Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 7. No. 1-2, Series A

On the cellular decompositions of unitary groups

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(Received March 31. 1956)

1. Introduction

The Betti numbers of compact Lie groups are determined by quite algebraic method. Among compact Lie groups, by making use of the spectral method, A. Borel [2], [3], [4] determined the homology structures with integral coefficient (or any coefficient of field) of the classical groups, G_2 and F_4. It seems to be, however, the more primitive and elementary method to give cellular decompositions of these groups. As for the special orthogonal group $SO(n)$, J.H.C. Whitehead [12] determined its structure as a cell complex. These cells were closely connected with the real projective space P. C. E. Miller [7] computed the homology structures of $SO(n)$ by making use of the above cell structure. As for the universal cover-group $spin(n)$ of $SO(n)$, S. Araki [1] recently gave a cellular decomposition.

In this paper, we shall give a cellular decomposition of the special unitary group $SU(n)$ (see § 7). These cells are closely connected with the suspended space $E(M)$ of the complex projective space M. Using this cell structure, homology group of $SU(n)$ can be computed very easily. In § 8 and § 9, we shall calculate the cup products and the Steenrod's reduced powers in this group.

I am deeply grateful to prof. H. Toda for his kind advices during the preparation of this paper.

(Errata. The results on $Sp(n)$ of my earlier note: *On the cell structures of* $SU(n)$ *and* $Sp(n)$, proc. Japan Acad. vol. 31 (1955), are false. The cellular decomposition of $Sp(n)$ will appear in the forthcoming paper.)

2. Notations

We denote by $H(X)$ $(H^*(X))$ the integral homology group (integral cohomology algebra) of a polyhedron X. If $f: X \longrightarrow Y$ is a continuous map, then we denote by f_*: $H(X) \longrightarrow H(Y)$ $(f^*: H^*(Y) \longrightarrow H^*(X))$ the homomorphism induced by f. In the following, we shall treat only the spaces (which are finite cell complexes) in which the boundary and the coboundary homomorphisms are trivial in all dimensions, so that we may identify $H(X)$ with the chain group $(H^*(X)$ with the cochain group) of X. If e^k is a cell of X, e^k also denotes the homology class containing the cell e^k, Let $[e^k]$ be the cocycle which assigns 1 to only e^k. If there ocurrs no confusion, $[e^k]$ is also denoted by e^k.

3. Unitary group $U(n)$ and special unitary group $SU(n)$

Let C^n be a vector space of dimension n over the field of complex numbers, and e_i be the element of C^n whose i-th component is 1 and whose other components are 0. The elements $e_1, e_2, \cdots\cdots, e_n$ form an orthonormal[1] base of C^n.

Let $U(n)$ be the group of all unitary linear transformations in C^n. In matrix notation, (n, n)-matrix A with complex coefficients is unitary if and only if

$$AA^* = A^*A = I_n{}^{2)}.$$

Let $SU(n)$ be the group of all special unitary linear transformations in C^n. Namely, $SU(n)$ is a subgroup of $U(n)$ composed of all unitary matrices whose determinants are 1. Define a map $\eta: U(n) \longrightarrow SU(n) \times S^1$, where S^1 is a 1-dimensional sphere of all complex numbers whose norms are 1, by

$$\eta(A) = A\begin{pmatrix} detA^{-1} & \\ & I_{n-1} \end{pmatrix} \times detA,$$

then η is a homeomorphism. So that, to consider the topology of $U(n)$, it is sufficient to treat $SU(n)$. Hence, in the following, we shall only consider $SU(n)$.

Embed C^{n-1} in C^n as a subspace whose last component is 0. Let S^{2n-1} be the unit sphere in C^n. Then, embedding $C^{n-1} \subset C^n$ gives rise to an embedding $S^{2n-3} \subset S^{2n-1}$. $SU(n-1)$ may be regarded as a subgroup of $SU(n)$ by extending a matrix A of $SU(n-1)$ to $SU(n)$ by requirement that $Ae_n = e_n$. Thus we have a sequence $I_n = SU(1) \subset SU(2) \subset \cdots\cdots \subset SU(n)$.

For integers n and m such that $n > m$, let $W_{n,m}$ be the complex Stiefel manifold of orthonormal m vectors $\boldsymbol{a} = (a_1, a_2, \cdots\cdots, a_m)$ in C^n. $W_{n-1, m-1}$ can be embedded in $W_{n, m}$ by regarding an element $(a_1, a_2, \cdots\cdots, a_{m-1})$ of $W_{n-1, m-1}$ as an element $\boldsymbol{a} = (a_1, a_2, \cdots\cdots, a_{m-1}, e_n)$ of $W_{n, m}$.

For $A \in SU(n)$, set

$$p_m(A) = (Ae_{n-m+1}, \cdots\cdots, Ae_{n-1}, Ae_n).$$

Then, by the map p_m, $SU(n)$ operates on $W_{n, m}$ transitively and the subgroup $SU(n-m)$ consists of all elements which fix the point $(e_{n-m+1}, \cdots\cdots, e_{n-1}, e_n)$. Hence we have $SU(n)/SU(n-m) = W_{n, m}$. Consequently, we have $W_{n, 1} = S^{2n-1}$ and $W_{n, n-1} = SU(n)$. Especially, we have $SU(n)/SU(n-1) = S^{2n-1}$ with projection $p = p_1$.

