$

where $y_{\rho,q}(x)=\chi_{\rho,q}x$. Moreover such $y_{\rho,s}(x)$ coincides with the one in (10•5), *and $y_{\rho,pq}(x)=x\cup x\cup\cdots\cup x$.*

Proof. Apply γ_ρ to the both sides of (10•5). Since $\gamma_\rho\kappa_\rho(x\times x\times\cdots\times x)=-\vartheta_\rho i^*(x\times x\times\cdots\times x)=-\Gamma^\rho_1(x\cup x\cup\cdots\cup x)$, we have $\sum_{s=q}^{pq}\Gamma^\rho_{pq-s+1}(y_{\rho,s}(x))=0$. Next suppose that there exist two systems $\{y_{\rho,s}(x)\}$, $\{\bar y_{\rho,s}(x)\}$ satisfying the equation. Then we have $\sum_{s=q+1}^{pq}\Gamma^\rho_{pq-s+1}(y_{\rho,s}(x)-\bar y_{\rho,s}(x))=0$. Since $[(pq+1+p-1)/p]=q+1$, this is the canonical decomposition of $0\in{}^{\bar\rho}N^{pq+1}(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)$. Therefore it follows from (10•4) that $y_{\rho,s}(x)-\bar y_{\rho,s}(x)=0$ for $q+1\leqq 1\leqq pq$. This completes the proof of (10•6).

Let us denote by $N^r(\mathfrak{Z}(K),\mathfrak{d}(K);Z_p)$ the kernel of the homomorphism $\pi^*:H^r(\mathfrak{Z}(K),\mathfrak{d}(K);Z_p)\longrightarrow H^r(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)$ induced by π. Obviously $\mathrm{I}^*:N^r(\mathfrak{Z}(K),\mathfrak{d}(K);Z_p)\approx{}^\sigma N^r(\mathfrak{X}(K),\mathfrak{D}(K);Z_p)$. Thus (10•4) for $\rho=\tau$ is rewritten as follows:

THEOREM (10•7). $N^r(\mathfrak{Z}(K),\mathfrak{d}(K);Z_p)=\oplus\sum_{p=[(r+p-1)/p]}^{r-1}E_{r-s}H^s(K;Z_p)$.

As translations of (10•5) for $\rho=\sigma$ and (10•6) for $\rho=\tau$, we have

THEOREM (10•8). *Let $x\in H^q(K;Z_p)$, then there exists a unique system of $(p-1)q$ elements $\{y_{\sigma,s}(x)\}(s=q,q+1,\cdots,pq-1)$ with $y_{\sigma,s}(x)\in H^s(K;Z_p)$ such that*

$$\phi_0^*(x\times x\times\cdots\times x)=\sum_{s=q}^{pq-1}E_{pq-s}(y_{\sigma,s}(x))\ ;$$

it holds that $y_{\sigma,q}(x)=\chi_{\sigma,q}x$.

THEOREM (10•9). *Let $x\in H^q(K;Z_p)$, then there exists a unique system of $(p-1)q$ elements $\{y_{\tau,s}(x)\}(s=q+1,q+2,\cdots,pq)$ with $y_{\tau,s}\in H^s(K;Z_p)$ satisfying an equation*

$$\sum_{s=q}^{pq}E_{pq-s+1}(y_{\tau,s}(x))=0.$$

where $y_{\tau,q}(x)=\chi_{\tau,q}(x)$.

11. Cohomology of the p-fold cyclic products

We shall retain the use of the notations in § 7. Especially recall the definitions of the sets $\Omega^*(K;Z_p)$, $B_{\mathfrak{t}}(\Omega^*(K;Z_p))$ and of the vector subspaces $V'^r(\Omega^*(K;Z_p))$, $V''(\Omega^*(K;Z_p))\subset H^r(\mathfrak{X}(K)\,;Z_p)$; the set $\{\iota^{*j}(w)\mid w\in B_{\mathfrak{t}}'^r(\Omega^*(K;Z_p)),0\leqq j\leqq p-1\}$ is a base of $V'^r(\Omega^*(K;Z_p))$.

THEOREM (11・1).

$$H^r(\mathfrak{Z}(K),\mathfrak{d}(K)\,;Z_p)=N^r(\mathfrak{Z}(K),\mathfrak{d}(K)\,;Z_p)\oplus\phi_0^*(V'^r(\Omega^*(K;Z_p))\,;$$

the kernel of $\phi_0^*: V'^r(\Omega^*(K;Z_p))\longrightarrow H^r(\mathfrak{Z}(K),\mathfrak{d}(K)\,;Z_p)$ *is* $\tau^*V'^r(\Omega^*(K;Z_p))$.

Proof. Let $z\in V''^r(\Omega^*(K;Z_p))$, then it follows from (10・2) that $\kappa_\sigma(z)\in{}^\sigma N^r(\mathfrak{X}(K),\mathfrak{D}(K)\,;Z_p)$. This implies $\mathrm{I}^{*^{-1}}\kappa_\sigma(z)\in N^r(\mathfrak{Z}(K),\mathfrak{d}(K)\,;Z_p)$. Hence $\phi_0^*(V''^r(\Omega^*(K;Z_p))\subset N^r(\mathfrak{Z}(K),\mathfrak{d}(K)\,;Z_p)$. Thus it follows from (6・4) for $\rho=\sigma$ that

$$H^r(\mathfrak{Z}(K),\mathfrak{d}(K)\,;Z_p)=N^r(\mathfrak{Z}(K),\mathfrak{d}(K)\,;Z_p)+\phi_0^*(V'^r(\Omega^*(K;Z_p)).$$

We shall next prove that this decomposition is direct. Assume that $c+\phi_0^*(b)=0$ for $c\in N^r(\mathfrak{Z}(K),\mathfrak{d}(K)\,;Z_p)$ and $b\in V'^r(\Omega^*(K;Z_p))$, and consider the following commutative diagram

$$\begin{tikzcd}
{}^\sigma H^r(\mathfrak{X}(K),\mathfrak{D}(K)\,;Z_p) \arrow[r, "\alpha_\tau"] & H^r(\mathfrak{X}(K),\mathfrak{D}(K)\,;Z_p) \arrow[r, "\beta_\tau"] \arrow[dr, "j^*"] & {}^\tau H^r(\mathfrak{X}(K),\mathfrak{D}(K)\,;Z_p) \\
H^r(\mathfrak{Z}(K),\mathfrak{d}(K)\,;Z_p) \arrow[u, "\mathrm{I}^*", "\wr"'] & H^r(\mathfrak{X}(K)\,;Z_p) \arrow[l, "\phi_0^*"] \arrow[u, "\sigma_0^*"'] \arrow[ul, "\kappa_\sigma"] \arrow[r, "\sigma^*"'] & H^r(\mathfrak{X}(K)\,;Z_p)
\end{tikzcd}$$

Since $\alpha_\tau\mathrm{I}^*(c)=\pi^*(c)=0$, we have

$$\begin{aligned}\sigma^*(b)&=j^*\sigma_0^*(b)=j^*\alpha_\tau\kappa_\sigma(b)=j^*\alpha_\tau\mathrm{I}^*\phi_0^*(b)\\&=-j^*\alpha_\tau\mathrm{I}^*(c)=0.\end{aligned}$$

Since $b\in V'^r(\Omega^*(K;Z_p))$, it follows from (7・4) that there is an element $d\in V'^r(\Omega^*(K;Z_p))$ such that $b=\tau(d)$. Thus $c+\phi_0^*\tau^*(d)=0$. Since $\phi_0^*\tau^*=\mathrm{I}^{*^{-1}}\kappa_\sigma\tau^*=0$, we conclude $c=0$. This proves that the decompostion is direct.

Put $c=0$ in the above proof, then the argument shows that the kernel of ϕ_0^* is $\tau^*V'^r(\Omega^*(K;Z_p))$. This completes the proof of (11・1).

Let G be a field, then we shall denote by $R_r(X,A\,;G)$ the rank of the group $H^r(X,A\,;G)$ (*i.e.* the dimension of the vector space $H^r(X,A\,;G)$). The following can be obtained from (11・1) and (10・4) by simple calculations.

THEOREM (11・2). $R_r(\mathfrak{Z}(K),\mathfrak{d}(K)\,;Z_p)$

$$=\sum_{s=[(r+p-1)/p]}^{r-1}R_s(K;Z_p)+\frac{1}{p}\{R_r(\mathfrak{X}(K)\,;Z_p)-R_{r/p}(K;Z_p)\},$$

where it is to be understood $R_{r/p}(K;Z_p)=0$ *if* r *is not divisible by* p. *The union of the set* $\{E_{r-s}(b)\mid[(r+p-1)/p]\leqq s\leqq r-1,\ b\in\Omega^*(K;Z_p),\ \dim b=s\}$ *and the set* $\{\phi_0^*(w)\mid w\in B_{\mathfrak{t}}'^r(\Omega^*(K;Z_p))\}$ *is a base for* $H^r(\mathfrak{Z}(K),\mathfrak{d}(K)\,;Z_p)$.

We shall next consider $H^r(\mathfrak{Z}(K)\,;Z_p)$. Since $E_1=\delta^*\mathrm{I}^{*^{-1}}d_0^*$, it is obvious from

(9·1) that $\delta^*: H^{r-1}(\mathfrak{d}(K); Z_p) \longrightarrow H^r(\mathfrak{Z}(K), \mathfrak{d}(K); Z_p)$ is isomorphic into for any $r>1$. Thus we have

LEMMA (11·3). *The sequence*

$$0 \longrightarrow H^{r-1}(\mathfrak{d}(K); Z_p) \xrightarrow{\delta^*} H^r(\mathfrak{Z}(K), \mathfrak{d}(K); Z_p) \xrightarrow{j^*} H^r(\mathfrak{Z}(K); Z_p) \longrightarrow 0$$

is exact for $r>1$.

In virtue of this lemma, the following is obvious from (11·2) and (2·13)

THEOREM (11·4). *Let* $r \geqq 1$, *then*

$$R_r(\mathfrak{Z}(K); Z_p) = \sum_{s=[(r+p-1)/p]}^{r-2} R_s(K; Z_p) + \frac{1}{p}\{R_r(\mathfrak{X}(K); Z_p) - R_{r/p}(K; Z_p)\}.$$

The union of the set $\{j^* E_{r-s}(b) \mid [(r+p-1)/p] \leqq s \leqq r-2,\ b \in \Omega^*(K; Z_p),\ dim\ b=s\}$ *and the set* $\{\phi^*(w) \mid w \in B_t^{\prime r}(\Omega^*(K; Z_p))\}$ *is a base for* $H^r(\mathfrak{Z}(K); Z_p)$.

The following can be obtained from (1·11) by easy calculations.

THEOREM (11·5). *Let G be a field of characteristic q, not a divisor of p. Then*

$$R_r(\mathfrak{Z}(K); G) = \frac{1}{p}\{R_r(\mathfrak{X}(K); G) + (p-1) R_{r/p}(K; G)\}$$

$$= if\ p \geqq 3\ or\ if\ p=2\ and\ r \not\equiv 2\ (mod\ 4),$$

$$= \frac{1}{2}\{R_r(\mathfrak{X}(K); G) - R_{1/2}(K; G)\}$$

$$if\ p=2\ and\ r \equiv 2\ (mod\ 4),$$

where it is to be understood that $R_{r/p}(K; G)=0$ *if r is not divisible by p. Moreover it holds that* $H^*(\mathfrak{Z}(K); G) = \phi^* H^*(\mathfrak{X}(K); G)$.

REMARK. (1) The relations (11·4) is known by Richardson-Smith [11].

(2) If we take as G the rational field, (11·5) gives relations among the Betti numbers of $\mathfrak{Z}(K), \mathfrak{X}(K)$ and K.

We shall next study the reduced powers, the Bockstein homomorphism and the cup products in the groups $H^*(\mathfrak{Z}(K), \mathfrak{d}(K); Z_p)$ and $H^*(\mathfrak{Z}(K); Z_p)$. In virtue of (11·3), the results on $H^*(Z(K); Z_p)$ can be obtained at once from those on $H^*(\mathfrak{Z}(K), \mathfrak{d}(K); Z_p)$. Hence we will not write the former explicitely.

THEOREM (11·6). *Let* $x_j \in H^*(K; Z_p)$ *for* $j=1, 2, \cdots, p$.

I) $\Delta_p \phi_0^*(x_1 \times x_2 \times \cdots \times x_p) = \phi_0^* \Delta_p(x_1 \times x_2 \times \cdots \times x_p) - E_1(x_1 \cup x_2 \cup \cdots \cup x_p)$.

II) (i) $\mathcal{P}^s \phi_0^*(x_1 \times x_2 \times \cdots \times x_p)$

$$= \phi_0^* \mathcal{P}^s(x_1 \times x_2 \times \cdots \times x_p) + \sum_{j=1}^s (-1)^{j+1} E_{2j(p-1)} \mathcal{P}^{s-j}(x_1 \cup x_2 \cup \cdots \cup x_p),$$

(ii) $\mathrm{Sq}^s \phi_0^*(x_1 \times x_2) = \phi_0^* \mathrm{Sq}^s(x_1 \times x_2) + \sum_{j=1}^s E_j \mathrm{Sq}^{s-j}(x_1 \cup x_2)$.

