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Dirichlet problem relative to a family of functions

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In their paper[1] Beckenbach and Jackson considered some properties of subfunctions for a dominating family of functions and studied a Dirichlet problem relative to that family of functions. In this paper the author follows them, develops their researches and gives some applications to partial differential equations of elliptic type.

1. Notations, definitions, postulates and theorems that they set or obtained are reproduced here with slight modifications for the later use.

Let D be a plane region (non null connected open set) and $\mathfrak{K}$ a family of circles κ such that

a) κ lies with its interior K in D, that is,

$$\bar{K}=\kappa+K\subset D,$$

b) for any point P of D, every small circle with center at P and every small circle through P belong to $\mathfrak{K}$.[2]

Let $\mathfrak{F}$ be a family of functions whose domains of definition are $\bar{K}\,(\kappa\in\mathfrak{K})$ and which satisfy the following postulates.

POSTULATE 1. For any member $\kappa\in\mathfrak{K}$ and any continuous boundary value function $h(P)$ on κ, there is a unique function $F(P)\equiv\mathfrak{F}(P;h,\kappa)\in\mathfrak{F}$ such that

a) $$F(P)=h(P) \qquad \text{on } \kappa,$$

b) $F(P)$ is continuous in $\mathfrak{D}(F)=\bar{K}$, where $\mathfrak{D}(F)$ denotes the (closed) domain of definition of F.

POSTULATE 2. For each constant $M\geqq 0$, if

$$F_i\in\mathfrak{F},\ \mathfrak{D}(F_i)\supseteqq\bar{K} \qquad (i=1,2)$$

and
$$F_1(P)\leqq F_2(P)+M \qquad \text{on } \kappa,$$

then
$$F_1(P)\leqq F_2(P)+M \qquad \text{in } \bar{K};$$

further, if the strict inequality holds at a point of κ, then the strict inequality hold throughout K.

REMARK. If $F\in\mathfrak{F}$ and $\mathfrak{D}(F)=\bar{K}$, then

$$\mathfrak{F}(P;F,\gamma)=F(P) \qquad \text{in } \bar{\Gamma},$$

for any $\gamma\in\mathfrak{K}$ such that γ lies together with its interior Γ in K.

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1) E. F. Beckenbach and L. K. Jackson, Subfunctions of several variables, Pacific Journal of Mathematics (1953).

2) In the sequel the elements of $\mathfrak{K}$ are denoted by $\kappa,\gamma,\ldots.$ and their interiors by $K,\Gamma,\ldots.$, respectively.

DEFINITION 1. A continuous function $g(P)$ defined in a region Ω, contained in D, is called a sub-$\mathfrak{F}$ function in Ω, if it holds

$$g(P) \leqq \mathfrak{F}(P; g, \kappa) \qquad \text{in } K,$$

for any $\kappa \in \mathfrak{K}$ such that $\bar{K} \subset \Omega$.

THEOREM 1. *A function $g(P)$, continuous in Ω, is a sub-$\mathfrak{F}$ function in Ω if, and only if, corresponding to each P of Ω, there exists a sequence of circles $\kappa_n \in \mathfrak{K}$ with center at P and of radii $\rho_n \to 0$, such that*

$$g(P) \leqq \mathfrak{F}(P; g, \kappa_n).$$

THEOREM 2. *If $g_n(P)$ is a sub-$\mathfrak{F}$ function in Ω for $n=1, 2, \dots.$ and $g_n \to g$ uniformly on each compact set of Ω, then $g(P)$ is also a sub-$\mathfrak{F}$ function in Ω.*

THEOREM 3. *If $g_1(P), g_2(P), \cdots, g_n(P)$ are sub-$\mathfrak{F}$ functions in Ω, then the function $g(P)$ defined by*

$$g(P) = \max[g_1(P), g_2(P), \cdots, g_n(P)]$$

is also a sub-$\mathfrak{F}$ function in Ω.

THEOREM 4. *If $g(P)$ is a sub-$\mathfrak{F}$ function in Ω, then for any $\kappa \in \mathfrak{K}$ with $\bar{K} \subset \Omega$, the function $g_\kappa(P)$ defined by*

$$g_\kappa(P) = \begin{cases} g(P) & \text{for } P \in \Omega - K, \\ \mathfrak{F}(P; g, \kappa) & \text{for } P \in K, \end{cases}$$

is also a sub-$\mathfrak{F}$ function in Ω.

DEFINITION 2. Super-$\mathfrak{F}$ functions are defined by reversing the inequality in Definition 1. It is easy to show that results analogues to Theorems 1~4, with suitable alternations, hold for super-$\mathfrak{F}$ functions: in addition to writing "super-$\mathfrak{F}$ function" for "sub-$\mathfrak{F}$ function," we reverse the inequality in Theorem 1 and replace "max" by "min" in Theorem 3.

DEFINITION 3. We shall say that a function $F(P)$, which is continuous in Ω and satisfies

$$F(P) = \mathfrak{F}(P; F, \kappa) \qquad \text{in } K,$$

for each $\kappa \in \mathfrak{K}$ with $\bar{K} \subset \Omega$, is an $\mathfrak{F}$-function in Ω.

Hence F is an $\mathfrak{F}$-function if and only if F is both sub-$\mathfrak{F}$ and super-$\mathfrak{F}$.

REMARK. An $\mathfrak{F}$-function need not belong to $\mathfrak{F}$, but a function $F \in \mathfrak{F}$ is an $\mathfrak{F}$-function in the interior of $\mathfrak{D}(F)$.