4. Complex projective space M_{n-1} and its suspended space $E(M_{n-1})$

Let M_{n-1} be the $(n-1)$-dimensional projective space over the field of complex numbers. If a point x of M_{n-1} has a representative $x = [x_1, x_2, \cdots\cdots, x_n]$, where $x_1, x_2,$

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1) If $x = \sum_{i=1}^n e_i x_i$ and $y = \sum_{i=1}^n e_i x_i$ are vectors in C^n, the inner product (x, y) is defined by $(x, y) = \sum_{i=1}^n \bar{x}_i y_i$. Tow vectors x and y are called to be orthonormal if $(x, y) = 0$ and $(x, x) = (y, y) = 1$.

2) A^* is the transposed conjugate matrix of A. I_n is the unit (n, n)-matrix.

$\cdots\cdots, x_n$ are, not all zero, complex numbers, then the other representatives are $x=[x_1a, x_2a, \cdots\cdots, x_na]$, where a is any non zero complex number. Hence, we may choose a representative $x=[x_1, x_2, \cdots\cdots, x_n]$ such that $|x_1|^2+|x_2|^2+\cdots+|x_n|^2=1$.

Another definition of projective space is due to matrix method [13]. Let $\mathfrak{J}$ be the set of all hermitian (n, n)-matrix (*i.e.* $X^*=X$) with complex coefficients. Define the Jacobi multiplication in $\mathfrak{J}$ by $X\circ Y=\frac{1}{2}(XY+YX)$. If X is an element of $\mathfrak{J}$, then the following conditions are equivalent to each other:

4\. 1) X is an irreducible idempotent element, *i.e.*, if $X=X^2\neq 0$[3] and $X=X_1+X_2$, where $X_i\in\mathfrak{J}$, $X_i=X_i^2$ $(i=1, 2)$ and $X_1\circ X_2=0$, then $X_1=0$ or $X_2=0$.

4\. 2) $tr(X)=tr(X^2)=\cdots=tr(X^n)=1$.

4\. 3) $X=X^2$ and $tr(X)=1$.

4\. 4) $X=UE_nU^*$, where E_n is the (n, n)-matrix whose (n, n)-coefficient is 1 and whose other coefficients are 0, and $U\in U(n)$.

4\. 5) $X=(x_{ij})$ satisfies

$$\begin{cases} x_{ik}x_{kj}=x_{kk}x_{ij} & \text{for } n\geqq i, j, k\geqq 1, \\ x_{11}+x_{22}+\cdots+x_{nn}=1. \end{cases}$$

Let M_{n-1}^* be the set of all elements X of $\mathfrak{J}$ satisfying one of the above conditions 1)-5). Then the usual space M_{n-1} and the above space M_{n-1}^* are equivalent by the correspondence $\zeta: M_{n-1}\longrightarrow M_{n-1}^*$ such that $\zeta(x)=X$, where $x=[x_1, x_2, \cdots\cdots, x_n]$ is an element of M_{n-1} such that $|x_1|^2+|x_2|^2+\cdots+|x_n|^2=1$, and

$$X=\begin{pmatrix} |x_1|^2 & x_1\bar{x}_2 & \cdots\cdots & x_1\bar{x}_n \\ x_2\bar{x}_1 & |x_2|^2 & \cdots\cdots & x_2\bar{x}_n \\ & \cdots\cdots\cdots\cdots & & \\ x_n\bar{x}_1 & x_n\bar{x}_2 & \cdots\cdots & |x_n|^2 \end{pmatrix}.$$

In the following, we shall identify x with X and M_{n-1} with M_{n-1}^*.

We shall regard M_{n-2} as a subcomplex of M_{n-1} whose last component is 0.

As well known, M_{n-1} is a cell complex composed of n cells whose dimensionalities are 0, 2, 4, $\cdots\cdots$, $2n-4$, and $2n-2$.

Let $E(M_{n-1})$ be the suspended space of M_{n-1}. This definition is the following. Let E be the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $E(M_{n-1})$ is the space formed from $E\times M_{n-1}$ by shrinking $-\frac{\pi}{2}\times M_{n-1}$, $\frac{\pi}{2}\times M_{n-1}$ and $E\times[1, 0, \cdots\cdots, 0]$ to a single point of $E(M_{n-1})$. Denote by $\rho: E\times M_{n-1}\longrightarrow E(M_{n-1})$ the shrinking map.

As easily verified, $E(M_{n-1})$ is a cell complex composed of n cells whose dimensionalities are 0, 3, 5, $\cdots\cdots$, $2n-3$ and $2n-1$.