Proof. I) is obvious from (3·7). The proof of (i) of II) is as follows: It follows from (3·14) and (2·19) that

$$\begin{aligned} &\mathcal{P}^s \phi_0^*(x_1 \times x_2 \times \cdots \times x_p) \\ &= \sum_{j=0}^s (-1)^j \mu^{j(p-1)} \phi_0^* \mathcal{P}^{s-j}(x_1 \times x_2 \times \cdots \times x_p) \\ &= \phi_0^* \mathcal{P}^s(x_1 \times x_2 \times \cdots \times x_p) + \sum_{j=1}^s (-1)^j \mu^{j(p-1)-1} \mu \phi_0^* \mathcal{P}^{s-j}(x_1 \times x_2 \times \cdots \times x_p) \\ &= \phi_0^* \mathcal{P}^s(x_1 \times x_2 \times \cdots \times x_p) + \sum_{j=1}^s (-1)^{j+1} \mu^{j(p-1)-1} \nu \delta^* \pi^{*^{-1}} i^* \mathcal{P}^{s-j}(x_1 \times x_2 \times \cdots \times x_p) \end{aligned}$$

$$=\phi_0^*\mathcal{P}^s(x_1\times x_2\times\cdots\times x_p)+\sum_{j=1}^s(-1)^{j+1}E_{2j(p-1)}\mathcal{P}^{s-j}(x_1\cup x_2\cup\cdots\cup x_p).$$

The proof of (ii) in II) is similar.

THEOREM (11·7). *Let* $x\in H^*(K;Z_p)$.

I) (i) $\Delta_pE_{2\alpha+1}(x)=-E_{2\alpha+1}(\Delta_px)$,

(ii) $\Delta_pE_{2\alpha+2}(x)=E_{2\alpha+3}(x)+E_{2\alpha+2}(\Delta_px)$.

II) (i) $\mathcal{P}^sE_{2\alpha+1}(x)=\sum_{j=0}^s{}_\alpha C_{s-j}E_{2(s-j)(p-1)+2\alpha+1}\mathcal{P}^j(x)$,

(ii) $\mathcal{P}^sE_{2\alpha+2}(x)=\sum_{j=0}^s{}_\alpha C_{s-j}E_{2(s-j)(p-1)+2\alpha+2}\mathcal{P}^j(x)$,

(iii) $\mathrm{Sq}^sE_{\alpha+1}(x)=\sum_{j=0}^s{}_\alpha C_{s-j}E_{\alpha+1+s-j}\mathrm{Sq}^j(x)\quad (s,\alpha\geqq 0)$,

where ${}_\alpha C_\beta$ *is the binomial coefficient with the usual conventions.*

Proof of I). Since $\Delta_p\delta^*=-\delta^*\Delta_p$, we have $\Delta_pE_1(x)=-E_1\Delta_p(x)$. Thus (i) follows from (3·8) and (9·8):

$$\Delta_pE_{2\alpha+1}(x)=\Delta_p\mu^\alpha E_1(x)=\mu^\alpha\Delta_pE_1(x)=-\mu^\alpha E_1\Delta_p(x)=E_{2\alpha+1}\Delta_p(x).$$

(ii) is obtained from (9·8), (3·8) and above (i):

$$\begin{aligned}\Delta_pE_{2\alpha+2}(x)&=\Delta_p\nu E_{2\alpha+1}(x)=\mu E_{2\alpha+1}(x)-\nu\Delta_pE_{2\alpha+1}(x)\\&=E_{2\alpha+3}(x)+E_{2\alpha+2}(\Delta_px).\end{aligned}$$

Proof of II). We shall prove (i) by mathematical induction on $s+\alpha$. If $s+\alpha=0$ and hence $s=\alpha=0$, it is obvious that the left and the right sides of (i) are both $E_1(x)$. Therefore we have (i). Assume that (i) holds for any (s,α) such that $s+\alpha\leqq l$. We shall now prove (i) for $s+\alpha=l+1$.

Case: $\alpha\geqq 1$. Let $s+\alpha=l+1$. Then it follows from (9·8) and (3·9) that

$$\mathcal{P}^sE_{2\alpha+1}(x)=\mathcal{P}^s\mu E_{2\alpha-1}(x)=\mu\mathcal{P}^sE_{2\alpha-1}(x)+\mu^p\mathcal{P}^{s-1}E_{2\alpha-1}(x).$$

Applying the hypothesis of induction for $\mathcal{P}^sE_{2\alpha-1}(x)$ and $\mathcal{P}^{s-1}\,E_{2\alpha-1}(x)$, we have

$$\begin{aligned}&\mathcal{P}^sE_{2\alpha+1}(x)\\&=\mu\sum_{j=0}^s{}_{\alpha-1}C_{s-j}\,E_{\beta-2}\mathcal{P}^j(x)+\mu^p\sum_{j=0}^{s-1}{}_{\alpha-1}C_{s-1-j}\,E_{\beta-2p}\mathcal{P}^j(x)\\&=\sum_{j=0}^s{}_{\alpha-1}C_{s-j}\,E_{2(s-j)}E_\beta\mathcal{P}^j(x)+\sum_{j=0}^{s-1}{}_{\alpha-1}C_{s-j-1}\,E_\beta\mathcal{P}^j(x)\\&=\sum_{j=0}^{s-1}({}_{\alpha-1}C_{s-j}+{}_{\alpha-1}C_{s-j-1})\,E_\beta\mathcal{P}^j(x)+{}_{\alpha-1}C_0E_{2\alpha+1}\mathcal{P}^s(x)\\&=\sum_{j=0}^s{}_\alpha C_{s-j}E_\beta\mathcal{P}^j(x),\end{aligned}$$

where we put $\beta=2(s-j)(p-1)+2\alpha+1$.

Case: $\alpha=0$ $(s=l+1)$. In this case, it is obvious that

$$\sum_{j=0}^s{}_\alpha C_{s-j}E_{2(s-j)(p-1)+2\alpha+1}\mathcal{P}^j(x)=E_1\mathcal{P}^{l+1}(x).$$

On the other hand, since $E_1=\delta^*\pi^{*^{-1}}d_0^{*^{-1}}$, it follows from the properties (I) and (II) of $\mathcal{P}^s$ that

$$\mathcal{P}^sE_{2\alpha+1}(x)=\mathcal{P}^{l+1}E_1(x)=E_1\mathcal{P}^{l+1}(x).$$

Thus we have (i) for $s+\alpha=l+1$, and complete the proof of (i). (ii) follows at once from (9·8), (3·9) and above (i), and the proof of (iii) is similar as in I).

THEOREM (11·8). *Let* $x, y, x_\alpha, y_\beta\in H^*(K;Z_p)$.

(i) $\phi_0^*(x_1\times x_2\times\cdots\times x_p)\cup\phi_0^*(y_1\times y_2\times\cdots\times y_p)$

$$=\sum_{j=1}^p(-1)^{\varepsilon_j}\phi_0^*((x_1\cup y_j)\times(x_2\cup y_{j+1})\times\cdots\times(x_p\cup y_{j-1})),$$

where $\varepsilon_j=(1+\sum_{\alpha=1}^p dim\ y_\alpha)(\sum_{\alpha=1}^j dim\ y_\alpha)+\sum_{\alpha=0}^{p-2}\sum_{\beta=\alpha+2}^p(dim\ y_{\alpha+j})(dim\ x_\beta)$.

(ii) $E_s(x) \cup \phi_0^*(x_1 \times x_2 \times \cdots \times x_p) = 0$ *for* $s \geqq 1$.

(iii) $E_s(x) \cup E_t(y) = 0$ *for* $s, t \geqq 1$.

Proof. (i) follows at once (3·3). To prove (ii), observe first that

$$\begin{aligned} &E_1(x) \cup \phi_0^*(x_1 \times x_2 \times \cdots \times x_p) \\ &= \delta^* \pi^{*^{-1}} d_0^{*^{-1}}(x) \cup \phi_0^*(x_1 \times x_2 \times \cdots \times x_p) \\ &= j^* \delta^* \pi^{*^{-1}} d_0^{*^{-1}}(x) \cup \phi_0^*(x_1 \times x_2 \times \cdots \times x_p) = 0. \end{aligned}$$

Then (ii) is obvious from (3·6). Since

$$\begin{aligned} E_1(x) \cup E_1(y) &= \delta^* \pi^{*^{-1}} d_0^{*^{-1}}(x) \cup \delta^* \pi^{*^{-1}} d_0^{*^{-1}}(y) \\ &= j^* \delta^* \pi^{*^{-1}} d_0^{*^{-1}}(x) \cup \delta^* \pi^{*^{-1}} d_0^{*^{-1}}(y) = 0, \end{aligned}$$

(iii) follows from (3·6). This completes the proof of (11·8).

12. Reduction formula——Axioms for the reduced powers

We shall first study relationships $y_{\sigma,s}(x)$ and $y_{\tau,s}(x)$ stated in (10·5) or (10·6). Let dim $x = q$.

Theorem (12·1). (i) *Let* $p \geqq 3$, *then we have* $y_{\sigma,s}(x) = y_{\tau,s}(x)$ *if* $pq - s$ *is even, and* $= 0$ *if* $pq - s$ *is odd.*

(ii) *Let* $p = 2$, *then* $y_{\sigma,s}(x) = y_{\tau,s}(x)$ *for any* s.

Proof. Apply ψ_σ to the both sides of (10·5) for $\rho = \tau$, and recall (2·9) and (2·21). Then we have

$$\kappa_\sigma(x \times x \times \cdots \times x) = \sum'' \Gamma^\tau_{pq-s}(y_{\tau,s}(x)) \quad (p \geqq 3),$$
$$\kappa_\sigma(x \times x) = \sum_{s=q}^{2q-1} \Gamma^\tau_{2q-s}(y_{\tau,s}(x)) \quad (p = 2),$$

where $\sum''$ stands for summation over the integers s such that $q \leqq s \leqq pq-1$ and $pq - s$ are even. From these and (10·5) for $\rho = \sigma$, we obtain (12·1).

Corollary (12·2). $\chi_{\sigma,q} \equiv \chi_{\tau,q}$, $\quad y_{\sigma,q+1}(x) = 0$ $(p \geqq 3)$.

We shall write briefly $\chi_q = \chi_{\rho,q}$ in the following.

Theorem (12·3). *If* $p \geqq 3$ *and* $pq - s$ *is odd, we have* $y_{\tau,s}(x) = \Delta_p y_{\sigma,s-1}(x)$.

Proof. Consider the elements $\Delta_p \phi_0^*(x \times x \times \cdots \times x)$ and $\phi_0^* \Delta_p(x \times x \times \cdots \times x)$. It follows from (10·8), (12·1), (11·7) and (11·1) that

$$\begin{aligned} &\Delta \phi_0^*(x \times x \times \cdots \times x) = \Delta_p\left(\sum_{s=q}^{pq-1} E_{pq-s}\, y_{\sigma,s}(x)\right) \\ &= \sum'' \Delta_p E_{pq-s}\, y_{\sigma,s}(x) = \sum'' E_{pq-s+1}\, y_{\sigma,s}(x) + \sum'' E_{pq-s} \Delta_p y_{\sigma,s}(x)), \\ &\phi_0^* \Delta_p(x \times x \times \cdots \times x) = \phi_0^*\left(\sum_{j=0}^{p-1} t^{j*}(\Delta_p x \times x \times \cdots \times x)\right) \\ &= p \phi_0^*(\Delta_p x \times x \times \cdots \times x) = 0, \end{aligned}$$

where $\sum''$ stands for summation over the integers s such that $q \leqq s \leqq pq-1$ and $pq - s$ are even.

On the other hand, it follows from (11·6), (10·9) and the definition of E_1 that

$$\begin{aligned} &\phi_0^* \Delta_p(x \times x \times \cdots \times x) - \Delta_p \phi_0^*(x \times x \times \cdots \times x) \\ &= E_1(x \cup x \cup \cdots \cup x) = -\sum_{s=q}^{pq-1} E_{pq-s+1}(y_{\tau,s}(x)). \end{aligned}$$

Thus we obtain

$$\sum'' E_{pq-s+1}\, y_{\sigma,s}(x) + \sum'' E_{pq-s}(\Delta_p\, y_{\sigma,s}(x)) = \sum_{s=q}^{pq-1} E_{pq-s+1}(y_{\tau,s}(x)),$$

and hence by (12·1)

$$0=\sum' E_{pq-s+1}(y_{\tau,s}(x))-\sum'' E_{pq-s}(\varDelta_p y_{\sigma,s}(x))$$
$$=\sum'' E_{pq-s}(y_{\tau,s+1}(x)-\varDelta_p\, y_{\sigma,s}(x)),$$

where $\sum'$ stands for summation over the integers s such that $q\leqq s\leqq pq-1$ and $pq-s$ are odd. Since $[(pq+1+p-1)/p]=q+1$, this is the canonical decomposition of $0\in N^{pq+1}(\mathfrak{Z}(K),\mathfrak{d}(K);Z_p)$. Therefore, by (10・7), we have $y_{\tau,s+1}(x)=\varDelta_p y_{\sigma,s}(x)$ if $pq-s$ is even. This completes the proof of (12・3).