Let $h(P)$ be a bounded, but non necessary continuous, function defined on the boundary ω of a bounded region Ω. We shall define $\bar{h}$, and $\underline{h}$ by

$$\bar{h}(Q) = \lim_{\delta \to 0} \sup_{\overline{PQ} < \delta} h(P), \quad \underline{h}(Q) = \lim_{\delta \to 0} \inf_{\overline{PQ} < \delta} h(P).$$

THEOREM 5. *Let $f(P)$ and $g(P)$ be a super-$\mathfrak{F}$ function and a sub-$\mathfrak{F}$ function in Ω, respectively. If*

$$\bar{g}(Q) \leqq \underline{f}(Q) \qquad \text{on } \omega,$$

then

$$g(P) \leqq f(P) \qquad \text{in } \Omega.$$

Let $H(P)$ be a function defined in Ω. We shall define $\bar{H}$, $\underline{H}$ on ω, by

$$\bar{H}(Q)=\lim_{\delta\to 0}\sup_{\overline{PQ}<\delta} H(P),\ \underline{H}(Q)=\lim_{\delta\to 0}\inf_{\overline{PQ}<\delta} H(P).$$

DEFINITION 4 (Dirichlet problem). By a solution of the Dirichlet problem for a bounded region Ω and a given boundary value function h on ω, relative to $\mathfrak{F}$, we shall mean a function $H(P)$ which is an $\mathfrak{F}$-function in Ω and satisfies

$$\underline{h}(Q)\leqq\underline{H}(Q)\leqq\bar{H}(Q)\leqq\bar{h}(Q) \qquad \text{on } \omega.$$

To construct such a solution of Dirichlet problem we shall use the so-called Poincaré-Perron method.

DEFINITION 5. A function $\Phi(P)$, bounded and sub-$\mathfrak{F}$ in Ω, is an under-function of first kind provided that

$$\bar{\Phi}(Q)\leqq h(Q) \qquad \text{on } \omega,$$

and an under-function of second kind provided that

$$\bar{\Phi}(Q)\leqq \underline{h}(Q) \qquad \text{on } \omega.$$

A function $\Psi(P)$, bounded and super-$\mathfrak{F}$ in Ω, is an over-function of first kind provided that

$$\underline{\Psi}(Q)\geqq h(Q) \qquad \text{on } \omega,$$

and an over-function of second kind provided that

$$\underline{\Psi}(Q)\geqq \bar{h}(Q) \qquad \text{on } \omega.$$

An under(over)-function of second kind becomes clearly an under(over)-function of first kind. $\mathfrak{U}(\mathfrak{U}^*)$ and $\mathfrak{O}(\mathfrak{O}^*)$ denote the families of under-functions and over-functions of first(second) kind, respectively.

POSTULATE 3. For each constant M and any Ω, there exist a bounded super-$\mathfrak{F}$ function $f(P)$ and a bounded sub-$\mathfrak{F}$ function $g(P)$ such that

$$g(P)<M<f(P) \qquad \text{in } \Omega.$$

From this it is clear that the families $\mathfrak{U}$, $\mathfrak{U}^*$, $\mathfrak{O}$ and $\mathfrak{O}^*$ are not empty, respectively.

THEOREM 6. *If $\Phi\in\mathfrak{U}$ and $\Psi\in\mathfrak{O}$, then*

$$\Phi(P)\leqq\Psi(P) \qquad \text{in } \Omega.$$

COROLLARY. *If $\Phi\in\mathfrak{U}^*$ and $\Psi\in\mathfrak{O}^*$, then $\Phi\leqq\Psi$ in Ω.*

POSTULATE 4. For any $\kappa\in\mathfrak{K}$ and for any collection $\{h_\nu\}$ of functions which are continuous and uniformly bounded on κ, the functions $\mathfrak{F}(P;h_\nu,\kappa)$ are equicontinuous in K.

DEFINITION 6. We shall define the functions $H^\circ(P)$, $H^*(P)$, $H_\circ(P)$ and $H_*(P)$ by

$$H^\circ(P)\equiv\inf_{\Psi\in\mathfrak{O}}\Psi(P),\ H^*(P)\equiv\inf_{\Psi\in\mathfrak{O}^*}\Psi(P),$$

$$H_\circ(P)\equiv\sup_{\Phi\in\mathfrak{U}}\Phi(P),\ H_*(P)\equiv\sup_{\Phi\in\mathfrak{U}^*}\Phi(P).$$

THEOREM 7. *The functions $H^\circ(P)$, $H^*(P)$, $H_\circ(P)$ and $H_*(P)$ are $\mathfrak{F}$-functions* *in Ω and it holds*

$$H_*(P) \leqq H_\circ(P) \leqq H^\circ(P) \leqq H^*(P) \qquad \textit{in } \Omega.$$

$H^\circ(H^*)$ and $H_\circ(H_*)$ are called the over-solution of first(second) kind and the under-solution of first(second) kind of the Dirichlet problem for Ω and h, relative to $\mathfrak{F}$, respectively.

REMARK. If h is continuous on ω, then

$$H_*(P) \equiv H_\circ(P),\ H^\circ(P) \equiv H^*(P) \qquad \text{in } \Omega.$$

DEFINITION 7 (Regular boundary point). A boundary point Q of Ω is regular (relative to $\mathfrak{F}$) provided that, for every continuous boundary value function h on ω, the functions $H^*(P)$ and $H_*(P)$ satisfy

$$\lim_{P\to Q} H^*(P) = \lim_{P\to Q} H_*(P) = h(Q).$$

THEOREM 8. *If a boundary point Q of Ω is regular, then*

$$\underline{h}(Q) \leqq \underline{H}_*(Q) \leqq \overline{H}^*(Q) \leqq \overline{h}(Q)$$

for every bounded boundary value function h on ω.

PROOF. Since $\overline{h}(P)$ is upper semi-continuous on ω, we can find a decreasing sequence of continuous functions $h_n(P)$ on ω, such that $h_n \downarrow \overline{h}$. Denoting by H_n^* the over-solution of second kind of the Dirichlet problem for Ω and h_n, we have

$$H^*(P) \leqq H_n^*(P) \qquad \text{in } \Omega.$$

Since Q is regular,

$$\overline{H}^*(Q) \leqq \lim_{P\to Q} H_n^*(P) = h_n(Q).$$

Hence we get by letting $n\to\infty$,

$$\overline{H}^*(Q) \leqq \overline{h}(Q).$$

The inequality $\underline{h}(Q) \leqq \underline{H}_*(Q)$ is shown similary.