5. Characteristic map f: $E(M_{n-1})\longrightarrow SU(n)$

Define a map $h: E\times M_{n-1}\longrightarrow U(n)$ by $h(\theta, X)=V$, where $\theta\in E$, $X\in M_{n-1}^*$ and

3\) $X^2=X\circ X$, $X^n=X\circ X^{n-1}$,

$$V=I_n-2exp(-\sqrt{-1}\theta)cos\theta X.^{4)}$$

It will be easily verified that V is unitary. Since the determinant of V is $-exp(-2\times\sqrt{-1}\theta)$, if a map $f': E\times M_{n-1}\longrightarrow SU(n)$ is defined by $f'(\theta, X)=VW$, where

$$W=\begin{pmatrix}-exp(2\sqrt{-1}\theta) & \\ & I_{n-1}\end{pmatrix}.$$

Then U is special unitary. If $\theta=\pm\frac{\pi}{2}$ or $x=[1, 0,\cdots\cdots, 0]$, then $U=I_n$. Therefore f' induces a map $f: E(M_{n-1})\longrightarrow SU(n)$ such that $f'=f\rho$. We shall call f the characteristic map of $E(M_{n-1})$ into $SU(n)$.

REMARK. We shall recall the case of $SO(n)$ [7]. Let P_{n-1} be the $(n-1)$-dimensional projective space over the field of real numbers and P_{n-1}^* be its matrix form. Define a map $h: P_{n-1}\longrightarrow O(n)$ (orthogonal group) by $h(X)=V$, where $X\in P_{n-1}^*$ and

$$V=I_n-2X.$$

(V is a reflection across the orthogonal complement of $x=\zeta^{-1}(X)$ in the n-dimensional euclidean space). Since its determinant is -1, if a map $f: P_{n-1}\longrightarrow SO(n)$ is defined by $f(X)=VW$, where

$$W=\begin{pmatrix}-1 & \\ & I_{n-1}\end{pmatrix},$$

then U is special orthogonal. By making use of this map, J.H.C. Whitehead and C.E. Miller obtained the cellular decompositions of $SO(n)$ and $V_{n,m}=SO(n)/SO(n-m)$.

6. Shrinking map $\xi: E(M_{n-1})\longrightarrow S^{2n-1}$

Define a map $\xi: E(M_{n-1})\longrightarrow S^{2n-1}$ by $\xi=pf$.

LEMMA 6. 1. *For $n\geqq2$, ξ maps $E(M_{n-2})$ to a point e_n of S^{2n-1} and $E(M_{n-1})-E(M_{n-2})$ homeomorphically onto $S^{2n-1}-e_n$.* Namely, ξ can be regarded a map which shrinks the boundary of $E(M_{n-1})$ to a point.

Proof. It is obvious that ξ maps $E(M_{n-2})$ to e_n. Given any point $a=(a_1, a_2, \cdots\cdots, a_{n-1}, \alpha+\sqrt{-1}\beta)$ of $S^{2n-1}-e_n$, where $a_1, a_2, \cdots\cdots, a_{n-1}$ are complex numbers and $\alpha\neq1$, β are real numbers such that $|a_1|^2+|a_2|^2+\cdots\cdots+|a_{n-1}|^2+\alpha^2+\beta^2=1$, it is sufficient to show the following equation can be solved continuously:

$$p(I_n-2exp(-\sqrt{-1}\theta)cos\theta X)=a,$$

i.e.

$$\begin{cases}-2exp(-\sqrt{-1}\theta)cos\theta x_1\bar{x}_n=a_1,\\ -2exp(-\sqrt{-1}\theta)cos\theta x_2\bar{x}_n=a_2,\end{cases}$$

4) $\lambda(a_{ij})=(\lambda a_{ij})$

$$\left\{\begin{array}{l}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ 1-2exp(-\sqrt{-1}\theta)cos\theta\,|x_n|^2=\alpha+\sqrt{-1}\beta\,.\end{array}\right.$$

From the last equation, we have

$$x_n=\frac{\sqrt{(1-\alpha)^2+\beta^2}}{\sqrt{2(1-\alpha)}}exp(\sqrt{-1}\varphi)\,,\qquad sin\theta=\frac{\beta}{\sqrt{(1-\alpha)^2+\beta^2}}\,,$$

where φ is an arbitrary real number. Thus x_n and θ are determined. From the other equations, $x_1,\cdots\cdots, x_{n-2}$, and x_{n-1} can be determined. Thus $x=[x_1, x_2,\cdots\cdots,x_n]$ are determined uniquely as a point of the projective space M_{n-1}.

REMARK. Define a map ϕ: $S^{2n-1}-e_n\longrightarrow U(n)$ by $\phi=h\xi^{-1}$, then we have

$$\phi(a_1,\cdots\cdots, a_{n-1}, a_n)=\begin{pmatrix} 1-\dfrac{|a_1|^2}{1-\bar{a}_n} & \cdots\cdots\cdots & -\dfrac{a_1\bar{a}_{n-1}}{1-\bar{a}_n} & a_1\\ \cdots & \cdots & \cdots & \cdots\\ -\dfrac{a_{n-1}\bar{a}_1}{1-\bar{a}_n} & \cdots\cdots\cdots & 1-\dfrac{|a_{n-1}|^2}{1-\bar{a}_n} & a_{n-1}\\ -\dfrac{(1-a_n)\bar{a}_1}{1-\bar{a}_n} & \cdots\cdots\cdots & -\dfrac{(1-a_n)\bar{a}_{n-1}}{1-\bar{a}_n} & a_n\end{pmatrix}.$$

This map coincides with ϕ used in [15, p. 125].