We shall next study relationships between $y_{\rho,s}(x)$ and the cohomology operations.

LEMMA (12・4).

(i) $E_{q(p-1)+2}(x)=\sum_{j=0}^{q/2-1}(-1)^{q/2+j+1}E_{2j(p-1)+2}\mathcal{P}^{q/2-j}(x)$, if $p\geqq 3$ *and even* q.

(ii) $E_{q(p-1)+p+1}(x)=\sum_{j=0}^{(q-1)/2}(-1)^{(q+1)/2+j+1}E_{2j(p-1)+2}\mathcal{P}^{(q+1)/2-j}(x)$, *if* $p\geqq 3$ *and odd* q.

(iii) $E_{q+1}(x)=\sum_{j=0}^{q-1}E_{j+1}\mathrm{Sq}^{q-j}(x)$, *if* $p=2$.

Proof. Put $\bar{x}=x\times\mathbf{1}\times\cdots\times\mathbf{1}$. Then it follows from (3・10), (3・11), (2・19) and the property (I) of $\mathcal{P}^s$ that

$$\mu\mathcal{P}^s\phi_0^*(\bar{x})=\sum_{j=0}^{s}(-1)^j\mu^{j(p-1)}\mathcal{P}^{s-j}\mu\phi_0^*(\bar{x})$$
$$=\sum_{j=0}^{s}(-1)^j\mu^{j(p-1)}\mu\phi_0^*\mathcal{P}^{s-j}(\bar{x})=\sum_{j=0}^{s}(-1)^{j+1}\mu^{j(p-1)}\nu\delta^*\pi^{*^{-1}}i^*\mathcal{P}^{s-j}(\bar{x})$$
$$=\sum_{j=0}^{s}(-1)^{j+1}\mu^{j(p-1)}\nu\delta^*\pi^{*^{-1}}d_0^{*^{-1}}\mathcal{P}^{s-j}(x)=\sum_{j=0}^{s}(-1)^{j+1}E_{2j(p-1)+2}\mathcal{P}^{s-j}(x).$$

It follows from the property (V) of $\mathcal{P}^s$ and (3・5) that

$$\mu\mathcal{P}^{q/2}\phi_0^*(\bar{x})=\mu=(\phi_0^*(\bar{x})\cup\phi_0^*(\bar{x})\cup\cdots\cup\phi_0^*(\bar{x}))=0 \qquad (q:\text{even}),$$
$$\mu\mathcal{P}^{(q+1)/2}\phi_0^*(\bar{x})=0 \qquad (q:\text{odd}).$$

Thus we have

$$0=\sum_{j=0}^{q/2}(-1)^{j+1}E_{2j(p-1)+2}\mathcal{P}^{q/2-j}(x) \qquad (q:\text{even}),$$
$$0=\sum_{j=0}^{(q+1)/2}(-1)^{j+1}E_{2j(p-1)+2}\mathcal{P}^{(q+1)/2-j}(x) \qquad (q:\text{odd}).$$

These are (i) and (ii). The proof of (iii) is similar. Q. E. D.

LEMMA (12・5). (i) $E_{q+1}(x)=\sum_{s=q+1}^{2q}E_{2q-s+1}(y_{\sigma,s}(x))$, $(p=2)$,

(ii) $E_{q(p-1)+2}(x)=-\chi_q^{-1}(\sum_{s=q+1}^{pq}E_{pq-s+2}(y_{\sigma,s}(x))$,

(iii) $E_{q(p-1)+p+1}(x)=-\chi_q^{-1}\sum_{s=q+1}^{pq}E_{pq-s+p+1}(y_{\sigma,s}(x))$.

Proof. These are obvious from (10・6) and (ii) of (12・1).

THEOREM (12・6).

(i) *Let* $p\geqq 3$, *then* $y_{\sigma,s}(x)=(-1)^j\chi_q\mathcal{P}^j(x)$ *if* $s=q+2j(p-1)$, *and* $=0$ *otherwise*;

(ii) *Let* $p=2$, *then* $y_{\sigma,s}(x)=\mathrm{Sq}^{s-q}(x)$.

Proof of (i). Case: q is even. It follows from (i) of (12・4) and (ii) of (12・5) that

$$\sum_{j=0}^{q/2-1}(-1)^{q/2+j+1}E_{2j(p-1)+2}\mathcal{P}^{q/2-j}(x)+\chi_q^{-1}\sum_{s=q+1}^{pq}E_{pq-s+2}(y_{\sigma,s}(x))=0.$$

Since $[(pq+2+p-1)/p]=q+1$, the left hand of this equation is the canonical decomposition of $0\in N^{pq+2}(\mathfrak{Z}(K),\mathfrak{d}(K);Z_p)$. Therefore it follows from (10・7) that

$$(-1)^{q/2+j+1}\mathcal{P}^{q/2-j}(x)+\chi_q^{-1}\,y_{\sigma,q+(p-1)(q-2j)}(x)=0,$$

and $y_{\sigma,s}(x)=0$ if $s-q\not\equiv 0 \bmod 2(p-1)$. This is (i) for even q.

Case: q is odd. Since $y_{\sigma,q+1}(x)=0$ from (12・2), it follows from (ii) of (12・4) and (iii) of (12・5) that

$$\sum_{j=0}^{(q-1)/2}(-1)^{(q+1)/2+j+1}E_{2j(p-1)+2}\mathcal{P}^{(q+1)/2-j}(x)+\chi_q^{-1}\sum_{s=q+2}^{pq}E_{pq-s+p+1}y_{\sigma,s}(x)=0.$$

Since $[(pq+p+1+p-1)/p]=q+2$, the left hand of this equation is the canonical decomposition of $0\in N^{pq+p+1}(\mathfrak{Z}(K),\ \mathfrak{b}(K);Z_p)$. Therefore we have (i) for odd q, by $(10\cdot 7)$.

Proof of (ii). From (iii) of $(12\cdot 4)$ and (i) of $(12\cdot 5)$, it follows that

$$\sum_{j=0}^{q-1}E_{j+1}\,\mathrm{Sq}^{q-j}(x)-\sum_{s=q+1}^{2q}E_{2q-s+1}(y_{\sigma,s}(x))=0.$$

The left hand of this equation is the canonical decomposition of $0\in N^{2q+1}(\mathfrak{Z}(K),\ \mathfrak{b}(K);\ Z_2)$, since $[(2q+1+1)/2]=q+1$. By $(10\cdot 7)$ this gives (ii). Q. E. D.

COROLLARY $(12\cdot 7)$. $\chi_q=(-1)^{q/2}$ *if q is even.*

From $(12\cdot 1)$, $(12\cdot 3)$ and $(12\cdot 6)$, we have

COROLLARY $(12\cdot 8)$. (i) *Let $p\geqq 3$, then $y_{\tau,s}(x)=(-1)^j\chi_q\mathcal{P}^j(x)$ if $s=q+2j(p-1)$, $=(-1)^j\chi_q\Delta_p\mathcal{P}^j(x)$ if $s=q+2j(p-1)+1$, and $=0$ otherwise.* (ii) *Let $p=2$, then $y_{\tau,s}(x)=\mathrm{Sq}^{s-q}(x)$.*

From $(12\cdot 6)$ and $(10\cdot 8)$, we have

THEOREM $(12\cdot 9)$ (reduction formula). *Let $x\in H^q(K;Z_p)$, then*

(i) $\phi_0^*(x\times x\times\cdots\times x)=\chi_q\sum_{0\leqq j<q/2}(-1)^jE_{(p-1)(q-2j)}\mathcal{P}^j(x)$, $(p\geqq 3)$,

(ii) $\phi_0^*(x\times x)=\sum_{j=0}^{q-1}E_{q-j}\mathrm{Sq}^j(x)$, $(p=2)$.

From $(12\cdot 8)$ and $(10\cdot 9)$, we have

THEOREM $(12\cdot 10)$ (reduction formula). *Let $x\in H^q(K;Z_p)$, then*

(i) $E_{(p-1)q+1}(x)=\sum_{0<j\leqq q/2}(-1)^{j+1}E_{(p-1)(q-2j)+1}\mathcal{P}^j(x)$
$+\sum_{0\leqq j<q/2}(-1)^{j+1}E_{(p-1)(q-2j)}\Delta_p\mathcal{P}^j(x)$, $(p\geqq 3)$,

(ii) $E_{q+1}(x)=\sum_{j=1}^{q}E_{q-j+1}\mathrm{Sq}^j(x)$, $(p=2)$.

A characterization of the reduced p-th power.

We shall now prove that the reduced p-th power is characterized by its properties (I)$-$(V) stated in § 3.

THEOREM $(12\cdot 11)$. *Suppose that an operation*

$$P^s:H^q(X,A;Z_p)\longrightarrow H^{q+2s(p-1)}(X,A;Z_p)$$

is given for any simplicial pair (X,A) and for any integers s and q, and that the properties (I)—(V) *replaced $\mathcal{P}^s$ by P^s are satisfied. Then we have $P^s=\mathcal{P}^s$.*

Proof. If A is not empty, we may consider a space K obtained from X by contracting A to a vertex v^0 of A. Then K is a finite cell complex which has v^0 as a vertex, so that K can be simplicially decomposed such that v^0 is a vertex. Let ζ: $(X,A)\longrightarrow(K,v^0)$ be the contraction. Then ζ maps $X-A$ onto $K-v^0$ homeomorphically, and hence we have by the excision property that

$$\zeta^*:H^r(K,v^0;Z_p)\approx H^r(X,A;Z_p)\qquad\text{for any } r.$$

Let $j:K\longrightarrow(K,v^0)$ be the inclusion, then it is obvious that

$$j^*:H^r(K,v^0;Z_p)\approx(K;Z_p)\qquad\text{for } r\geqq 1.$$

Thus we have

$$j^*\zeta^{*-1}: H^r(X, A; Z_p) \approx H^r(K; Z_p) \qquad \text{for } r \geqq 1.$$

Furthermore if A is not empty then $H^0(X, A; Z_p)=0$.[18] Therefore, in virtue of the property (I) of $\wp^s$ and P^s, it is sufficient to prove $P^s=\wp^s$ on the absolute cohomology groups. However, since the properties of the reduced power used in the proof of (12·6) are only (I)–(V), we have also (12·6) replaced $\wp^s$ by P^s. This shows that $P^s(x)=\wp^s(x)$ for any complex K and any element $x \in H^q(K; Z_p)$. Q. E. D.

By the same method, it can be proved that the squaring operation Sq^s is characterized by the properties (I)–(V).[19] For a characterization of the squaring operation, see also J-P. Serre [12].

REMARK. 1. We do not assume the linearity of the operation P^s. However, since it is easily seen that $y_{\sigma,s}(x)$ in (10·8) is linear with respect to x, it follows from (12·6) that P^s is a homomorphism.

REMARK. 2. Since the reduced power and squaring operation are not used in the proof of (10·8) or (10·9), it follows from (12·6) or (12·8) that we may adopt the unique solution $\{y_{\rho,s}(x)\}$ of the equation of variable x in (10·8) or (10·9) as a definition of the reduced power and the squaring operation.

13. Cyclic products of special complexes

The cohomology of the p-fold cyclic product of an n-sphere S^n is especially interest. In this section we shall first record the results for this special case. Some of these results are obtained by S. D. Liao [5], by using of the different methods from ours. Next, we shall determine the integral cohomology groups[20] of the p-fold cyclic products of S^n and $Y^{n+1}(p^m)$, where $Y^{n+1}(p^m)$ denotes a complex obtained by attaching an $(n+1)$-cell e^{n+1} to S^n by a map of degree p^m.

Let e_n be a generator of $H^n(S^n; Z_p)$, and write

$$a_s=j^*E_{s-n}(e_n) \in H^s(\mathfrak{Z}(S^n); Z_p)$$

for $n+2 \leqq s \leqq np$. Let $1 \leqq q \leqq p$, and $\{\alpha_1,\alpha_2, \cdots, \alpha_q\}$ a set of q different integers mod p. Then we shall write

$$g_{nq}(\alpha_1,\alpha_2, \cdots, \alpha_q)=\phi^*(x_1 \times x_2 \times \cdots \times x_p) \in H^{nq}(\mathfrak{Z}(S^n); Z_p),$$

where $x_j=e_n$ if $j \equiv \alpha_1,\alpha_2, \cdots, \alpha_q \bmod p$, and $=\mathbf{1}$ otherwise. Then (11·4) and (12·9) yield the following

THEOREM (13·1). *a_s and g_{nq} $(\alpha_1, \alpha_2, \cdots, \alpha_q)$ are non-zero elements of $H^*(\mathfrak{Z}(S^n); Z_p)$; $g_{nq}(\alpha_1, \alpha_2, \cdots, \alpha_q)=\pm g_{nq}(\beta_1, \beta_2, \cdots, \beta_q)$ if and only if there is an integer k such*

18) Without loss of generality, we may assume that X and A are connected.