THEOREM 9. *If all boundary points of Ω are regular, an $\mathfrak{F}$-function $H(P)$ in Ω, such that*

$$H_*(P) \leqq H(P) \leqq H^*(P) \qquad \textit{in } \Omega,$$

is a solution of the Dirichlet problem for Ω and h. Especially if h is continuous on ω, the Dirichlet problem has the unique solution $H_*(P) \equiv H^*(P)$.

EXAMPLE. Consider the case where $\mathfrak{F}$ consists of harmonic functions. Take for Ω the unit circle: $|z| = |x+iy| < 1$, and let $h(e^{i\theta})$, $0 \leqq \theta < 2\pi$, be given as follows:

$$h(e^{i\theta}) = \begin{cases} 1 & \text{for irrational } \theta, \\ 0 & \text{for rational } \theta. \end{cases}$$

Then

$$H_*(P) \equiv 0,\ H_\circ(P) \equiv H^\circ(P) \equiv H^*(P) \equiv 1.$$

DEFINITION 8 (Barrier). For a boundary point Q of Ω, a circle $\kappa \in \mathfrak{K}$ with center at Q and with $\bar{K} \subset D$, and constants $\varepsilon > 0$, M and N, a function

$$s(P) \equiv s(P;\ \kappa, \varepsilon, M, N)$$

is a sub-barrier provided that

a) $s(P)$ is a sub-$\mathfrak{F}$ function in $\Omega \cap K$,

b) $\underline{s}(Q) \geqq N-\varepsilon$,

c) $\bar{s}(P) \leqq N+\varepsilon$ on $\omega \cap \bar{K}$,

d) $\bar{s}(P) \leqq M$ on $\kappa \cap \Omega$.

A function

$$S(P) \equiv S(P; \kappa, \varepsilon, M, N)$$

is a super-barrier provided that

a) $S(P)$ is a super-$\mathfrak{F}$ function in $\Omega \cap K$,

b) $\bar{S}(Q) \leqq N+\varepsilon$,

c) $\underline{S}(P) \geqq N-\varepsilon$ on $\omega \cap \bar{K}$,

d) $\underline{S}(P) \geqq M$ on $\kappa \cap \Omega$.

THEOREM 10. *For a boundary point Q of Ω and for each set of constants $\varepsilon > 0$, M and N, there exists a sequence of circles $\kappa_n \in \mathfrak{K}$ with center at Q and radii $\rho_n \to 0$ for which super-barriers $S(P; \kappa_n, \varepsilon, M, N)$ and sub-barriers $s(P; \kappa_n, \varepsilon, M, N)$ exist, then Q is a regular boundary point.*

2. Let $\kappa, \gamma \in \mathfrak{K}$ be two circles crossing at two different points, and $\{h_n(P)\}$ a sequence of non-negative continuous functions defined on κ, such that

$$h_n(P) \leqq h_{n+1}(P) \quad \text{on } \kappa,$$
$$h_n(P) = 0 \quad \text{on } \kappa - \Gamma,$$
$$\lim_{n \to \infty} h_n(P) = 1 \quad \text{on } \kappa \cap \Gamma.$$

Then by Postulate 2 the functions $\mathfrak{F}(P; h_n, \kappa)$ tend to a limiting function in K, independent of choice of $\{h_n\}$. Let $\mathfrak{F}(P, \kappa, \gamma)$ denote this limiting function. We shall now set up the following postulates.

POSTULATE 5. If $F_i \in \mathfrak{F}$, $\mathfrak{D}(F) \supseteqq \bar{K}$ $(i=1, 2)$ and

$$F_1(P) \leqq F_2(P) \quad \text{on } \kappa,$$

then
$$F_2(P) \leqq F_1(P) + \mathfrak{G}(P; F_2 - F_1, \kappa) \quad \text{in } \bar{K},$$

where $\mathfrak{G}$ is a family of functions which satisfy Postulates 1~2 and further

a)
$$\mathfrak{G}(P; \lambda h, \kappa) = \lambda \mathfrak{G}(P; h, \kappa)$$

for any $\kappa \in \mathfrak{K}$, any continuous function h on κ and any positive number λ,

b)
$$\mathfrak{G}(P; \kappa, \gamma) < q \quad \text{on } \gamma \cap K,$$

where q is a positive constant less than 1, dependent only upon κ and γ.[1)]

POSTULATE 6. Let Q be a point of D, and P be a point $(\neq Q)$ in the interior K of a circle $\kappa \in \mathfrak{K}$ with center at Q, then there exist a super-$\mathfrak{F}$ function f and a sub-$\mathfrak{F}$ function g in K, such that

$$|f(Q) - N| < \varepsilon, \ f(P) < M$$

and
$$|g(Q) - N| < \varepsilon, \ g(P) > M,$$

for each set of constants $\varepsilon > 0$, M and N.

1) The family of all harmonic functions forms evidently a $\mathfrak{G}$-family.

THEOREM 11 (Poincaré's condition). *If for a boundary point Q of Ω there exists a closed triangle Δ with vertex at* Q, *such* that

$$\Delta\cap\bar{\Omega}=Q,$$

then Q is a regular boundary point.

PROOF. We shall give the construction of a sub-barrier for a suitable small circle κ with center at Q. The existence of super-barriers can be treated similarly.

Let $\kappa_0\in\mathfrak{K}$ be a circle with center at Q. Then there exists a super-$\mathfrak{F}$ function $f(P)$ in K, such that

$$|f(Q)-N|<\frac{\varepsilon}{2}.$$

By continuity there is a circle $\kappa\subset K_0$, concentric with κ_0, such that

$$f(P)<N+\varepsilon \qquad \text{in } K.$$

Now define
$$M_*=\min[M, N].$$
Then by Postulate 3 there is a sub-$\mathfrak{F}$ function $g(P)$ such that

$$g(P)<M_* \qquad \text{in } K.$$

Let QT_1, QT_2 be two sides of Δ and QT_0 the line-segment bisecting the outer angle of Q. Take three points Q_0, Q_1 and Q_2 on QT_0, QT_1 and QT_2, respectively, so that

$$\rho=\overline{QQ_0}=\overline{QQ_1}=\overline{QQ_2}.$$

Let κ' be the circle through Q, Q_0, Q_1, and κ'' the circle through Q, Q_0, Q_2. If we choose ρ sufficiently small, κ' and κ'' belong to $\mathfrak{K}$ and lie together with their interiors in K. Let σ_i be the arc $\widehat{Q_0Q_i}(i=1,2)$ not containing Q and set $\sigma=\sigma_1\cup\sigma_2$.