7. Cells of $W_{n,m}$ and $SU(n)$

In the preceding section, we saw that f mapped $\varepsilon^{2k-1}=E(M_{k-1})-E(M_{k-2})$ homeomorphically into $SU(k)\subset SU(n)$ for $n\geqq k\geqq 1$. Set $e^{2k-1}=f(\varepsilon^{2k-1})$. We shall call e^{2k-1} $(2k-1)$-dimensional primitive cell of $SU(n)$. Thus we have n-1 primitive cells whose dimensionalities are 3, 5,$\cdots\cdots$, $2n-3$ and $2n-1$.

For integers $n\geqq k_1>k_2>\cdots\cdots>k_j\geqq 2$, extend f to a map $\bar{f}$: $E(M_{k_1-1})\times E(M_{k_2-1})\times\cdots\cdots\times E(M_{k_j-1})\longrightarrow SU(n)$ by $\bar{f}(z_1\times z_2\times\cdots\cdots\times z_j)=f(z_1)f(z_2)\cdots\cdots f(z_j)$. Put

$$e^{2k_1-1,\ 2k_2-1,\cdots,2k_j-1}=\bar{f}(\varepsilon^{2k_1-1}\times\varepsilon^{2k_2-1}\times\cdots\cdots\times\varepsilon^{2k_j-1}),$$

and

$$e^0=I_n.$$

Furthermore, define a map $\bar{f}_m$: $E(M_{k_1-1})\times E(M_{k_2-1})\times\cdots\cdots\times E(M_{k_j-1})\longrightarrow W_{n,m}$ by $\bar{f}_m=p_m\bar{f}$. Put

$$e_m^{2k_1-1,\ 2k_2-1,\cdots,2k_j-1}=\bar{f}_m(\varepsilon^{2k_1-1}\times\varepsilon^{2k_2-1}\times\cdots\cdots\times\varepsilon^{2k_j-1}),$$

and

$$e_m^0=(e_{n-m+1}, e_{n-m+2},\cdots\cdots, e_n).$$

Now, we shall show that $W_{n,m}$ is a cell complex composed of e_m^0 and $e_m^{2k_1-1,\ 2k_2-1,\cdots,\ 2k_j-1}$ with $n\geqq k_1>k_2>\cdots\cdots>k_j\geqq n-m+1$.

First of all, we shall show that $W_{n,m}$ is the union of these cells. As $W_{n-m+1,1}=S^{2n-2m+1}$ is the union of e_m^0 and $e^{2n-2m+1}$, we shall assume that the above assertion is true for $W_{s,t}$, where $s-t=n-m$ and $m>t\geqq 1$. Given $\boldsymbol{a}=(a_1,\cdots\cdots a_{m-1}, a_m)\in W_{n,m}$ but $\boldsymbol{a}\notin W_{n-1,m-1}$, then $a_m\neq e_n$. So that, we can choose a point $z\in\varepsilon^{2n-1}$ uniquely such that $\xi(z)=a_m$. Put $U=f(z)$, then $U^*\boldsymbol{a}=(U^*a_1,\cdots, U^*a_{m-1}, U^*a_m)=(U^*a_1,\cdots, U^*a_{m-1}, e_n)\in W_{n-1,\ m-1}$. Hence $U^*\boldsymbol{a}$ belongs to a certain cell $e_m^{2k_2-1,\ 2k_3-1,\cdots,\ 2k_j-1}$, with $n-1\geqq k_2>k_3>\cdots\cdots>k_j\geqq n-m+1$ by the induction. Therfore $\boldsymbol{a}$ belongs to a cell $e_m^{2n-1, 2k_2-1,\ 2k_3-1,\cdots,\ 2k_j-1}$.

Next, we shall show that $\overline{f_m}$ maps $\varepsilon^{2k_1-1}\times\varepsilon^{2k_2-1}\times\cdots\cdots\times\varepsilon^{2k_j-1}$ homeomorphically onto $e_m^{2k_1-1,\ 2k_2-1,\cdots,2k_j-1}$ and these cells are disjoint to each other. In fact, let $\boldsymbol{a}\in e_m^{2k_1-1,\ 2k_2-1,\cdots,2k_s-1}\cap e_m^{2l_1-1,\ 2l_2-1,\cdots,2l_t-1}$, namely there exist $U,\ V\in SU(n)$ such that

$$\boldsymbol{a}=(U_1U_2\cdots U_s e_{n-m+1},\ U_1U_2\cdots U_s e_{n-m+2},\cdots\cdots,\ U_1U_2\cdots U_s e_n)$$
$$=(V_1V_2\cdots V_t e_{n-m+1},\ V_1V_2\cdots V_t e_{n-m+2},\cdots\cdots,\ V_1V_2\cdots V_t e_n),$$

where $U_i\in e_m^{2k_i-1}$ and $V_i\in e_m^{2l_i-1}$. If

$$U_1U_2\cdots U_s e_p=V_1V_2\cdots V_t e_p=e_p \qquad \text{for } p=n-q+1,\cdots\cdots,\ n,$$

and

$$U_1U_2\cdots U_s e_{n-q}=V_1V_2\cdots V_t e_{n-q}\neq e_{n-q}\,,$$

then this means

$$U_1 e_{n-q}=V_1 e_{n-q}\,.$$

Since ξ is homeomorphic, it follows $U_1=V_1$. Hence

$$U_2U_3\cdots U_s e_p=V_2V_3\cdots V_t e_p \qquad \text{for } p=n-m+1,\cdots\cdots,\ n\,.$$