19) Thom [18] does not assume the property (I) in the characterization of squaring operation. However it seems to me that the property (I) is used in the Thom's proof, so that (I) is needed in the characterization.

20) We do not know the integral cohomology groups of $\mathfrak{Z}(K)$ for any complex K and any prime number p. For $p=2$, see [17]. Recently T. Yoshioka obtained the results for $p=3$, 5 and 7 by making use of the same method as in [17].

that $\{\alpha_1+k, \alpha_2+k, \cdots, \alpha_q+k\}=\{\beta_1, \beta_2, \cdots, \beta_q\}$; *there is* ${}_pC_q/p$ *different* $g_{nq}(\alpha_1, \alpha_2, \cdots, \alpha_q)$ *for a given* q ; $H^*(\mathfrak{Z}(S^n); Z_p)$ *is generated by the elements* $\mathbf{1}$, a_s $(n+2 \leqq s \leqq np)$ *and* $g_{nq}(\alpha_1, \alpha_2, \cdots, \alpha_q)$ *for* $1 \leqq q \leqq p-1$ *and every set* $\{\alpha_1, \alpha_2, \cdots, \alpha_q\}$; $g_{nq}(1, 2, \cdots, p)=\chi_n a_{np}$ $(\chi_n \not\equiv 0 \bmod p)$.

The following relations are obtained from (11·6) and (11·7).

THEOREM (13·2). (i) $\Delta_p g_{nq}(\alpha_1, \alpha_2, \cdots, \alpha_q)=0$; $\Delta_p a_{n+2\alpha+1}=0$, $\Delta_p a_{n+2\alpha+2}=a_{n+2\alpha+3}$.
(ii) $\mathcal{P}^s g_n(1)=(-1)^{s+1}a_{n+2s(p-1)}$ *if* $s \neq 0$, $\mathcal{P}^s g_{nq}(\alpha_1, \alpha_2, \cdots, \alpha_q)=0$ *if* $q>1$ *and* $s \neq 0$;
$\mathcal{P}^s a_{n+2\alpha+1}={}_\alpha C_s a_{n+2s(p-1)+2\alpha+1}$, $\mathcal{P}^s a_{n+2\alpha+2}={}_\alpha C_s a_{n+2s(p-1)+2\alpha+2}$.
(iii) $\mathrm{Sq}^s g_n(1)=a_{n+s}$ $(s \geqq 2)$, $\mathrm{Sq}^s a_{n+\alpha+1}={}_\alpha C_s\, a_{n+\alpha+s+1}$.

The structure of cohomology ring $H^*(\mathfrak{Z}(S^n); Z_p)$ is also determined easily by (11·8). For example we have

THEOREM (13·3). *Let* $p \geqq 3$, *then* $g_n(1) \cup g_n(1)=2\sum_{k=2}^{(p+1)/2} g_{2n}(1, k)$ *if* n *is even, and* $=0$ *if* n *is odd. Let* $p=2$, *then* $g_n(1) \cup g_n(1)=a_{2n}$.

Denote by $\beta_r(Y)$ the Betti number of a complex Y. Then, by (11·5) we have

THEOREM (13·4). (i) *Let* $p \geqq 3$, $\beta_r(\mathfrak{Z}(S^n))={}_pC_q/p$ *for* $r=nq$ $(1 \leqq q<p)$; $=1$ *for* $r=np$ *and* 0 ; *and* $=0$ *for any other* r. (ii) *Let* $p=2$. $\beta_r(\mathfrak{Z}(S^n))=1$ *for* $r=2n$ $(n:$ *even*$)$, n *and* 0 ; *and* $=0$ *for any other* r.

Let A be an abelian group, and q a prime number. Then we shall denote by $C(A, q)$ the q-primary component of A, and by $C(A, \infty)$ the free component of A. Let us denote by $J\{A, r\}$ the direct sum of r groups each of which is isomorphic with A. Then we have

THEOREM (13·5). (i) $C(H^s(\mathfrak{Z}(S^n); Z), q)=0$ *for any* s *and* $q \neq p, \infty$. (ii) $C(H^s(\mathfrak{Z}(S^n); Z), \infty) \approx Z$ *if* $s=0$ *and* pn *with* $(p-1)n=$ *even*, $\approx J\{Z, {}_pC_q/p\}$ *if* $s=nq$ *with* $1 \leqq q \leqq p-1$, *and* $=0$ *for any other* s. (iii) $C(H^s(\mathfrak{Z}(S^n); Z), p) \approx Z_p$ *if* $s-n$ *is odd and* $3 \leqq s-n \leqq (p-1)n$, *and* $=0$ *for any other* s.

Proof. Consider the exact sequence

$$ {}^\tau H^{s-1}(\mathfrak{X}(S^n); Z) \xrightarrow{\gamma_\tau} {}^{\tau^{-1}}H^s(\mathfrak{X}(S^n); Z) \xrightarrow{\alpha_\tau} H^s(\mathfrak{X}(S^n); Z). $$

Then $H^s(\mathfrak{X}(S^n); Z)$ is free abelian, and it follows from (1·7) that the image of γ_τ consists of elements of order p. Thus (i) is obvious. (ii) and (iii) follows from (13·1) and (13·4), by the universal coefficient theorem [3]. This completes the proof.

Let $e_n^* \in H^n(S^n; Z)$ be a generator, and consider the homomorphism

$$ E_{2\alpha+1}^*=\mu^\alpha \delta^* \pi^{*^{-1}} d_0^{*^{-1}} : H^n(S^n; Z) \longrightarrow H^{n+2\alpha+1}(\mathfrak{Z}(S^n), \mathfrak{d}(S^n); Z), $$
$$ \phi^* : H^{nq}(\mathfrak{X}(S^n); Z) \longrightarrow H^{nq}(\mathfrak{Z}(S^n); Z). $$

(See § 2.) Then the following is obvious from (13·1) and (13·5).

THEOREM (13·6). $j^* E_{2\alpha+1}^*(e_n^*)$ *is a generator of* $C(H^{n+2\alpha+1}(\mathfrak{Z}(S^n); Z), p)$ *for* $1 \leqq \alpha \leqq (pn-n-1)/2$; $C(H^{nq}(\mathfrak{Z}(S^n); Z), \infty)=\phi^* H^{nq}(\mathfrak{X}(S^n); Z)$.

In order to state the integral cohomology groups of the p-fold cyclic product of $Y^{n+1}(p^m)$, we shall first introduce the functions $\xi_n(r)$, $\eta_n(r)$ and $\mu_n(r)$ defined for each r and each integer n. These function are defined as follows:

(i) $\xi_n(r) = 1$ if $r-np$ is odd and $1 \leqq r-np \leqq p-1$,
$= 0$ otherwise.

(ii) $\eta_n(r) = (-1)^{r-np}(1-p)/p - \xi_n(r)$ if $1 \leqq r-np \leqq p$,
$= 0$ otherwise.

(iii) $\mu_n(r) = 1$ if $r-n$ is odd and $n+3 \leqq r \leqq pn$,
$= 0$ otherwise.

THEOREM (13·7). *The r-dimensional integral cohomology group of* $\mathfrak{Z}(Y^{n+1}(p^m))$ $(r \neq 0)$ *is isomorphic with*

$$J\{Z_{p^{m+1}}, \xi_n(r)\} \oplus J\{Z_{p^m}, \eta_n(r) + R^r/p\} \oplus J\{Z_p, \mu_n(r) + \mu_{n+1}(r)\}$$

if $(p-1)n$ *is even, and is isomorphic with*

$$J\{Z_{p^m}, R^r/p - \xi_n(r) - \eta_n(r)\} \oplus J\{Z_p, \mu_n(r) + \mu_{n+1}(r)\}$$

if $(p-1)n$ *is odd, where* R^r *denotes the rank of the r-dimensional integral cohomology group of* $\mathfrak{X}(Y^{n+1}(p^m))$.

We prove this theorem by making use of the method of Nakamura [7], which will be explained in later Appendix. For this purpose, calculate the number ${}_lR^r = {}_lR^r(\mathfrak{Z}(Y^{n+1}(p^m)); Z_p)$ defined there. Then, in virtue of (16·4) in Appendix, the theorem (13·7) follows by easy calculations. As for the number ${}_lR^r$, we have the following results.

LEMMA (13·8). *Let* $r \neq 0$. *We have*

(i) ${}_1R^r = {}_2R^r = \cdots = {}_{m-1}R^r$

$$= \frac{1}{p}\{R_r(\mathfrak{X}(K); Z_p) + (p-1)R_{r/p}(K; Z_p)\} \qquad \text{if } p \geqq 3,$$

$$= \frac{1}{2}\{R_r(\mathfrak{X}(K); Z_2) + (-1)^{r/2}R_{r/2}(K; Z_2)\} \qquad \text{if } p = 2.$$

(ii) *If* $p \geqq 3$, ${}_mR^r = 1$ *for* $np \leqq r \leqq (n+1)p - 2$; *if* $p = 2$ *and* n *is even*, ${}_mR^r = 1$ *for* $r = 2n$ *and* $2n+1$. *Otherwise* ${}_mR^r = 0$.

(iii) ${}_{m+1}R^r = {}_{m+2}R^r = \cdots = {}_{\infty}R^r = 0$.

This lemma can be proved by making use of the results in §11 and §12 and of the theorem (16·5) in Appendix. The verification needs some cumbersome calculations. We shall omit to denote it here.

CHAPTER III. COHOMOLOGY OF SYMMETRIC PRODUCTS OF SPHERES

14. Cohomology mod 2 of $S^n * S^n * S^n$

Let K be a space, and consider a space obtained by identifying any two points $(x_1, x_2, \cdots, x_p)$, $(y_1, y_2, \cdots, y_p)$ of $\mathfrak{X}_{(p)}(K)$ into a single point whenever $y_i = x_{\gamma(i)}$ for some permutation γ of letters 1, 2, ⋯, p. Such a space is called usually the *p-fold symmetric product* of K. We shall denote this space by $\mathfrak{S}_{(p)}(K)$ or $K * K * \cdots * K$.

In this chapter, we shall determine the cohomology of the 3-fold symmetric product of an n-sphere S^n. Since no confusion arise, we shall abbreviate $\mathfrak{X}_{(3)}(S^n)$, $\mathfrak{Z}_{(3)}(S^n)$, $\mathfrak{S}_{(3)}(S^n)$ ⋯ as $\mathfrak{X}$, $\mathfrak{Z}$, $\mathfrak{S}$ ⋯ respectively.

Let

$$\mathfrak{t}, \mathfrak{s}, : \mathfrak{X} \longrightarrow \mathfrak{X}$$

be transformations given by

$$\mathfrak{t}(x_1, x_2, x_3) = (x_2, x_3, x_1), \qquad \mathfrak{s}(x_1, x_2, x_3) = (x_2, x_1, x_3)$$

$(x_i \in S^n)$ respectively. Then $\mathfrak{t}$ (resp. $\mathfrak{s}$) is a periodic transformation of period 3 (resp. 2). The orbit space $O(\mathfrak{X}, \mathfrak{t})$ is the 3-fold cyclic product $\mathfrak{Z}$. Since

$$(14\cdot1) \qquad \mathfrak{t}\mathfrak{s} = \mathfrak{s}\mathfrak{t}^2, \qquad \mathfrak{t}^2\mathfrak{s} = \mathfrak{s}\mathfrak{t},$$

the transformation $\mathfrak{s}$ induces a transformation $\tilde{\mathfrak{s}} : \mathfrak{Z} \longrightarrow \mathfrak{Z}$ such that the commutativity $\pi\mathfrak{s} = \tilde{\mathfrak{s}}\pi$ holds, where $\pi : \mathfrak{X} \longrightarrow \mathfrak{Z}$ is the projection. Then $\tilde{\mathfrak{s}}$ is a periodic transformation on $\mathfrak{Z}$ with period 2. Suppose that $\mathfrak{Z}$ is simplicially decomposed and is locally ordered as in § 7. Then it is easily seen that $\tilde{\mathfrak{s}}$ is a transformation satisfying the conditions a) and b) in § 1. Thus we may apply the results in Chapter I with $p = 2$, $W = \mathfrak{Z}$, $W' = \emptyset$ and $t = \tilde{\mathfrak{s}}$.