We can then construct, by Postulate 6 and Theorem 3, a super-$\mathfrak{F}$ function F in K, such that

$$|F(Q)-N|<\varepsilon,\ F(P)<g(P) \qquad \text{on } \sigma.$$

Define the function $F^*(P)$ by $F^*(P)=\min[f(P), F(P)]$, then $F^*(P)$ is a super-$\mathfrak{F}$ function in K and satisfies

$$\begin{aligned}&|F^*(Q)-N|<\varepsilon,\\ &F^*(P)<g(P) \qquad \text{on } \sigma,\\ &F^*(P)<N+\varepsilon \qquad \text{in } K.\end{aligned}$$

We now define $H_n'(P)$ and $H_n''(P)$ on $\bar{K}'$ and $\bar{K}''$, respectively, and $F_n(P)$ in $\bar{K}'\cup\bar{K}''$ as follows:

$$H_1'(P)=\mathfrak{F}(P;F^*,\kappa') \qquad \text{in } \bar{K}',$$

$$H_1''(P)=\begin{cases}F^*(P) & \text{on } \kappa''-K',\\ H_1'(P) & \text{on } \kappa''\cap K',\\ \mathfrak{F}(P;H_1'',\kappa'') & \text{in } K'',\end{cases}$$

....

$$H_n'(P)=\begin{cases}F^*(P) & \text{on } \kappa'-K'',\\ H_{n-1}''(P) & \text{on } \kappa'\cap K'',\\ \mathfrak{F}(P;H_n',\kappa') & \text{in } K',\end{cases}$$

$$H_n''(P)=\begin{cases}F^*(P) & \text{on } \kappa''-K', \\ H_n'(P) & \text{on } \kappa''\cap K', \\ \mathfrak{F}(P; H_n'', \kappa'') & \text{in } K'',\end{cases}$$

for $n=2, 3, \cdots$, and

$$F_{2k-1}(P)=\begin{cases}H_k'(P) & \text{in } \bar{K}', \\ F_{2k-2}(P) & \text{in } \bar{K}''-\bar{K}',\end{cases}$$

$$F_{2k}(P)=\begin{cases}H_k''(P) & \text{in } \bar{K}'', \\ F_{2k-1}(P) & \text{in } \bar{K}'-\bar{K}'',\end{cases}$$

for $k=1, 2, \cdots$, where $F_0(P)\equiv F^*(P)$.

Since $F^*(P)$ is a super-$\mathfrak{F}$ function, $\{F_n(P)\}$ forms a decreasing sequence of super-$\mathfrak{F}$ functions. Let

$$L=\min_{P\in\bar{K}'\cup\bar{K}''} F^*(P).$$

By Postulate 3, there is a sub-$\mathfrak{F}$ function $g'(P)$ such that

$$g'(P)<L \qquad \text{in } \bar{K}'\cup\bar{K}''.$$

Then
$$g'(P)<F_n(P)\leqq F^*(P) \qquad \text{in } \bar{K}'\cup\bar{K}'',$$

for $n=1, 2, \cdots$, and

$$F_{2k-1}(P)=\mathfrak{F}(P; F_{2k-1}, \kappa') \qquad \text{in } K',$$
$$F_{2k}(P)=\mathfrak{F}(P; F_{2k}, \kappa'') \qquad \text{in } K'',$$

for $k=1, 2, \cdots$. Hence, by Theorem 2 and Postulate 4, the limiting function $U(P)$ of $F_n(P)$ exists and is an $\mathfrak{F}$-function in $K'\cup K''$. In view of Postulates 1 and 2, we can show easily that $U(P)$ is continuous in $\bar{K}'\cup\bar{K}''$ except possibly at Q and Q_0.

We next consider the sequence of non-negative functions

$$W_n(P)=F_n(P)-F_{n+2}(P).$$

By our construction

$$W_1(P)=0 \qquad \text{on } \kappa'-K'',$$
$$0\leqq W_1(P)\leqq 2C \qquad \text{on } \kappa'\cap K'',$$

where
$$C=\max_{P\in\bar{K}'\cup\bar{K}''}[|F^*(P)|, |g'(P)|]>0.$$

By Postulate 5,

$$0\leqq W_1(P)\leqq 2C\mathfrak{G}(P; \kappa', \kappa'')$$
$$<2Cq' \qquad \text{on } \kappa''\cap K',$$

so that
$$0\leqq W_1(P)<2Cq' \qquad \text{on } \kappa''\cap\bar{K}',$$

where q' is a suitable positive constant less than 1. Also

$$W_2(P)=0 \qquad \text{on } \kappa''-K',$$
$$0\leqq W_2(P)<2Cq' \qquad \text{on } \kappa''\cap K',$$

so that
$$0\leqq W_2(P)<2Cq'q'' \qquad \text{on } \kappa'\cap\bar{K}'',$$

where q'' is a suitable positive constant less than 1. Consequently

$$0\leqq W_1(P)<2Cq \qquad \text{on } \kappa''\cap\bar{K}',$$

$$0 \leqq W_2(P) < 2Cq^2 \qquad \text{on } \kappa' \cap \bar{K}'',$$

where $$q = \min[q', q''].$$

Repeating this argument we obtain

$$0 \leqq W_{2k-1}(P) < 2Cq^{2k-1} \qquad \text{on } \kappa'' \cap \bar{K}',$$
$$0 \leqq W_{2k}(P) < 2Cq^{2k} \qquad \text{on } \kappa' \cap \bar{K}'',$$

for $k=1, 2, \cdots$. From this it follows that the functions $F_n(P)$ converge uniformly on $\kappa' \cap \bar{K}''$ and $\kappa'' \cap \bar{K}'$. Hence on account of Postulate 2, we see that $U(P)$ is continuous at Q and Q_0.