Similarly $U_2=V_2$ and so on. Consequently $s=t$. Therefore these cells are disjoint. The above proof also gives that $\overline{f_m}$ is one-to-one. The fact that $\overline{f_m}$ is a homeomorphism is obvious from the continuity of the group multiplication and the homeomorphism of ξ.

Thus we have the following results.

THEOREM 7. 1. *The complex Stiefel manifold* $W_{n,\ m}=SU(n)/SU(n-m)$ *is a cell complex composed of* 2^m *cells* e_m^0 *and* $e_m^{2k_1-1,\ 2k_2-1,\cdots,\ 2k_j-1}$ *with* $n\geqq k_1>k_2>\cdots>k_j\geqq n-m+1$. *The dimension of* $e_m^{2k_1-1,\ 2k_2-1,\cdots,\ 2k_j-1}$ *is* $(2k_1-1)+(2k_2-1)+\cdots\cdots+(2k_j-1)$.

Especially,

THEOREM 7. 2. *The special unitary group* $SU(n)$ *is a cell complex composed of* 2^{n-1} *cells* e^0 *and* $e^{2k_1-1,\ 2k_2-1,\cdots,\ 2k_j-1}$ *with* $n\geqq k_1>k_2\cdots\cdots>k_j\geqq 2$. *The dimension of* $e^{2k_1-1,\ 2k_2-1,\cdots,\ 2k_j-1}$ *is* $(2k_1-1)+(2k_2-1)+\cdots\cdots+(2k_j-1)$. *Especially* e^{2k-1}, *called the* $(2k-1)$*-dimensional primitive cell of* $SU(n)$, *is obtained as the image of the interior of the suspended space* $E(M_{k-1})$ *of* $(k-1)$*-dimensional complex projective space* M_{k-1} *by the characteristic map* $f\colon E(M_{k-1})\longrightarrow SU(k)\subset SU(n)$.

REMARK. As obviously $p_m\colon SU(n)\longrightarrow W_{n,\ m}$ is a cellular map.

8. Homology and cohomology groups of $W_{n,\ m}$ and $SU(n)$

With respect to the preceding cell structure, the boundary homomorphisms are trivial in all dimensions. Hence we can compute homology groups very easily. In fact, the Betti number for m-dimension is the number of the cells whose dimensions are m.

THEOREM 8. 1. *$W_{n,m}$ and $SU(n)$ have no torsion groups in all dimensions, and their Poincaré polynomials are*

$$P_{W_{n,m}}(t)=(1+t^{2n-2m+1})(1+t^{2n-2m+3})\cdots\cdots(1+t^{2n-1}),$$

and

$$P_{SU(n)}(t)=(1+t^3)(1+t^5)\cdots\cdots(1+t^{2n-1}).$$

Let X be a space and k be a field. Denote by $D^*(X;k)$ the subgroup of the cohomology group $H^*(X;k)$ with coefficient k generated by the elements of the form $a\cup b$[5], where a and b are elements of degree>0 in $H^*(X;k)$. Let v be a homogeneous element of the homology group $H(X;k)$ with coefficient k and *dim* $v>0$. We shall call v a *homological primitive element* (or minimal element) of X with respect to coefficient k if and only if v is orthogonal to $D^*(X;k)$ [5], [6]. Remark that if the space X has no torsions, the above definition is also applicable to the case of the primitive element of X with integral coefficient.

LEMMA 8. 1 (*Invariance theorem*). *Let $f: X\longrightarrow Y$ be a map, then for any homological primitive element v of X, the image $f_*(v)$ is also a homological primitive element of Y.*

Proof. $(f_*(v), a\cup b)=(v, f^*(a\cup b))=(v, f^*(a)\cup f^*(b))=0.$ q.e.d.

THEOREM 8. 2. *Let e_m^{2k-1} be the element of $H(W_{n,m})$ containing the cell e_m^{2k-1}. Then, e_m^{2k-1} $(n\geqq k\geqq n-m-1)$ is a homological primitive element of $W_{n,m}$.*

Proof. Since all cup products are trivial in the space $E(M_{k-1})$, ε^{2k-1} of $H(E(M_{k-1}))$ is a homological primitive element of $E(M_{k-1})$. Therefore e_m^{2k-1} is also primitive as the image of ε^{2k-1} by the map $f_m: E(M_{k-1})\longrightarrow W_{n,m}$. q.e.d.

As for the cup product $\cup$, we have the following results.