Denote by $F(t)$ the fixed points set under a transformation t. Then we have

$$F(\mathfrak{t}) = \{(x, x, x) \mid x \in S^n\}, \ F(\mathfrak{s}) = \{(x, x, y) \mid x, y \in S^n\},$$
$$F(\mathfrak{s}\mathfrak{t}) = \{x, y, x) \mid x, y \in S^n\}, \ F(\mathfrak{s}\mathfrak{t}^2) = \{(y, x, x) \mid x, y \in S^n\}.$$

Obviously it holds for $i = 0, 1, 2$ that $F(\mathfrak{t}) = \mathfrak{D}_{(3)}(S^n) \subset F(\mathfrak{s}\mathfrak{t}^i)$, $F(\mathfrak{s}\mathfrak{t}^i) \cap F(\mathfrak{s}\mathfrak{t}^j) = F(\mathfrak{t})$ if $i \neq j$, and $\mathfrak{t} : F(\mathfrak{s}\mathfrak{t}^i) \longrightarrow F(\mathfrak{s}\mathfrak{t}^{i+1})$ is a homeomorphism. Let $\mathfrak{F} = F(\mathfrak{s}) \cup F(\mathfrak{s}\mathfrak{t}) \cup \mathfrak{F}(\mathfrak{s}\mathfrak{t}^2)$, then $\mathfrak{F}$ is a $\mathfrak{t}$-invariant subcomplex, which contains $F(\mathfrak{t})$, of $\mathfrak{X}$. Furthermore it is easily seen that $\tilde{\mathfrak{F}} = O(\mathfrak{F}, \mathfrak{t})$ is the fixed points set under the map $\tilde{\mathfrak{s}}$. Let $h : \mathfrak{F} \longrightarrow S^n \times S^n$ be a map defined by

$$\begin{aligned} h(x, x, y) &= (x, y) && \text{for } (x, x, y) \in F(\mathfrak{s}), \\ h(x, y, x) &= (x, y) && \text{for } (x, y, x) \in F(\mathfrak{s}\mathfrak{t}), \\ h(y, x, x) &= (x. y) && \text{for } (y, x, x) \in F(\mathfrak{s}\mathfrak{t}^2). \end{aligned}$$

Then h is continuous. Since $h\mathfrak{t} = h$ and $h\mathfrak{t}^2 = h$, h induces a map $\tilde{h} : \tilde{\mathfrak{F}} \longrightarrow S^n \times S^n$ such that $\tilde{h}\pi = h$. Then $\tilde{h}$ is obviously homeomorphic onto. Thus we have proved

LEMMA (14·2). *The fixed points set under $\tilde{\mathfrak{s}} : \mathfrak{Z} \longrightarrow \mathfrak{Z}$ is the orbit space $\tilde{\mathfrak{F}}$ over $F(\mathfrak{s}) \cup F(\mathfrak{s}\mathfrak{t}) \cup F(\mathfrak{s}\mathfrak{t}^2)$ relative to $\mathfrak{t}$, The map $\tilde{h}$ is a homeomorphism of $\tilde{\mathfrak{F}}$ onto $S^n \times S^n$.*

It is obvious that the orbit space $O(\mathfrak{Z}, \tilde{\mathfrak{s}})$ is the symmetric product $\mathfrak{S} = S^n * S^n * S^n$. Let us denote by $\tilde{\pi} : \mathfrak{Z} \longrightarrow \mathfrak{S}$ the projection, and by $\tilde{\mathfrak{f}}$ the image of $\tilde{\mathfrak{F}}$ by $\tilde{\pi}$. We shall denote by $\tilde{\phi}_0$, $\tilde{\nu}$, $\tilde{\mu}$ $\cdots$ the homomorphisms ϕ_0, ν, μ $\cdots$ for the complex $\mathfrak{Z}$ with the transformation $\tilde{\mathfrak{s}}$.

By (13·1) we have

$$(14\cdot3) \qquad \begin{aligned} H^r(\mathfrak{Z}; Z_2) &\approx Z_2 && \text{for } r = jn \ (j = 0, 1, 2, 3) \\ &= 0 && \text{for any other } r; \end{aligned}$$

and it follows from (14·2) that

$$(14\cdot4) \qquad \begin{aligned} H^r(\tilde{\mathfrak{F}}; Z_2) &\approx Z_2 && \text{for } r = 0,\ 2n, \\ &\approx Z_2 \oplus Z_2 && \text{for } r = n, \\ &= 0 && \text{for any other } r. \end{aligned}$$

LEMMA (14·5). *The inclusion homomorphism* $i^*: H^r(\mathfrak{Z}; Z_2) \longrightarrow H^r(\tilde{\mathfrak{F}}; Z_2)$ *is isomorphic into for* $r=n$, *and is trivial for* $r=2n$.

Proof. Case $1: r=n$. It is sufficient to prove that $i_*; H_r(\tilde{\mathfrak{F}}; Z_2) \longrightarrow H_r(\mathfrak{Z}; Z_2)$ is onto for the homology groups. Let s_n be an n-cycle of S^n representing the generator of $H_n(S^n; Z_2)$, and let $f: S^n \longrightarrow \mathfrak{X}$ be a map defined by $f(x)=(x, x_0, x_0)$, where $x_0 \in S^n$ is a base point. Then it is obvious that $H_n(\mathfrak{Z}; Z_2)$ is generated by a class containing the singular cycle $(s_n, \pi f)$. This is also a cycle in $\tilde{\mathfrak{F}}$, since $f(S^n) \subset \tilde{\mathfrak{F}}$. Thus i_* is onto.

Case $2: r=2n$. It is sufficient to prove that $i_*: H_{2n}(\tilde{\mathfrak{F}}; Z_2) \longrightarrow H_{2n}(\mathfrak{Z}; Z_2)$ is trivial. Let e_{2n} be a $2n$-cycle of $S^n \times S^n$ representing the generator of $H^{2n}(S^n \times S^n; Z_2)$. Let $g_1, g_2: S^n \times S^n \longrightarrow \mathfrak{X}$ be maps defined by $g_1(x, y)=(x, x_0, y)$ and $g_2(x, y)=(x_0, y, x)$, and let $h': S^n \times S^n \longrightarrow \tilde{\mathfrak{F}}$ be a map defined by $h'(x, y)=(x, x, y)$. Then, for the singular cycles (e_{2n}, g_1), (e_{2n}, g_2) and (e_{2n}, h'), we have a relation

$$(e_{2n}, h') \sim (e_{2n}, g_1)+(e_{2n}, g_2) \quad \mathrm{mod}\ 2$$

in $\mathfrak{X}$, where $\sim$ denotes to be homologous. Therefore we have

$$(e_{2n}, \pi h') \sim 2(e_{2n}, \pi g_1) \equiv 0 \quad \mathrm{mod}\ 2$$

in $\mathfrak{Z}$, because of $\pi g_1 = \pi g_2$. However it is obvious from (14·2) that $H_{2n}(\tilde{\mathfrak{F}}; Z_2)$ is generated by the class containing the singular cycle $(e_{2n}, \pi h')$. This shows that i_* is trivial. Q. E. D.

Consider the following diagram:

(14·6)
$$\begin{tikzcd}
 & & \arrow[dl, "\eta_{r-1}"'] \\
H^{r-2}(\tilde{\mathfrak{f}}; Z_2) \arrow[r, "\delta^*_{r-2}"] & H^{r-1}(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2) \arrow[d, "\zeta_{r-1}"] & \\
H^r(\tilde{\mathfrak{f}}; Z_2) & H^r(\mathfrak{S}; Z_2) \arrow[l, "i^*_r"'] \arrow[r, "\xi_r"] & H^r(\mathfrak{Z}; Z_2) \arrow[dl, "\eta_r"] \\
H^{r-1}(\tilde{\mathfrak{f}}; Z_2) \arrow[r, "\delta^*_{r-1}"] & H^r(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2) \arrow[u, "j^*_r"'] \arrow[d, "\zeta_r"] & \\
 & &
\end{tikzcd}$$

where $\xi_r = \tilde{\pi}^*$, $\eta_r = \tilde{\phi}_0^*$ and $\zeta_{r-1} = j^* \tilde{\nu}$. Then the sequence $\cdots \eta_{r-1}, \zeta_{r-1}, \xi_r, \eta_r, \zeta_r \cdots$ is exact and $\xi^*_r j^*_r \eta_r = \tilde{\sigma}^*$, as is stated in (2·22). Note moreover that $\cdots \delta^*_{r-1}$, j^*_r, i^*_r, $\delta^*_r \cdots$ is the exact sequence for the pair $(\mathfrak{S}, \tilde{\mathfrak{f}})$. Using this diagram, we shall determine the cohomology groups $H^r(S^n * S^n * S^n; Z_2)$.

LEMMA (14·7). (i) $H^r(\mathfrak{S}; Z_2) \approx Z_2$ *for* $r=0$, *and* $=0$ *for* $1 \leqq r \leqq n-1$. (ii) $H^r(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)=0$ *for* $0 \leqq r \leqq n-1$.

Proof. Since $H^r(\mathfrak{Z}; Z_2) = H^r(\tilde{\mathfrak{f}}; Z_2)=0$ for $0<r<n$, (14·7) is obtained easily by the consideration of (14·6).

LEMMA (14·8). $H^n(\mathfrak{S}; Z_2) \approx Z_2$, $H^n(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)=0$. *Moreover* $\tilde{\pi}^*: H^n(\mathfrak{S}; Z_2) \approx H^n(\mathfrak{Z}; Z_2)$.

Proof. Since $H^{n-1}(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)=0$ by (14·7), $\xi_n=\tilde{\pi}^*$ is isomorphic into. Consider the diagram

$$\begin{array}{ccc} H^n(\mathfrak{S}; Z_2) & \xrightarrow{\tilde{\pi}^*=\xi_n} & H^n(\mathfrak{Z}; Z_2) \\ \downarrow{\scriptstyle i_n^*} & & \downarrow{\scriptstyle i_n^*} \\ H^n(\tilde{\mathfrak{f}}; Z_2) & \xrightarrow{\tilde{\pi}^*} & H^n(\tilde{\mathfrak{Y}}; Z_2) \end{array}$$

then the right i_n^* is isomorphic into from (14·5), and both $\tilde{\pi}^*$ are isomorphic into. Hence the left i_n^* is isomorphic into. Consider (14·6) for $r=n$, then it follows that j_n^* is trivial. While, since $H^{n+1}(\tilde{\mathfrak{f}}; Z_2)=0$, j_n^* is isomorphic into. Therefore $H^n(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)=0$. This shows that ξ_n is onto. Thus ξ_n is an isomorphism. Since $H^n(\mathfrak{Z}; Z_2)\approx Z_2$, we have $H^n(\mathfrak{S}; Z_2)\approx Z_2$. Q. E. D.

Lemma (14·9). $H^{n+1}(\mathfrak{S}; Z_2)=0$. $H^{n+1}(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)\approx Z_2$.

Proof. Consider (14·6) for $r=n+1$. Since $H^n(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)=0$, ξ_{n+1} is isomorphic into. While $H^{n+1}(\mathfrak{Z}; Z_2)=0$, and hence $H^{n+1}(\mathfrak{S}; Z_2)=0$. From this, it follows that δ_n^* is onto. Thus we have

$$H^{n+1}(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)\approx H^n(\tilde{\mathfrak{f}}; Z_2)/i_n^*H^n(\mathfrak{S}; Z_2).$$

Since $H^n(\tilde{\mathfrak{f}}; Z_2)\approx Z_2\oplus Z_2$ and i_n^* is isomorphic into, we have $H^{n+1}(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)\approx Z_2$.

Lemma (14·10). $H^r(\mathfrak{S}; Z_2)\approx H^r(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)\approx Z_2$ *for* $n+2\leqq r\leqq 2n-1$.

Proof. Since $H^r(\mathfrak{Z}; Z_2)=H^r(\tilde{\mathfrak{f}}; Z_2)=0$ for $n<r<2n$, we have by using of (14·6)

$$H^r(\mathfrak{S}, \tilde{\mathfrak{f}}. Z_2)\overset{\zeta_r}{\approx} H^{r+1}(\mathfrak{S}; Z_2)\overset{j_{r+1}^*}{\approx} H^{r+1}(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)$$

for $n+1\leqq r\leqq 2n-2$. Therefore (14·10) follows from (14·9).

Lemma (14·11). $H^{3n}(\mathfrak{S}; Z_2)\approx H^{3n}(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)\approx Z_2$.

Proof. Consider (14·6) for $r=3n$. Since $H^{3n-1}(\tilde{\mathfrak{f}}; Z_2)=H^{3n}(\tilde{\mathfrak{f}}; Z_2)=0$, j_{3n}^* is isomorphic onto. Since $H^{3n+1}(\mathfrak{S}; Z_2)=0$, η_{3n} is onto. While $H^{3n}(\mathfrak{Z}; Z_2)\approx Z_2$, and hence $H^{3n}(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)=0$, or $\approx Z_2$. Assume that the former holds, then ξ_{3n} is onto and $H^{3n}(\mathfrak{S}; Z_2)=0$. This implies $H^{3n}(\mathfrak{Z}; Z_2)=0$, which contradicts (14·3) Therefore it holds that $H^{3n}(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)\approx Z_2$. Thus we have (14·11).