Now we define the function $s(P)$ by

$$s(P) = \begin{cases} g(P) & \text{on } \Omega \cap (K - \bar{K}' \cup \bar{K}''), \\ \max[g(P), U(P)] & \text{on } \Omega \cap (\bar{K}' \cup \bar{K}''). \end{cases}$$

We shall show that $s(P)$ is a sub-barrier $s(P; \kappa, \varepsilon, M, N)$. Since $U < g$ on σ, s is continuous and sub-$\mathfrak{F}$ in $\Omega \cap K$, and

$$U(Q) = F^*(Q), \ |F^*(Q) - N| < \varepsilon,$$
$$g(P) < \min[M, N], \ U(P) \leqq F^*(P) < N + \varepsilon \qquad \text{in } K,$$

so that

$$\underline{s}(Q) = \lim_{P \to Q} s(P) > N - \varepsilon,$$
$$\bar{s}(P) \leqq N + \varepsilon \qquad \text{on } \omega \cap \bar{K},$$
$$\bar{s}(P) \leqq M \qquad \text{on } \kappa \cap \Omega.$$

Thus $s(P)$ satisfies all the conditions of Definition 8 and is a sub-barrier desired.[1)]

Postulate 6^*. Let Ω be a bounded region such that $\bar{\Omega} \subset D$, and let P, Q be two different points in Ω. Then, for any set of constants $\varepsilon > 0$, M and N, there exist a super-$\mathfrak{F}$ function f and a sub-$\mathfrak{F}$ function g in Ω, such that

$$|f(Q) - N| < \varepsilon, \ f(P) < M,$$
$$|g(Q) - N| < \varepsilon, \ g(P) > M.$$

Postulate 6 follows immediately from Postulate 6^*.

Theorem 12. *In order that a boundary point Q of Ω be regular, it is necessary and sufficient that, for any set of constants $\varepsilon > 0$, M and N, there exists a sequence of circles $\kappa_n \in \mathfrak{K}$ with center at Q and of radii $\rho_n \to 0$, for that super-barriers $S(P; \kappa_n, \varepsilon, M, N)$ and sub-barriers $s(P; \kappa_n, \varepsilon, M, N)$ exist.*

Proof. It suffices to prove that the condition is necessary. For this we shall construct a sub-barrier at Q. Let $f(P)$ be a super-$\mathfrak{F}$ function in a circle $\kappa' \in \mathfrak{K}$ with center at Q, such that

$$|f(Q) - N| < \frac{\varepsilon}{2}.$$

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1) The case where we can form a disk outside Ω in place of triangle in Theorem 11 was proved in the paper of Beckenbach-Jackson under different Postulates from ours. They do not set such a Postulate as P. 5, which is certainly undesirable. But the author can not support their proof. Indeed, the two limiting function u' and u'' in their proof do not coincide necessarily on $\kappa' \cap K''$, on $\kappa'' \cap K'$.

Then, in a suitable small circle $K \subset K'$, concentric with K', we have

$$|f(P)-N|<\varepsilon.$$

By Postulate 6^* and Theorem 3, we can find a super-$\mathfrak{F}$ function $f'(P)$ in a region $\Omega_0 \supset \bar{\Omega}$, such that

$$|f'(Q)-N|<\varepsilon,\ f'(P)<M_* \qquad \text{on } \kappa \cap \bar{\Omega},$$

where

$$M_* = \min[M, \min_{P \in \bar{K}} f(P)].$$

Let

$$h(P)=\begin{cases}\min[f(P), f'(P)] & \text{on } \omega \cap \bar{K}, \\ f'(P) & \text{on } \omega - \bar{K},\end{cases}$$

and let $H_0(P)$ be the under-solution (of first kind) of the Dirichlet problem for Ω and h. Since Q is regular and h is continuous at Q, we have

$$\lim_{P \to Q} H_0(P) = h(Q) > N - \varepsilon.$$

Let $H_0'(P)$ be the under-solution (of first kind) of the Dirichlet problem for $\Omega \cap K$[1] and h', where h' is defined by

$$h'(P)=\begin{cases}M_* & \text{on } \kappa \cap \Omega, \\ h(P) & \text{on } \omega \cap \bar{K}.\end{cases}$$

Then we have

$$H_0(P) \leqq H_0'(P) \qquad \text{in } \Omega \cap K,$$

for any under-function of the first problem is also an under-function of the second problem in $\Omega \cap K$. Hence

$$N-\varepsilon < \lim_{P \to Q} H_0(P) \leqq \underline{H}_0'(Q).$$

By our construction

$$H_0'(P) \leqq f(P) \qquad \text{in } K \cap \Omega,$$

so that

$$\bar{H}_0'(P) \leqq N+\varepsilon \qquad \text{on } \omega \cap \bar{\Omega},$$

and since every point of $\kappa \cap \Omega$ is regular,

$$H_0'(P) = M_* \leqq M \qquad \text{on } \kappa \cap \Omega.$$

Thus $H_0'(P)$ satisfies all the conditions of Definition 8 and is a sub-barrier desired. The existence of super-barriers can be shown similarly.

3. As an application we consider the partial differential equation of elliptic type:

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y, u),$$

where $f(x, y, u)$ is a continuous function defined for $(x, y) \in D$ and $-\infty < u < \infty$. Suppose for simplicity that $f(x, y, u)$ is continuous differentiable with respect to each variable and that f_u' is non-negative and bounded.

It is well known that, for any small circle $\kappa \in \mathfrak{K}$ contained with its interior K

1) When $\Omega \cap K$ is not connected, we mean by H_0' the collection of under-solutions of Dirichlet problem for each component.

in D and for any continuous function h defined on κ, there exists a unique regular function u which satisfies

$$\Delta u=f(x,y,u) \qquad \text{in } K,$$
$$u=h \qquad \text{on } \kappa.$$[1]

We take for $\mathfrak{K}$ a family of such circles, that is, circles for which the boundary value problems relative to $\Delta u=f(x,y,u)$ and relative to continuous boundary values are always solvable, and for $\mathfrak{F}$ the family of solutions of the above boundary value problems. Thus Postulate 1 is satisfied.