THEOREM 8. 3. *In the cohomology algebra $H^*(W_{n,m})$, we have*

$$e_m^{2k_1-1,\,2k_2-1}=e_m^{2k_1-1}\cup e_m^{2k_2-1} \qquad \textit{for } n\geqq k_1>k_2\geqq n-m+1.$$

Especially, in $H^*(SU(n))$, we have

$$e^{2k_1-1,\,2k_2-1}=e^{2k_1-1}\cup e^{2k_2-1} \qquad \textit{for } n\geqq k_1>k_2\geqq 2.$$

i.e., $H^*(W_{n,m})$ is the free exterior algebra generated by e_m^0 (which is a unit) and e_m^{2k-1} with $n\geqq k\geqq n-m+1$.

Proof. Since $p_m^*: H^*(W_{n,m})\longrightarrow H^*(SU(n))$ is isomorphic into and

$p_m^*(e^{2k_1-1,\,2k_2-1,\cdots,\,2k_j-1})=e^{2k_1-1,\,2k_2-1,\cdots,\,2k_j-1}$ for $n\geqq k_1>k_2>\cdots\cdots>k_j\geqq n-m+1$, and $p_m^*(e_m^{2k_1-1,\,2k_2-1,\cdots,\,2k_j-1})=0$ otherwise, we shall prove the formula for $H^*(SU(n))$. In order to prove this, it is sufficient to show

$$e^{2n-1,\,2k-1}=e^{2n-1}\cup e^{2k-1} \qquad \text{for } n\geqq k\geqq 2,$$

using the induction with respect to n of $SU(n)$.

Define a map $\nu: E(M_{n-1})\times SU(n-1)\longrightarrow SU(n)$ by

5) The symbol $\cup$ denotes the cup product.

$$\nu(z, A)=f(z)A.$$

Let $\nu_1^*: H^*(SU(n), SU(n-1)) \longrightarrow H^*(E(M_{n-1}) \times SU(n-1), E(M_{n-2}) \times SU(n-1))$ be the homomorphism induced by ν. Then ν_1^* is isomorphic into and

$$\nu_1^*(e^{2n-1, 2k-1})=\varepsilon^{2n-1} \times e^{2k-1}.$$

On the other hand,

$$\nu_1^*(e^{2n-1} \cup e^{2k-1})=\nu_1^*(e^{2n-1}) \cup \nu_1^*(e^{2k-1})=(\varepsilon^{2n-1} \times e^0) \cup \nu_1^*(e^{2k-1}),$$

where the symbols $\cup$ contained in the last tow expressions mean the relative cup product in $H^*(E(M_{n-1}) \times SU(n-1), E(M_{n-2}) \times SU(n-1))$[10]. Define tow maps j: $SU(n-1) \longrightarrow E(M_{n-1}) \times SU(n-1)$ by, $j(\mathrm{A})=(\varepsilon^0, \mathrm{A})$, and p: $E(M_{n-1}) \times SU(n-1) \longrightarrow SU(n-1)$ by $p(z, A)=A$, and let i: $SU(n-1) \longrightarrow SU(n)$ be the inclusion map.

Since $\nu j=i$, we have

$$j^* \nu_1^*(e^{2k-1})=i^*(e^{2k-1}).$$

Hence,

$$p^* j^* \nu_1^*(e^{2k-1})=p^* i^*(e^{2k-1})=p^*(e^{2k-1})=\varepsilon^0 \times e^{2k-1}.$$

Since the last expression is non zero, we have

$$\nu_1^*(e^{2k-1})=\varepsilon^0 \times e^{2k-1}.$$

Hence,

$$\begin{aligned} \nu_1^*(e^{2n-1} \cup e^{2k-1}) &=(\varepsilon^{2n-1} \times e^0) \cup (\varepsilon^0 \times e^{2k-1}) \\ &=(\varepsilon^{2n-1} \cup \varepsilon^0) \times (e^0 \cup e^{2k-1}) \\ &=\varepsilon^{2n-1} \times e^{2k-1}. \end{aligned}$$

Since ν_1^* is isomorphic into, we have

$$e^{2n-1, 2k-1}=e^{2n-1} \cup e^{2k-1}.$$

LEMMA 8. 2. *Let* ν^*: $H^*(SU(n)) \longrightarrow H^*(E(M_{n-1}) \times SU(n-1))$ *be the homomorphism induced by* ν *defined in the theorem* 8. 3. *Then* ν^* *is isomorphic into and we have*

$$\begin{cases} \nu^*(e^{2k-1})=\varepsilon^{2k-1} \times e^0+\varepsilon^0 \times e^{2k-1} & \text{for } n>k \geqq 2, \\ \nu^*(e^{2n-1})=\varepsilon^{2n-1} \times e^0. \end{cases}$$

Proof. As obviously, we have

$$\begin{aligned} &(\nu^*(e^{2k-1}), \quad \varepsilon^{2s-1} \times e^{2t_1-1, \cdots, 2t_j-1}) \\ &\quad=(e^{2k-1}, e^{2s-1} * e^{2t_1-1, \cdots, 2t_j-1}), \end{aligned}$$

where $k>s$, $k>t_1>\cdots>t_j \geqq 2$ and the symbol $*$ is the Pontrjagin product. If $e^{2s-2} * {}^{2t_1-1, \cdots, 2t_j-1}$ contains the homological primitive element, e^{2k-1} is generated by lower dimensional elements by the Pontrjagin product. This conclusion is, however, contradictry to the general homology theory of compact Lie groups [8], [9]. Hence we obtain

$$(\nu^*(e^{2k-1}),\ \varepsilon^{2s-1}\times e^{2t_1-1,\cdots,2t_j-1})=0.$$

On the other hand, we have

$$(\nu^*(e^{2k-1}),\ \ \varepsilon^{2k-1}\times e^0)=(\nu^*(e^{2k-1}),\ \ \varepsilon^0\times e^{2k-1})=1.$$

Thus the lemma is completed.