Lemma (14·12). $H^r(\mathfrak{S}; Z_2)\approx Z_2$ *for* $2n+2\leqq r\leqq 3n-1$.

$$H^r(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)\approx Z_2 \quad \textit{for} \quad 2n+1\leqq r\leqq 3n-1.$$

Proof. Since η_{3n} is isomorphic, we have that ζ_{3n-1} is onto. While $H^{3n-1}(\mathfrak{Z}; Z_2)=0$, and hence ζ_{3n-1} is isomorphic. Thus ζ_{3n-1}: $H^{3n-1}(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)\approx H^{3n}(\mathfrak{S}; Z_2)\approx Z_2$ from (14·11). Since $H^r(\mathfrak{Z}; Z_2)=H^r(\tilde{\mathfrak{f}}; Z_2)=0$ for $2n+1\leqq r\leqq 3n-1$, we have

$$H^r(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)\approx H^{r+1}(\mathfrak{S}; Z_2)\approx H^{r+1}(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)$$

for $2n+1\leqq r\leqq 3n-2$. This proves (14·12).

Lemma (14·13). $H^{2n+1}(\mathfrak{S}; Z_2)=0$.

$$H^{2n}(\mathfrak{S}; Z_2)\approx H^{2n}(\mathfrak{S}, \tilde{\mathfrak{f}}; Z_2)\approx Z_2.$$

Proof. Consider the diagram

$$\begin{array}{ccc} H^{2n}(\mathfrak{S};\ Z_2) & \xrightarrow{\xi_{2n}=\tilde{\pi}^*} & H^{2n}(\mathfrak{Z};\ Z_2) \\ \downarrow i^*_{2n} & & \downarrow i^*_{2n} \\ H^{2n}(\tilde{\mathfrak{f}};\ Z_2) & \xrightarrow{\tilde{\pi}^*} & H^{2n}(\tilde{\mathfrak{F}};\ Z_2), \end{array}$$

then the right i^*_{2n} is trivial from (14·5) and the lower $\tilde{\pi}^*$ is isomorphic onto. Hence the left i^*_{2n} is also trivial. Therefore, in (14·6) for $r=2n+1$, δ^*_{2n} is isomorphic into. While $H^{2n}(\tilde{\mathfrak{f}};\ Z_2)\approx Z_2$, and $H^{2n+1}(\mathfrak{S},\ \tilde{\mathfrak{f}};\ Z_2)\approx Z_2$ from (14·12). Hence it holds that δ^*_{2n} is isomorphic onto, so that j^*_{2n+1} is trivial. Since $H^{2n+1}(\tilde{\mathfrak{f}};Z_2)=0$, we have $H^{2n+1}(\mathfrak{S};\ Z_2)=0$.

Since i^*_{2n} is trivial and $H^{2n-1}(\tilde{\mathfrak{f}};\ Z_2)=0$, it follows that j^*_{2n} is an isomorphism. Since $H^{2n+1}(\mathfrak{S};\ Z_2)=0$, η_{2n} is onto. While $H^{2n}(\mathfrak{Z};\ Z_2)=Z_2$, and hence $H^{2n}(\mathfrak{S},\ \tilde{\mathfrak{f}};\ Z_2)=0$ or $\approx Z_2$. Assume that the former holds. Then we have $H^{2n}(\mathfrak{S};\ Z_2)=0$. However, since $H^{2n-1}(\mathfrak{Z};\ Z_2)=0$ and $H^{2n-1}(\mathfrak{S},\ \tilde{\mathfrak{f}};\ Z_2)\approx Z_2$ from (14·10), it follows that ζ_{2n-1} is isomorphic into and hence $H^{2n}(\mathfrak{S};Z_2)\approx H^{2n}(\mathfrak{S},\tilde{\mathfrak{f}};Z_2)\supset Z_2$. This is a contradiction. Thus we have $H^{2n}(\mathfrak{S},\ \tilde{\mathfrak{f}};\ Z_2)\approx Z_2$. This proves (14·13).

Summarizing (14·7)—(14·13), we have

THEOREM (14·14). *$H^r(S^n * S^n * S^n;\ Z_2)\approx Z_2$ for $r=0, n, n+2\leqq r\leqq 2n$ and $2n+2\leqq r\leqq 3n$; $=0$ for any other r.*

Since $S^n * S^n$ is the 2-fold cyclic product of S^n, it follows from § 13 that $H^*(S^n * S^n;\ Z_2)$ has as a base the elements $\mathbf{1}$, $g_n=g_n(1)$ and $a_{n+s}(2\leqq s\leqq n)$ such that

$$\text{(14·15)}\quad \begin{aligned} &\mathrm{Sq}^i g_n=a_{n+i}\ (2\leqq i\leqq n), \qquad \mathrm{Sq}^i a_{n+\alpha+1}={}_\alpha C_i a_{n+\alpha+i+1}\ (\alpha\geqq 1),\\ &g_n\cup g_n=a_{2n}, \qquad g_n\cup a_{n+i}=0 \qquad (2\leqq i\leqq n). \end{aligned}$$

Let

$$\begin{aligned} \pi_2 &: S^n\times S^n \longrightarrow S^n * S^n,\\ \pi_3 &: S^n\times S^n\times S^n \longrightarrow S^n * S^n * S^n,\\ \pi_{12} &: S^n\times S^n\times S^n \longrightarrow S^n\times(S^n * S^n),\\ \pi_0 &: S^n\times(S^n * S^n) \longrightarrow S^n * S^n * S^n \end{aligned}$$

be the natural projections. Then it is obvious that

$$\text{(14·16)}\qquad \pi_{12}=i\times\pi_2, \qquad \pi_0\pi_{12}=\pi_3, \qquad \pi_3=\tilde{\pi}\pi,$$

where i is the identity map.

For $i=0$, $2\leqq i\leqq n$, $n+2\leqq i\leqq 2n$, let h_{n+i} be the generators of $H^{n+i}(S^n * S^n * S^n;\ Z_2)$. Then we have

LEMMA (14·17). (i) *$\pi_0^*(h_n)=\mathbf{1}\times g_n+e_n\times\mathbf{1}$.*

(ii) *$\pi_0(h_{n+i})=\mathbf{1}\times a_{n+i}$, and $\mathrm{Sq}^i(h_n)=h_{n+i}$ if $2\leqq i\leqq n$.*

Proof. It follows from (14·16) that

$$\pi^*_{12}\ (\mathbf{1}\times g_n)=\mathbf{1}\times\pi^*_2(g_n)=(\mathbf{1}\times e_n\times\mathbf{1})+(\mathbf{1}\times\mathbf{1}\times e_n),$$

$$\pi_{12}^*(e_n\times 1)=e_n\times 1\times 1.$$

Since $H^n(S^n\times(S^n * S^n); Z_2)$ is generated by $1\times g_n$ and $e_n\times 1$, this shows that π_{12}^* is isomorphic into. Furthermore it follows from (14·16) and (14·8) that

$$\pi_3^*(h_n)=\pi^*\tilde{\pi}^*(h_n)=(e_n\times 1\times 1)+(1\times e_n\times 1)+(1\times 1\times e_n).$$

Therefore we have by (14·16)

$$\pi_{12}^*(1\times g_n+e_n\times 1)=\pi_3^*(h_n)=\pi_{12}^*\pi_0^*(h_n).$$

Since π_{12}^* is isomorphic into, this implies that $\pi_0^*(h_n)=1\times g_n+e_n\times 1$. Namely we have (i).

It follows from above (i) that

$$\pi_0^*\mathrm{Sq}^i(h_n)=\mathrm{Sq}^i\pi_0^*(h_n)=\mathrm{Sq}^i(1\times g_n+e_n\times 1)$$
$$=1\times \mathrm{Sq}^i g_n=1\times a_{n+i} \qquad (i\neq 0).$$

Since $1\times a_{n+i}\neq 0$ if $2\leqq i\leqq n$, it follows that $\pi_0^*\mathrm{Sq}^i(h_n)$ and hence $\mathrm{Sq}^i(h_n)$ is not zero for $2\leqq i\leqq n$. Therefore we have $\mathrm{Sq}^i(h_n)=h_{n+i}$. This proves (ii). Q. E. D.

LEMMA (14·18). *If $2\leqq i\leqq n$, we have*

$$h_n\cup h_{n+i}=h_{2n+i},\qquad \pi_0^*(h_{2n+i})=e_n\times a_{n+i}.$$

Proof. It follows from (14·17) and (14·15) that

$$\begin{aligned}\pi_0^*(h_n\cup h_{n+i})&=\pi_0^*h_n\cup\pi_0^*h_{n+i}\\&=(1\times g_n+e_n\times 1)\cup(1\times a_{n+i})\\&=1\times(g_n\cup a_{n+i})+e_n\times a_{n+i}\\&=e_n\times a_{n+i},\qquad (2\leqq i\leqq n).\end{aligned}$$

Since $e_n\times a_{n+i}\neq 0$, we have $h_n\cup h_{n+i}\neq 0$ and hence $h_n\cup h_{n+i}=h_{2n+i}$. This proves (14·18).

The following is obvious from (14·14), (14·17) and (14·18).

THEOREM (14·19). *The homomorphism*

$$\pi_0^*: H^r(S^n * S^n * S^n; Z_2)\longrightarrow H^r(S^n\times(S^n * S^n); Z_2)$$

is isomorphic into for any r.

Finally we have

THEOREM (14·20). *Let $h_{n+i}\in H^{n+i}(S^n * S^n * S^n; Z_2)$ be the generator, where $i=0$, $2\leqq i\leqq n$, $n+2\leqq i\leqq 2n$. Then we have the following:*

(i) $\mathrm{Sq}^i(h_n)=h_{n+i}$ *for* $2\leqq i\leqq n$.

(ii) *Let* $k=1, 2$, *and* $1\leqq j\leqq n-1$, *then*

$$\begin{aligned}\mathrm{Sq}^i(h_{kn+j+1})&={}_jC_i h_{kn+i+j+1}&&\textit{if}\ \ i+j\leqq n-1,\\&=0&&\textit{if}\ \ i+j>n-1.\end{aligned}$$

(iii) $h_n\cup h_{n+i}=h_{2n+i}$ *for* $2\leqq i\leqq n$.

(iv) $h_{n+i}\cup h_{n+j}=0$ *for* $2\leqq i, j\leqq n$.

Proof. (i) and (iii) are proved in (14·17) and (14·18) respectively. We shall prove (ii) for $k=1$. The proofs of (ii) for $k=2$ and of (iv) are similar.

It follows from (14·17) and (14·15) that

$$\begin{aligned}&\pi_0^*\mathrm{Sq}^i(h_{n+j+1})=\mathrm{Sq}^i\pi_0^*(h_{n+j+1})=\mathrm{Sq}^i(\mathbf{1}\times a_{n+j+1})\\&=\mathbf{1}\times\mathrm{Sq}^i(a_{n+j+1})=0 \text{ if } i+j+1>n, \text{ and}\\&=\mathbf{1}\times {}_jC_i a_{n+i+j+1}={}_jC_i\pi_0^*(h_{n+i+j+1}) \text{ if } i+j+1\leqq n.\end{aligned}$$

Since π_0^* is isomorphic into from (14・19), we have (ii) for $k=1$. This completes the proof of (14・20).

15. Cohomology mod 3 and integral homology of $S^n * S^n * S^n$

Let

$$\begin{aligned}&\Upsilon_\rho:\ {}^\rho H^r(\mathfrak{X},\mathfrak{D};Z_3)\longrightarrow {}^{\bar\rho}H^{r+1}(\mathfrak{X},\mathfrak{D};Z_3),\\&\psi_\rho:\ {}^{\bar\rho}H^r(\mathfrak{X},\mathfrak{D};Z_3)\longrightarrow {}^{\rho}H^r(\mathfrak{X},\mathfrak{D};Z_3)\end{aligned}$$

be the homomorphisms, defined in § 2, for the complex $\mathfrak{X}$ with the transformation $\mathfrak{t}$. Since the map $\mathfrak{t}:\mathfrak{X}\longrightarrow\mathfrak{X}$ is obviously $\mathfrak{t}$-equivariant, $\mathfrak{t}$ induces the homomorphism $\mathfrak{t}^*:\ {}^\rho H^r(\mathfrak{X},\mathfrak{D};Z_3)\longrightarrow {}^\rho H^r(\mathfrak{X},\mathfrak{D};Z_3)$ which commutes with Υ_ρ and ψ_ρ. (See the later part of § 2.) The map $\mathfrak{s}:\mathfrak{X}\longrightarrow\mathfrak{X}$ is not $\mathfrak{t}$-equivariant. However we can easily verified that

(15・1) $\quad\sigma\mathfrak{s}^{\#}=\mathfrak{s}^{\#}\sigma,\qquad \tau\mathfrak{s}^{\#}=-\mathfrak{t}^{\#}\mathfrak{s}^{\#}\tau,\qquad \mathfrak{s}^{\#}\tau=-\tau\mathfrak{s}^{\#}\mathfrak{t}^{\#}$,[21]

for the cochain map $\mathfrak{s}^{\#}$ induced by $\mathfrak{s}$. Therefore we can also easily prove that $\mathfrak{s}$ induces a homomorphism $\mathfrak{s}^*:\ {}^\rho H^r(\mathfrak{X},\mathfrak{D};Z_3)\longrightarrow {}^\rho H^r(\mathfrak{X},\mathfrak{D};Z_3)$.