Let L be a constant such that

$$z(x,y)\equiv L-x^2>0 \qquad \text{for } (x,y)\in \bar{K}.$$

By putting $u=zv$, the equation $\Delta u=f(x,y,u)$ is reduced to

$$\Delta v=\frac{1}{z}\left\{f(x,y,zv)+2v+4xv_x\right\}$$
$$\equiv\varphi(x,y,v)+\frac{4xv_x}{z}.$$

To verify Postulate 2 we suppose that $F_1\leqq F_2+M$ on κ and $F_1>F_2+M$ at a certain point of K. Then setting $F_i/z=V_i$ for $i=1,2$, we have

$$V_1-V_2-\frac{M}{z}\leqq 0 \qquad \text{on } \kappa,$$
$$V_1-V_2-\frac{M}{z}>0 \qquad \text{at a certain point of } K.$$

Hence there is a point $(x_0,y_0)\in K$, where $V_1-V_2-\dfrac{M}{z}$ attains its maximum and it holds at (x_0,y_0),

$$\frac{\partial}{\partial x}\left(V_1-V_2-\frac{M}{z}\right)=0, \qquad \Delta\left(V_1-V_2-\frac{M}{z}\right)\leqq 0;$$

that is,

$$\frac{\partial}{\partial x}\left(V_1-V_2\right)=\frac{2Mx}{z^2}, \qquad \Delta(V_1-V_2)\leqq 2M\left(\frac{1}{z^2}+\frac{4x^2}{z^3}\right) \text{ at } (x_0,y_0).$$

On the other hand, since F_1 and F_2 are solutions of $\Delta u=f(x,y,u)$, we have at (x_0,y_0),

$$\Delta(V_1-V_2)=\varphi(x,y,V_1)-\varphi(x,y,V_2)+\frac{4x}{z}\frac{\partial}{\partial x}(V_1-V_2)$$
$$=(V_1-V_2)[\varphi_v'(x,y,v)]_{v=\xi}+\frac{8Mx^2}{z^3}, \quad V_2\leqq\xi\leqq V_1.$$

Since $f_u'(x,y,u)\geqq 0$,

$$\varphi_v'(x,y,v)=f_u'(x,y,u)+\frac{2}{z}\geqq\frac{2}{z}>0$$

and
$$V_1-V_2>\frac{M}{z} \qquad \text{at } (x_0,y_0).$$

Therefore we obtain at (x_0,y_0),

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1) Picard, Leçons sur quelques problèmes aux limites de la théorie des équations différentielles (1930).

$$\Delta(V_1-V_2)>2M\left(\frac{1}{z^2}+\frac{4x^2}{z^3}\right).$$

This is a contradiction. Hence $F_1 \leqq F_2+M$ in K. The last part of Postulate 2 is proved similarly. Thus Postulate 2 is satisfied.

Before we verify Postulate 3 we remark that, if u is twice continuous differentiable with respect to x and y, and satisfies

$$\Delta u \geqq f(x,y,u)$$

or

$$\Delta u \leqq f(x,y,u),$$

then u is a sub-$\mathfrak{F}$ function or a super-$\mathfrak{F}$ function.

Let Ω be a bounded region such that $\bar{\Omega}\subset D$, and M be a given constant. Let

$$L=\max_{(x,y)\in\bar{\Omega}}|f(x,y,0)|,$$

then the function $u(x,y)$ defined by

$$u(x,y)=-\frac{L}{4}(x^2+y^2)+M^*,$$

where M^* is a constant, satisfies

$$\Delta u=-L\leqq 0$$

and

$$u(x,y)>\max[M,0]\geqq 0 \qquad \text{in } \Omega,$$

if M^* is large enough. Since $f_u' \geqq 0$, we have

$$\Delta u \leqq f(x,y,0)\leqq f(x,y,u).$$

Hence u is a bounded super-$\mathfrak{F}$ function and $u>M$ in Ω. The existence of a bounded sub-$\mathfrak{F}$ function u such that $u<M$ in Ω, is shown similarly. Thus Postulate 3 is satisfied.[1)]

We now go to verify Postulate 4. Let $\mathfrak{H}(x,y\,;\,h,\kappa)$ be the function, harmonic in K and continuous on $\bar{K}$, which coincide with h on κ. Then

$$\mathfrak{F}(x,y\,;\,h_\nu,\,\kappa)=\mathfrak{H}(x,y;\,h_\nu,\,\kappa)+w_\nu(x,y),$$

$$w_\nu(x,y)=-\frac{1}{2\pi}\iint_K G(x,y;\,\xi,\eta)f(\xi,\eta,\,w_\nu+\mathfrak{H}_\nu)\,d\xi d\eta,$$

where $\mathfrak{H}_\nu \equiv \mathfrak{H}(x,y;\,h_\nu,\kappa)$ and $G(x,y;\,\xi,\eta)$ is the Green's function of K with pole at $(\xi,\,\eta)$.

Assume that h_ν are uniformly bounded, then $\mathfrak{H}_\nu$ are equicontinuous in K as is well known, and by Postulate 3 and Theorem 5, $\mathfrak{F}(x,y;\,h_\nu,\,\kappa)$ are uniformly bounded, say $|\mathfrak{F}(x,y;\,h_\nu,\kappa)|<L$. Then

$$\left|\frac{\partial}{\partial x}w_\nu(x,y)\right| \leqq \frac{1}{2\pi}\iint_K\left|G_x(x,y;\,\xi,\eta)f(\xi,\eta;\,w_\nu+\mathfrak{H}_\nu)\right|d\xi d\eta$$

$$\leqq \frac{M}{2\pi}\iint_K\left|G_x(x,y\,;\,\xi,\eta)\right|d\xi d\eta \equiv W_1(x,y),$$

where

$$M=\max_{(\xi,\eta)\in K,\,|u|\leqq L}|f(\xi,\eta,u)|$$

1) We note that the function obtained above has the continuous derivatives of all orders.

Similarly

$$\left|\frac{\partial}{\partial y}w_\nu(x,y)\right| \leqq \frac{M}{2\pi}\int\int_K \left|G_y(x,y\,;\,\xi,\eta)\right| d\xi d\eta \equiv W_2(x,y).$$

Since W_1 and W_2 are bounded in K, $\frac{\partial}{\partial x}w_\nu$ and $\frac{\partial}{\partial x}w_\nu$ are so; hence w_ν are equicontinuous in K. Thus Postulate 4 is satisfied.