9. Steenrod's reduced powers in $W_{n,m}$

Let p be a fixed prime number, K be a finite complex and L be a subcomplex of K. The Steenrod's reduced powers $\wp_p^s$ are homomorphisms

$$\wp_p^s:\ H^q(K,\ L\ ;\ Z_p)\longrightarrow H^{q+2s(p-1)}(K,\ L\ ;\ Z_p)^{6)}$$

defined for all $s\geqq 0$ and all $q\geqq 0$. On the other hand, if $p=2$, there exist, as well known, Steenrod's square homomorphisms Sq^s

$$Sq^s:\ H^q(K,\ L\ ;\ Z_2)\longrightarrow H^{q+s}(K,\ L\ ;\ Z_2)$$

defined for all $s\geqq 0$ and $q\geqq 0$. These tow operations $\wp_p^s$ and Sq^s are combined by the relation $\wp_2^s=Sq^{2s}$.

The following formulae are well known.

9. 1) If f: $(K,\ L)\longrightarrow(K',\ L')$ is a map, then $\wp_p^s f^*=f^*\wp_p^s$ $(Sq^s f^*=f^*Sq^s)$.

9. 2) $\wp_p^0$ is the identity isomorphism (Sq^0 is also so).

9. 3) $\wp_p^s$ is trivial for $q<2S$ (Sq^s is trivial for $q<s$).

9. 4) $\wp_p^s(x)=x^p$ [7] for $x\in H^{2s}(K,\ L\ ;\ Z_p)$ $(Sq^s(x)=x^2$ for $x\in H^s(K,\ L\ ;\ Z_2))$.

9. 5) Let δ ; $H^q(L,\ Z_p)\longrightarrow H^{q-1}(K,\ L\ ;\ Z_p)$ be the coboudary homomorphism, then $\wp_p^s\delta=\delta\wp_p^s$ $(Sq^s\delta=\delta Sq^s)$.

9. 6) $\wp_p^s(x\cup y)=\sum_{i+j=s}\wp_p^i(x)\cup\wp_p^j(y)$ $(Sq^s(x\cup y)=\sum_{i+j=s}Sq^i(x)\cup Sq^j(y))$.

M_{n-1} has 0, 2, 4,$\cdots$, $2n-4$ and $2n-2$ dimensional cells. Let $u_{2k}\in H^{2k}(M_{n-1}\ ;\ Z_p)$ be the cohomology class reduced modulo p containing the $2k$-dimensional cell. As well known, if we orient these cells suitably, then we have $u_{2k}=u_2^k$.

LEMMA 9. 1. *In the complex projective space* M_{n-1}, *we have*

$$\wp_p^s(u_{2k})=\binom{k}{s}u_{2k+2s(p-1)},^{8),\,9)}$$

and

$$\begin{cases} Sq^{2s}(u_{2k})=\binom{k}{s}u_{2k+2s}\,, \\ Sq^{2s+1}(u_{2k})=0. \end{cases}$$

---

6) Z_p denotes a cyclic group of order p.

7) x^p denotes the p-times cup product of x.

8) $\binom{k}{s}$ is the binormial coefficient reduced modulo p.

9) The expressions in the right hand sides are zero if they have no means.

Proof. Suppose $p \neq 2$. The extreme case $s=0$ is obvious from formula 9. 2). For the general case, we proceed by induction on s.

$$\begin{aligned}\mathcal{P}_p^s(u_{2k}) &= \mathcal{P}_p^s(u_2^k) = \mathcal{P}_p^s(u_2 \cup u_2^{k-1}) = \mathcal{P}_p^0(u_2) \cup \mathcal{P}_p^s(u_2^{k-1}) + \mathcal{P}_p^1(u_2) \cup \mathcal{P}_p^{s-1}(u_2^{k-1}) \\ &= u_2 \cup \binom{k-1}{s} u_{2k-2+2s(p-1)} + u_2^p \cup \binom{k-1}{s-1} u_{2k-2+2(s-1)(p-1)} \\ &= \binom{k}{s} u_{2k+2s(p-1)}.\end{aligned}$$

In the case of $p=2$, $Sq^{2s+1}=0$. Therefore, $Sq^{2s}(u_{2k})$ is the special case of $\mathcal{P}_p^s(u_{2k})$. q. e. d.