LEMMA (15・2). (i) $\mathfrak{s}^*\Upsilon_\tau=-\Upsilon_\tau\mathfrak{t}^*\mathfrak{s}^*$. (ii) $\mathfrak{s}^*\Upsilon_\sigma=\Upsilon_\sigma\mathfrak{s}^*$.
(iii) $\mathfrak{s}^*\psi_\sigma=-\psi_\sigma\mathfrak{s}^*$.

Proof. Let $a\in{}^\tau H^r(\mathfrak{X},\mathfrak{D};Z_3)$, and let τu be a representative cocycle of a. Then $\mathfrak{s}^*\Upsilon_\tau(a)$ is represented by $\mathfrak{s}^{\#}\delta u$. On the other hand, it follows from $\mathfrak{t}^{\#}\mathfrak{s}^{\#}\tau=-\tau\mathfrak{s}^{\#}$ that $\Upsilon_\tau\mathfrak{t}^*\mathfrak{s}^*(a)$ is represented by $-\delta\mathfrak{s}^{\#}u$. Thus we have (i). (ii) follows from $\sigma\mathfrak{s}^{\#}=\mathfrak{s}^{\#}\sigma$ easily. Using the above notations, $\mathfrak{s}^*\psi_\sigma(a)$ is represented by $\mathfrak{s}^{\#}\sigma u$. On the other hand, it follows from $\mathfrak{s}^{\#}\tau=-\tau\mathfrak{s}^{\#}\mathfrak{t}^{\#}$ that $\psi_\sigma\mathfrak{s}^*(a)$ is represented by $-\sigma\mathfrak{s}^{\#}\mathfrak{t}^{\#}u$. However $-\sigma\mathfrak{s}^{\#}\mathfrak{t}^{\#}=-\mathfrak{s}^{\#}\sigma$. Therefore we have (iii). Q. E. D.

Let $\tilde{\mathfrak{s}}^*:H^r(\mathfrak{Z},\mathfrak{d};Z_3)\longrightarrow H^r(\mathfrak{Z},\mathfrak{d};Z_3)$ be the homomorphism induced by the map $\tilde{\mathfrak{s}}^*:\mathfrak{Z}\longrightarrow\mathfrak{Z}$, and let

$$\begin{aligned}&\mu:\ H^r(\mathfrak{Z},\mathfrak{d};Z_3)\longrightarrow H^{r+2}(\mathfrak{Z},\mathfrak{d};Z_3),\\&\nu:\ H^r(\mathfrak{Z},\mathfrak{d};Z_3)\longrightarrow H^{r+1}(\mathfrak{Z},\mathfrak{d};Z_3),\\&\phi^*:\ H^r(\mathfrak{X};Z_3)\longrightarrow H^r(\mathfrak{Z};Z_3)\end{aligned}$$

be the homomorphisms, defined in § 2, for the complex $\mathfrak{X}$ with the transformation $\mathfrak{t}$. Then we have

LEMMA (15・3). (i) $\tilde{\mathfrak{s}}^*\mu^\alpha=(-1)^\alpha\mu^\alpha\tilde{\mathfrak{s}}^*$. (ii) $\tilde{\mathfrak{s}}^*\nu=-\nu\tilde{\mathfrak{s}}^*$.
(ii) $\phi^*\mathfrak{s}^*=\tilde{\mathfrak{s}}^*\phi^*$.

Proof. We shall prove (i) by mathematical induction on α. If $\alpha=1$, it follows from (15・2) that

$$\tilde{\mathfrak{s}}^*\mu=\tilde{\mathfrak{s}}^*\mathrm{I}^{*^{-1}}\Upsilon_\tau\Upsilon_\sigma\mathrm{I}^*=\mathrm{I}^{*^{-1}}\mathfrak{s}^*\Upsilon_\tau\Upsilon_\sigma\mathrm{I}^*=-\mathrm{I}^{*^{-1}}\Upsilon_\tau\mathfrak{t}^*\mathfrak{s}^*\Upsilon_\sigma\mathrm{I}^*$$

---

21) Of course, we write $\sigma=1+\mathfrak{t}^{\#}+\mathfrak{t}^{2\#}$ and $\tau=1-\mathfrak{t}^{\#}$.

$$=-\mathrm{I}^{*^{-1}}\gamma_\tau\gamma_\sigma\mathfrak{t}^*\mathfrak{s}^*\mathrm{I}^*=-\mathrm{I}^{*^{-1}}\gamma_\tau\gamma_\sigma\mathrm{I}^*\tilde{\mathfrak{s}}^*=-\mu\tilde{\mathfrak{s}}^*.$$

Assume that (i) holds for $\alpha\leqq l$, then we have

$$\tilde{\mathfrak{s}}^*\mu^{l+1}=\tilde{\mathfrak{s}}^*\mu^l\mu=(-1)^l\mu^l\tilde{\mathfrak{s}}^*\mu=(-1)^{l+1}\mu^{l+1}\tilde{\mathfrak{s}}.$$

This is (i) for $\alpha=l+1$. Thus the proof of (i) is complete.

(ii) is obtained from (15・2) as follows:

$$\tilde{\mathfrak{s}}^*\nu=\tilde{\mathfrak{s}}^*\mathrm{I}^{*^{-1}}\psi_\sigma\gamma_\sigma\mathrm{I}^*=\mathrm{I}^{*^{-1}}\mathfrak{s}^*\psi_\sigma\gamma_\sigma\mathrm{I}^*=-\mathrm{I}^{*^{-1}}\psi_\sigma\mathfrak{s}^*\gamma_\sigma\mathrm{I}^*$$
$$=-\mathrm{I}^{*^{-1}}\psi_\sigma\gamma_\sigma\mathfrak{s}^*\mathrm{I}^*=-\mathrm{I}^{*^{-1}}\psi_\sigma\gamma_\sigma\mathrm{I}^*\tilde{\mathfrak{s}}^*=-\nu\tilde{\mathfrak{s}}^*.$$

(iii) follows at once from $\sigma\mathfrak{F}=\mathfrak{s}\sigma$. Q. E. D.

The cohomology group $H^*(\mathfrak{Z}\,;\,Z_3)$ is generated by the elements $\mathbf{1}$, $g_n(1)$, $g_{2n}(1,2)$ and a_{n+s} $(2\leqq s\leqq 2n)$. We shall study the image of these elements by the homomorphism $\tilde{\sigma}^*=1+\tilde{\mathfrak{s}}^*:\ H^r(\mathfrak{Z}\,;\,Z_3)\longrightarrow H^r(\mathfrak{Z}\,;\,Z_3)$.

Lemma (15・4) (i) $\tilde{\sigma}^*(g_n(1))=-g_n(1)$.

(ii) $\tilde{\sigma}^*(g_{2n}(1,\,2))=0$ *if n is odd, and* $=-g_{2n}(1,\,2)$ *if n is even.*

(iii) $\tilde{\sigma}^*(a_{n+2\alpha+2})=-a_{n+2\alpha+2}$ *if α is odd, and* $=0$ *if α is even.*

(iv) $\tilde{\sigma}^*(a_{n+2\alpha+1})=0$ *if α is odd, and* $=-a_{n+2\alpha+1}$ *if α is even.*

Proof. It follows from (16・3) that

$$\tilde{\mathfrak{s}}^*(g_n(1))=\tilde{\mathfrak{s}}^*(\phi^*(e_n\times\mathbf{1}\times\mathbf{1}))=\phi^*\mathfrak{s}^*(e_n\times\mathbf{1}\times\mathbf{1})$$
$$=\phi^*(\mathbf{1}\times e_n\times\mathbf{1})=\phi^*(\mathfrak{t}^{*2}(e_n\times\mathbf{1}\times\mathbf{1}))=g_n(1).$$

From this, (i) is obvious. The proof of (ii) is similar. The homomorphism $\mathfrak{s}^*$: $H^r(\mathfrak{D}\,;\,Z_3)\longrightarrow H^r(\mathfrak{D}\,;\,Z_3)$ is obviously the identity. This, together with (15・3), implies that

$$\tilde{\mathfrak{s}}^*(a_{n+2\alpha+2})=\tilde{\mathfrak{s}}^*j^*\mu^\alpha\nu\,\delta^*\,\pi^{*-1}d_0^*(e_n)$$
$$=j^*\tilde{\mathfrak{s}}^*\mu^\alpha\nu\delta^*\pi^{*-1}d_0^{*-1}(e_n)$$
$$=(-1)^{\alpha+1}j^*\mu^\alpha\nu\delta^*\tilde{\mathfrak{s}}^*\pi^{*-1}d_0^{*-1}(e_n)$$
$$=(-1)^{\alpha+1}j^*\mu^\alpha\nu\delta^*\pi^{*-1}d_0^{*-1}\mathfrak{s}^*(e_n)$$
$$=(-1)^{\alpha+1}a_{n+2\alpha+2}.$$

From this, we obtain (iii). The proof of (iv) is similar. This completes the proof of (15・4).

Since it follows from (1・11) that

$$\tilde{\pi}^*:\ H^r(S^n*S^n*S^n\,;\,Z_3)\approx\tilde{\sigma}^*H^r(\mathfrak{Z}\,;\,Z_3),$$

we have from (16・4) by easy calculations the following

Theorem (15・5). (i) $H^r(S^n*S^n*S^n\,;\,Z_3)=0$ *for* $0<r<n$, $r=n+1$, $r=n+4k-2$ *with* $1\leqq k\leqq[(n+1)/2]$ *and* $k\neq[(n+2)/4]$, $r=n+4k-1$ *with* $1\leqq k\leqq[(2n+1)/4]$ *and* $k\neq[(n+1)/4]$, $r=2n$ *with* $n\equiv-1$ *(mod 4), and* $r>3n$.

(ii) $H^r(S^n*S^n*S^n\,;\,Z_3)\approx Z_3$ *for* $r=0$, $r=n$, $r=n+4k$ *with* $1\leqq k\leqq[n/2]$ *and* $k\neq[n/4]$, $r=n+4k+1$ *with* $1\leqq k\leqq[(2n-1)/4]$ *and* $k\neq[(n-1)/4]$, *and* $r=2n$ *with* $n\equiv-2$ *or* 1 *(mod 4)*.

(iii) $H^r(S^n * S^n * S^n; Z_3) \approx Z_3 \oplus Z_3$ *for* $r=2n$ *with* $n \equiv 0$ (*mod* 4).

Consider the homomorphisms

$$\tilde{\pi}^*: H^r(S^n * S^n * S^n; Z_3) \longrightarrow H^r(\mathfrak{Z}; Z_3),$$
$$\tilde{\phi}^*: H^r(\mathfrak{Z}; Z_3) \longrightarrow H^r(S^n * S^n * S^n; Z_3).$$

Then we see that $\tilde{\pi}^*\tilde{\phi}^* = \tilde{\sigma}^*$ and that $\tilde{\pi}^*$ is isomorphic into. (See (2·10) and (1·11).) Therefore if we write

$$(15\cdot 6)\qquad \begin{aligned} &\tilde{g}_n = \tilde{\phi}^* g_n(1), \quad \tilde{g}_{2n} = \tilde{\phi}^* g_{2n}(1, 2) \text{ for even } n, \\ &\tilde{a}_{n+4\alpha+1} = \tilde{\phi}^* a_{n+4\alpha+1} \text{ for } 1 \leqq \alpha \leqq [(2n-1)/4], \\ &\tilde{a}_{n+4\alpha} = \tilde{\phi}^* a_{n+4\alpha} \quad \text{ for } 1 \leqq \alpha \leqq [n/2], \end{aligned}$$

then it follows from (15·4) that

$$(15\cdot 7)\qquad \begin{aligned} &\tilde{\pi}^*\tilde{g}_n = -g_n(1), \quad \tilde{\pi}^*\tilde{g}_{2n} = -g_{2n}(1, 2), \\ &\tilde{\pi}^*\tilde{a}_{n+4\alpha+1} = -a_{n+4\alpha+1}, \quad \tilde{\pi}^*\tilde{a}_{n+4\alpha} = -a_{n+4\alpha}. \end{aligned}$$

Thus we have

THEOREM (15·8). *The element* **1** *and all the elements of* (15·6) *compose a base for the vector space* $H^*(S^n * S^n * S^n; Z_3)$.

As for the reduced powers, the Bockstein homomorphisms and the cup products in $H^*(S^n * S^n * S^n; Z_3)$, we have

THEOREM (15·9). (i) $\mathcal{P}^i\tilde{g}_n = (-1)^{i+1}\tilde{a}_{n+4i}$ $(i \neq 0)$, $\mathcal{P}^i\tilde{g}_{2n} = 0$ $(i \neq 0)$, $\mathcal{P}^i\tilde{a}_{n+4\alpha+1} = {}_{2\alpha}C_i\tilde{a}_{n+4(\alpha+i)+1}$ $(\alpha+i \leqq [(2n-1)/4])$, $\mathcal{P}^i\tilde{a}_{n+4\alpha} = {}_{2\alpha-1}C_i\tilde{a}_{n+4(\alpha+i)}$ $(\alpha+i \leqq [n/2])$.