Postulate 5 is verified as follows. Since $f_u' \geqq 0$ and $F_1 \leqq F_2$ in K (Postulate 2),

$$\Delta(F_2-F_1) = f(x,y,F_2) - f(x,y,F_1) \geqq 0 \qquad \text{in } K.$$

This shows that $F_2 - F_1$ is subharmonic. Hence

$$F_2(P) - F_1(P) \leqq \mathfrak{H}(P\,;F_2-F_1,\ \kappa) \qquad \text{in } \bar{K}.$$

Finally we shall verify Postulate 6*. Assume without loss of generality that $Q=(0,0)$, $P=(a,0)$, $a \neq 0$, and that

$$\max_{(x,y)\in\Omega}[|x|^2,\ |x-a|^2] < R < \infty.$$

Since f_u' is bounded, there exists a positive constant L such that, for any v,

$$|f(x,y,u) - f(x,y,u+v)| < L|v|, \qquad (x,y) \in \Omega.$$

Let u be a regular function such that

$$u(0,0) < N,$$
$$\Delta u \geqq f(x,y,u) \qquad \text{in } \Omega.$$

Indeed such a function exists as proved above.[1] Set

$$g(x,y) = u(x,y) + v(x,y),\ v(x,y) = \lambda(x-a)^{2n} + \mu x^{2n},$$

where λ, μ are positive numbers and n is a positive integer such that

$$LR < 2n(2n-1).$$

Then for any λ and μ,

$$\begin{aligned}\Delta g = \Delta u + \Delta v &\geqq f(x,y,u) + L|v| \\ &> f(x,y\,g).\end{aligned}$$

Thus g is a sub-$\mathfrak{F}$ function desired in Postulate 6* provided that λ and μ are so chosen that

$$g(0,0) = u(0,0) + \lambda a^{2n} = N,$$
$$g(a,0) = u(a,0) + \mu a^{2n} > M.$$

The existence of super-$\mathfrak{F}$ functions is shown similarly.

EXAMPLES. $\Delta u = c(x,y)u + \varphi(x,y),\ c(x,y) \geqq 0\,;$

$\Delta u = c(x,y)(u + \sin u) + \varphi(x,y),\ c(x,y) \geqq 0.$

4. Next we consider the elliptic equation

$$\Delta u = f(x,y,u)$$

under different conditions: f and all its first partial derivatives are continuous for $(x,y) \in D$ and $-\infty < u < \infty$, and further we suppose that $f \geqq 0$, $f_u' \geqq 0$ and f_u' is increasing with respect to u.

Then for any small circle $\kappa \in \mathfrak{K}$ contained with its interior K in D and for

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1) See the foot-note of page 11,

any continuous function h defined on κ, there exists a unique regular function u which satisfies

$$\begin{aligned} \Delta u &= f(x, y, u) && \text{in } K, \\ u &= h && \text{on } \kappa. \end{aligned}$$

We take for $\mathfrak{R}$ a family of such circles, that is, circles for which the boundary value problem relative to $\Delta u = f(x, y, u)$ and relative to continuous boundary values are always solvable, and for $\mathfrak{F}$ the family of solvations of the above boundary value problems. Then Postulates 1∼5 are satisfied as in the previous paragraph. Hence Theorems 1∼10 hold for this case. Postulate 6 will not be verified but we can show that Theorem 11 remains true.

LEMMA 1. *Let Ω be a bounded region such that $\bar{\Omega} \subset D$, and P, Q be two different points in Ω. Then, for any set of constants M and N, there exists a super-$\mathfrak{F}$ functions h in Ω, such that*

$$h(Q) = N, \qquad h(P) < M.$$

PROOF. Assume without loss of generality that $Q = (0, 0)$, $P = (a, 0)$, $a \neq 0$, and that

$$\max_{(x, y) \in \bar{\Omega}} [|x|^2, \ |x-a|^2] < R < \infty.$$

Let w be a regular function, continuous in $\bar{\Omega}$, such that

$$\begin{aligned} &\Delta w \leqq f(x, y, w) \qquad \text{in } \Omega, \\ &w(0, 0) > N. \end{aligned}$$

Such a function exists by Postulate 3.[1] Set

$$h(x, y) = w(x, y) - v(x, y), \ v(x, y) = \lambda (x-a)^{2n} + \mu x^{2n},$$

where λ, μ are positive numbers and n is a positive integer such that

$$L = \max_{(x, y) \in \bar{\Omega}} |w(x, y)|,$$

$$R \max_{(x, y) \in \bar{\Omega}} [f'_u(x, y, u)]_{u=L} < 2n(2n-1).$$

Then for any λ and μ,

$$\begin{aligned} \Delta h = \Delta w - \Delta v &\leqq f(x, y, w) - v[f'_u(x, y, u)]_{u=L} \\ &\leqq f(x, y, w-v) = f(x, y, h), \end{aligned}$$

since $v \geqq 0$ and f'_u is increasing with respect to u.

Thus h is a super-$\mathfrak{F}$ function desired in Lemma provided that λ and μ are so chosen that

$$\begin{aligned} h(0, 0) &= w(0, 0) - \lambda a^{2n} = N, \\ h(a, 0) &= w(a, 0) - \mu a^{2n} < M. \end{aligned}$$

LEMMA 2. *Let Q be a point of D. Then for any set of constants N and M $(>N)$, there exist a circle $\kappa \in \mathfrak{R}$ with center at Q and a function g, such that*

a) *$g(P)$ is continuous in $\bar{K}$ and sub-$\mathfrak{F}$ in K,*

1) See the footnote of page 11.

b) $g(Q)=N, g(P)\geqq N$ *in* $\bar{K}$ *and* $g(P)=M$ *on* κ.