$E(M_{n-1})$ has $0, 3, 5, \ldots, 2n-3$ and $2n-1$ dimensional cells. Let $v_{2k-1} \in H^{2k-1}(E(M_{n-1})\,;\, Z_p)$ be the cohomology class reduced modulo p containing the cell ε^{2k-1}. Let S^*: $H^q(M_{n-1}\,;\, Z_p) \longrightarrow H^{q+1}(E(M_{n-1})\,;\, Z_p)$ be the suspended isomorphism. Then S^* commute with the operations $\mathcal{P}_p^s$ and Sq^s. If we orient these cells suitably, then we have

$$v_{2k-1} = S^*(u_{2k-2}).$$

LEMMA 9. 2. *In $E(M_{n-1})$, we have*

$$\mathcal{P}_p^s(v_{2k-1}) = \binom{k-1}{s} v_{2k-1+2s(p-1)},$$

and

$$\begin{cases} Sq^{2s}(v_{2k-1}) = \binom{k-1}{s} v_{2k-1+2s}, \\ Sq^{2s+1}(v_{2k-1}) = 0. \end{cases}$$

Proof. These are obvious from the lemme 9. 1.

This result enables us to compute the reduced powers in $W_{n,m}$, especially $SU(n)$ rather simply.

THEOREM 9. 1. *In $W_{n,m}$, the reduced powers are given by*

$$\begin{cases} \mathcal{P}_p^s(e_m^{2k-1}) = \binom{k-1}{s} e_m^{2k-1+2s(p-1)}, \\ \mathcal{P}_p^s(e_m^{2k_1-1,\, 2k_2-1, \cdots,\, 2k_j-1}) \\ = \displaystyle\sum_{i_1+i_2+\cdots+i_j=s} \binom{k_1-1}{i_1}\binom{k_2-1}{i_2} \cdots \binom{k_j-1}{i_j} e_m^{2k_1-1+2i_1(p-1),\, 2k_2-1+2i_2(p-1), \cdots, 2k_j+i_j(p-1)}, \end{cases}$$

and

$$\begin{cases} Sq^{2s}(e_m^{2k-1}) = \binom{k-1}{s} e_m^{2k-1+2s}, \\ Sq^{2s}(e_m^{2k_1-1,\, 2k_2-1, \cdots, 2k_j-1}) \\ = \displaystyle\sum_{i_1+i_2+\cdots+i_j=s} \binom{k_1-1}{i_1}\binom{k_2-1}{i_2} \cdots \binom{k_j-1}{i_j} e_m^{2k_1-1+2i_1,\, 2k_2-1+2i_2, \cdot,\, 2k_j-1+2i_j}, \\ Sq^{2s+1} = 0. \end{cases}$$

Proof. If $n=2$, the theorem is trivial. For $n>2$, we proceed inductively, supposing the theorem is valid for $W_{n-1,t}$, especially $SU(n-1)$. Furthermore, it is sufficient to prove the formulae for $SU(n)$. Let ν^*: $H^*(SU(n)\,;\, Z_p) \longrightarrow H^*(E(M_{n-1}) \times SU(n-1)$;

Z_p) be the isomorphism into defined in § 8. If $n > k$, then

$$\begin{aligned}\nu^* \mathcal{P}_p^s(e^{2k-1}) &= \mathcal{P}_p^s(\nu^*(e^{2k-1})) \\ &= \mathcal{P}_p^s(v_{2k-1} \times e^0 + v_0 \times e^{2k-1}) \\ &= \binom{k-1}{s} v_{2k-1+2s(p-1)}) \times e_0 + v_0 \times \binom{k-1}{s} e^{2k-1+2s(p-1)} \\ &= \binom{k-1}{k} \nu^*(e^{2k-1+2s(p-1)}).\end{aligned}$$

Since ν^* is isomorphic into, so we have the first formula. If $n=k$, $\mathcal{P}_p^s(e^{2n-1})=0$.

$$\mathcal{P}_p^s(e^{2k_1-1,\, 2k_2-1, \cdots,\, 2k_j-1}) = \mathcal{P}_p^s(e^{2k_1-1} \cup e^{2k_2-1, \cdots,\, 2k_j-1})$$

$$= \sum_{l+m=s} \mathcal{P}_p^l(e^{2k_1-1}) \cup \mathcal{P}_p^m(e^{2k_2-1, \cdots, 2k_j-1})$$

$$= \sum_{l+m=s} (\binom{k_1-1}{l} e^{2k_1-1+2l(p-1)} \cup (\sum_{i_2+\cdots+i_j=m} \binom{k_2-1}{i_2} \cdots \binom{k_j-1}{i_j} e^{2k_2-1+2i_2(p-1), \cdots,\, 2k_j-1+2i_j(p-1)}))$$

$$= \sum_{i_1+i_2+\cdots+i_j=s} \binom{k_1-1}{i_1} \binom{k_2-1}{i_2} \cdots \binom{k_j-1}{i_j} e^{2k_1-1+2i_1(p-1),\, 2k_2-1+2i_2(p-1), \cdots,\, 2k_j-1+2i_j(p-1)}.$$

The other formulae are obtained quite similarly. q. e. d.

REMARK. This results coincide with those of A. Borel and J. P. Seere [4]. In fact, due to result of S. Mukohda and S. Sawaki [8], we have that

$$b_p^{k,\, j} \equiv \binom{j-1-k(p-1)}{k} \qquad (mod\ p).$$

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