(ii) $\Delta_3\tilde{g}_n = 0$, $\Delta_3\tilde{g}_{2n} = 0$, $\Delta_3\tilde{a}_{n+4\alpha+1} = 0$, $\Delta_3\tilde{a}_{n+4\alpha} = \tilde{a}_{n+4\alpha+1}$.

(iii) $\tilde{g}_n \cup \tilde{g}_n = \tilde{g}_{2n}$ *for even* n, *and* $=0$ *for odd* n; $\tilde{g}_n \cup \tilde{g}_{2n} = (-1)^{n/2+1}\tilde{a}_{3n}$ *for even* n, *and* $=0$ *for odd* n; $\tilde{g}_{jn} \cup \tilde{a}_{n+4\alpha+\varepsilon} = 0$ *for* $j=1, 2$ *and* $\varepsilon = 0, 1$; $\tilde{a}_{n+4\alpha+\varepsilon} \cup \tilde{a}_{n+4\beta+\varepsilon'} = 0$ *for* ε, $\varepsilon' = 0, 1$.

Proof. It follows from (15·7) and (13·2) that

$$\begin{aligned} \tilde{\pi}^*\mathcal{P}^i\tilde{g}_n &= \mathcal{P}^i\tilde{\pi}^*\tilde{g}_n - \mathcal{P}^i g_n(1) \\ &= (-1)^{i+2}a_{n+4i} = (-1)^{i+1}\tilde{\pi}^*\tilde{a}_{n+4i}. \end{aligned}$$

Since $\tilde{\pi}^*$ is isomorphic into, we have $\mathcal{P}^i\tilde{g}_n = (-1)^{i+1}\tilde{a}_{n+4i}$. The proofs of the other results are similar. Q. E. D.

THEOREM (15·10). *Let* G *be a field of characteristic* $q \neq 2, 3$. *Then* $H^r(S^n*S^n*S^n; G) \approx G$ *for* $r=0, n, 2n$ *with even* n, $3n$ *with even* n. *For any other* r, $H^r(S^n*S^n*S^n; G) = 0$.

Proof. It follows from (11·5) that $H^*(\mathfrak{Z}; G)$ is generated by the elements $\phi^*(e_n \times \mathbf{1} \times \mathbf{1})$, $\phi^*(e_n \times e_n \times \mathbf{1})$ and $\phi^*(e_n \times e_n \times e_n)$, where $e_n \in H^n(S^n; G)$ is a generator. Furthermore, since $\tilde{\mathfrak{s}}^*\phi^* = \phi^*\mathfrak{s}^*$, it holds that

$$\begin{aligned} \tilde{\mathfrak{s}}^*\phi^*(e_n \times \mathbf{1} \times \mathbf{1}) &= \phi^*\mathfrak{s}^*(e_n \times \mathbf{1} \times \mathbf{1}) \\ &= \phi^*\mathfrak{t}^{*2}(e_n \times \mathbf{1} \times \mathbf{1}) = \phi^*(e_n \times \mathbf{1} \times \mathbf{1}), \\ \tilde{\mathfrak{s}}^*\phi^*(e_n \times e_n \times \mathbf{1}) &= \phi^*\mathfrak{s}^*(e_n \times e_n \times \mathbf{1}) \\ &= (-1)^n\phi^*(e_n \times e_n \times \mathbf{1}), \end{aligned}$$

$$\tilde{\mathfrak{s}}^*\phi^*(e_n\times e_n\times e_n)=\phi^*\mathfrak{s}^*(e_n\times e_n\times e_n)$$
$$=(-1)^n\phi^*(e_n\times e_n\times e_n).$$

Therefore it follows that $\tilde{\sigma}^*H^n(\mathfrak{Z};G)\approx G$, and that both $\tilde{\sigma}^*H^{2n}(\mathfrak{Z};G)$ and $\tilde{\sigma}^*H^{3n}(\mathfrak{Z};G)$ are isomorphic with G if n is even, and $=0$ if n is odd. This, together with (1・11), proves (15・10). Q. E. D.

THEOREM (15・11). *For the integral homology groups $H_r(S^n*S^n*S^n;Z)$, we have the following*:

(i) $C(H_r(S^n*S^n*S^n;Z),\infty)\approx Z$ *for* $r=0,n,2n$ *with even* n, $3n$ *with even* n; *and* $=0$ *for any other* r.

(ii) $C(H_r(S^n*S^n*S^n;Z),2)\approx Z_2$ *for* $r=jn+2k$ *with* $1\leqq k\leqq[(n-1)/2]$ *and* $j=1,2$; *and* $=0$ *for any other* r.

(iii) $C(H_r(S^n*S^n*S^n;Z),3)\approx Z_3$ *for* $r=n+4k$ *with* $1\leqq k\leqq[(2n-1)/4]$, *and* $=0$ *for any other* r.

(iv) $C(H_r(S^n*S^n*S^n;Z),q)=0$ *for odd prime* $q\neq 3$ *and any* r.

Proof. Consider the Smith-Richardson sequence

$$^{\tilde{\tau}}H^r(\mathfrak{Z};Z)\xrightarrow{\alpha_{\tilde{\tau}}}{}^{\tilde{\tau}^{-1}}H^r(\mathfrak{Z};Z)\xrightarrow{\gamma_{\tilde{\tau}}}H^r(\mathfrak{Z};Z)$$

for the complex $\mathfrak{Z}$ with the transformation $\tilde{\mathfrak{s}}$, where $\tilde{\tau}=1-\tilde{\mathfrak{s}}$. Then it follows from (1・7) that $2\gamma_{\tilde{\tau}}\,{}^{\tilde{\tau}}H^r(\mathfrak{Z};Z)=0$. On the other hand, we see from (13・5) that $H^*(\mathfrak{Z};Z)$ has only free component and 3-primary component. Therefore it follows that ${}^{\tilde{\tau}^{-1}}H^r(\mathfrak{Z};Z)\approx H^r(S^n*S^n*S^n;Z)$ is isomorphic with a direct sum of some Z, Z_2 and Z_3. Now (15・11) can be obtained from (14・14), (15・5) and (15・10) by the universal coefficient theorem [3]. Q. E. D.

THEOREM (15・12). *The 3-fold symmetric product of an n-sphere and the Eilenberg-MacLane complex $K(Z,n)$ are of the same $(n+4)$-type.*

Proof. This follows from (14・20), (15・9) and (15・11) by similar arguments as in [10], §4.

REMARK. The symmetric group of degree 3 is solvable. This is the first reason for which we can apply the theory in Chapter I to the determination of cohomology of the 3-fold symmetric products. Since the symmetric group of degree 4 is also solvable, we shall be able to apply the similar arguments as in this chapter to determine the cohomology of the 4-fold symmetric product.

APPENDIX

16. Calculating method of integral cohomology groups

T. Nakamura gives in his paper [7] a method to calculate the integral cohomology groups from the cohomology with coefficients in fields. Different from the original exposition, we shall here explain it as an easy application of the theory of *exact couple* due to W. S. Massey [6].

We shall first recall some definitions and properties. Let

$$(E)\qquad \begin{tikzcd} A \arrow[rr, "f"] & & A \arrow[dl, "g"] \\ & C \arrow[ul, "h"] & \end{tikzcd}$$

be an exact couple (*i. e.* a system consisting of two abelian groups, A and C, and three homomorphisms f, g, h such that the following exactness conditions hold: image $f=$kernel g, image $g=$kernel h, image $h=$kernel f). Then, define

$$(E')\qquad \begin{tikzcd} A' \arrow[rr, "f'"] & & A' \arrow[dl, "g'"] \\ & C' \arrow[ul, "h'"] & \end{tikzcd}$$

as follows: $A'=f(A)$, $C'=(\text{kernel}\, d)/(\text{image}\ d)$, $f'(a)=f(a)$, $g'(a)=gf^{-1}(a)$ $(a\in A')$ and $h'(\bar{c})=h(c)(\bar{c}\in C')$, where $d=gh$ and $\bar{c}$ denotes the element of C' containing c. Then (E') is also an exact couple, which is called the *derived couple*. Define the l-th derived couple

$$({}_lE)\qquad \begin{tikzcd} {}_lA \arrow[rr, "{}_lf"] & & {}_lA \arrow[dl, "{}_lg"] \\ & {}_lC \arrow[ul, "{}_lh"] & \end{tikzcd}$$

by $({}_0E)=(E)$ and $({}_lE)=({}_{l-1}E)'$. Denote by ${}_l\kappa$ the natural homomorphism of a subgroup of C onto ${}_lC$. In (E), let A and C be graded and let f, g, h be homogeneous homomorphisms of degree 0, 0, $+1$ respectively. Then it is verified easily that the same holds in $({}_lE)$.

Let $K=\{C_q(K),\ \partial\}$ be a chain complex such that each $C_q(K)$ is a finitely generated free abelian group and $C_q(K)=0$ if $q<0$. Take an exact sequence

$$0\longrightarrow Z\xrightarrow{\ \xi\ } Z\xrightarrow{\ \eta\ } Z_p\longrightarrow 0,$$

where ξ is the homomorphism defined by $\xi(r)=pr$ $(r\in Z)$ and η is the natural projection. Then, as is well known, we have the exact couple of cohomology groups:

$$\begin{tikzcd} H^*(K;Z) \arrow[rr, "\xi_*"] & & H^*(K;Z) \arrow[dl, "\eta_*"] \\ & H^*(K;Z_p) \arrow[ul, "\tilde{\Delta}_p"] & \end{tikzcd}$$

where ξ_* and η_* are the homomorphisms induced by ξ and η respectively, and $\tilde{\Delta}_p$ is the Bockstein homomorphism. Consider the l-th derived couple of this exact couple:

$$\begin{tikzcd} {}_lH^*(K;Z) \arrow[rr, "{}_l\xi_*"] & & {}_lH^*(K;Z) \arrow[dl, "{}_l\eta_*"] \\ & {}_lH^*(K;Z_p) \arrow[ul, "{}_l\tilde{\Delta}_p"] & \end{tikzcd}$$

Then the following is obvious from the definitions.

LEMMA (16·1). *${}_lH^*(K;Z)=p^lH^*(K;Z)$, and ${}_l\xi_*$ is the homomorphism sending p^la ($a\in H^*(K;Z)$) to $p(p^la)$.*

Since $H^q(K;Z)$ is finitely generated, $H^q(K;Z)$ can be written as a direct sum of cyclic groups whose orders are infinite or a power of a prime number. Denote by $b_q(K)$ the number of Z, and by $t^q(K;p^h)$ the number of Z_{p^h}, in this direct decomposition of $H^q(K;Z)$. Then the following is obvious from (16·1).

LEMMA (16·2). *The kernel of ${}_l\xi_*:{}_lH^q(K;Z)\longrightarrow{}_lH^q(K;Z)$ is isomorphic with $J\{Z_p,\ \Sigma_{h\geqq l+1}t^q(K;p^h)\}$, and the cokernel is isomorphic with $J\{Z_p,\ b^q(K)+\Sigma_{h\geqq l+1}t^q(K;p^h)\}$.*

Therefore, by the exactness of the l-th derived couple, we obtain

LEMMA (16·3). ${}_lR^q(K;Z_p)=b^q(K)+\Sigma_{h\geqq l+1}t^{q+1}(K;p^h)+\Sigma_{h\geqq l+1}t^q(K;p^h)$, *where ${}_lR^q(K;Z_p)$ denotes the rank of the group ${}_lH^q(K;Z_p)$.*

From this, we have

$$ {}_lR^q(K;Z_p)-{}_{l+1}R^q(K;Z_p)=t^{q+1}(K;p^{l+1})+t^q(K;p^{l+1}). $$

Thus we have

THEOREM (16·4).

$$ t^r(K;p^{l+1})=\Sigma_{q=0}^{r-1}(-1)^{r-q+1}\{{}_lR(K;Z_p)-{}_{l+1}R^q(K;Z_p)\}. $$

This theorem shows that if we know ${}_lH^q(K;Z_p)$ for every prime p, then the integral cohomology groups can be calculated immediately.

To the calculations of ${}_lH^q(K;Z_p)$, we may use the following theorem which is obvious from the definition.

THEOREM (16·5). *Let ${}_la\in{}_lH^q(K;Z_p)$, and let $a\in H^q(K;Z_p)$ be an element such that ${}_l\kappa(a)={}_la$. Let further α be an integral cochain such that α mod p represents a. Then the image of ${}_la$ by the homomorphism ${}_ld={}_lf_lh:{}_lH^q(K;Z_p)\longrightarrow{}_lH^{q+1}(K;Z_p)$ is represented by the cohomology class containing $(1/p^{l+1})\delta\alpha$.*

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