Proof. Let us assume that Q is the origin of coordinates. Let κ_0 be a fixed circle with center at Q and of radii ρ, such that $\bar{K}_0\subset D$. Then there are a positive number $a(<\rho)$ and a function $u(x)$, such that

a) $u(x)$ is a solution of the ordinary differential equation

$$\frac{d^2u}{dx^2}=F(u)\equiv \max_{(x,y)\in \bar{K}} f(x,y,u) \qquad \text{in } |x|\leqq a,$$

b) $u(x)$ is increasing in $|x|\leqq a$ and $u(0)=N,\ u(a)=M$. Indeed such a function is given by the form

$$x=\int_N^u \frac{dt}{\sqrt{\text{Const.} +2\int_N^t F(\xi)d\xi}} \quad .$$

Putting

$$g_\theta(x,y)=u(x\cos\theta+y\sin\theta),\ x^2+y^2\leqq a^2,$$

we have $\quad \Delta g_\theta(x,y)=F(g_\theta)\geqq f(x,y,g_\theta).$

Then the function g defined by

$$g(x,y)=\sup_{0\leqq\theta<2\pi} g_\theta(x,y)=u(\sqrt{x^2+y^2})$$

is a sub-$\mathfrak{F}$ function desired in Lemma.

Proof of theorem 11. By Lemma 1 the existence of sub-barriers at Q is clear.[1] We shall give the construction of a super-barrier for a suitable small circle κ with center at Q and for any set of constants $\varepsilon>0$, M and N.

By Lemma 2 there exist a circle $\kappa\in\mathfrak{K}$ with center at Q, such that $\bar{K}\subset D$, and a function g such that

a) $g(P)$ is continuous in $\bar{K}$ and sub-$\mathfrak{F}$ in K,

b) $g(Q)=N,\ g(P)\geqq N$ in K and $g(P)=\max[M,N]+1$ on κ.

Choose a infinite sequence of circles $\kappa_i\in\mathfrak{K}$ such that

$$\bar{K}_i\subset K-\Delta,\ \bigcup_{i=1}^{\infty}\bar{K}_i=K-\Delta,$$

and define the functions $S_i(P)$ by

$$S_i(P)=\begin{cases}\mathfrak{F}(P;S_{i-1},\kappa_{j(i)}) & \text{for } P\in K_{j(i)},\\ S_{i-1}(P) & \text{for } P\in \overline{K-\Delta-K_{j(i)}},\end{cases}$$

where $S_0\equiv g$ and $\{j(i)\}=1,2,1,2,3,1,2,3,4,1,2,\cdots\cdots\cdots$ for $i=1,2,3,\cdots\cdots\cdots$. Then $S_i(P)$ are sub-$\mathfrak{F}$ in $K-\Delta$ and

$$S_{i-1}(P)\leqq S_i(P) \qquad \text{in } \overline{K-\Delta}.$$

We note that

$$\Delta S_i=f(x,y,S_i)\geqq 0 \qquad \text{in } K_{j(i)},\ P=(x,y),$$

that is, $S_i(P)$ are subharmonic in $K_{j(i)}$.

Now we define

1) See the proof of Theorem 11 in Paragraph 2.

$$H_i(P)=\begin{cases}\mathfrak{H}(P\,;H_{i-1},\ \kappa_{j(i)}) & \text{for } P\in K_{j(i)},\\ H_{i-1}(P) & \text{for } P\in \overline{K-\Delta-K_{j(i)}},\end{cases}$$

for $i=1, 2, 3, \ldots\ldots$, where $H_0\equiv g$. Then in Potential theory* we know that

$$\lim_{i\to\infty} H_i(P)=H(P)$$

is harmonic in $K-\Delta$ and, since Q is regular relative to the harmonic equation,

$$\lim_{P\to Q} H(P)=g(Q)=N.$$

Comparing S_i and H_i, we obtain

$$g(P)\leqq S_i(P)\leqq H_i(P) \qquad \text{in } K-\Delta.$$

From this it follows that

$$\lim_{i-\infty} S_i(P)=S(P)$$

exists and is an $\mathfrak{F}$-function in $K-\Delta$. Also

$$g(P)\leqq S(P)\leqq H(P) \qquad \text{in } K-\Delta,$$

therefore

$$\overline{S}(Q)=N,$$

$$S(P)\geqq N \qquad \text{in } K-\Delta,$$

$$\underline{S}(P)\geqq \max[M,N]+1>M \qquad \text{on } \kappa\cap(K-\Delta).$$

This shows that $S(P)$ is a super-barrier $S(P\,;\kappa,\ \varepsilon,\ M,N)$ in $K\cap\Omega$, since Δ is outside Ω. Thus the proof of Theorem 11 is completed.

By replacing $K-\Delta$ by $K\cap\Omega$ in the above proof we can show the existence of super-barriers $S(P\,;\kappa,\ \varepsilon,\ M,N)$ provided that Q is regular relative to the Laplace equation $\Delta u=0$. Hence we obtain

THEOREM 13. *If a boundary point Q of Ω is regular relative to the Laplace equation, then we have*

$$\overline{H}^*(Q)\leqq \overline{h}(Q)$$

for every bounded boundary value function h.

Finally we add the following theorem.

THEOREM 14. *If all boundary points of Ω are regular relative to the Laplace equation, then we have*

$$H^*(P)\equiv H_*(P)$$

for every continuous boundary value function.

PROOF. By Theorem 13 we have

$$\overline{H}^*(Q)\leqq h(Q) \qquad \text{on } \omega,$$

for every continuous boundary value function h. Since $H^*(P)$ is an $\mathfrak{F}$-function in Ω, it is an under-function, so that

$$H^*(P)\leqq H_*(P),$$

therefore

$$H^*(P)\equiv H_*(P).$$

EXAMPLE. $\Delta u=c(x,y)e^u+\varphi(x,y)$, $c(x,y)\geqq 0$, $\varphi(x,y)\geqq 0$.

REMARK. We return to the general case as treated in Paragraphs 1~2. Then Theorems 13~14 remain valid if we add the following postulate.

POSTULATE 7. If $F \in \mathfrak{F}$ and $\mathfrak{D}(F)=\bar{K}$, then F is subharmonic in K.

Hence Theorems 1∼14 hold for the elliptic equation

$$\Delta u=f(x, y, u),$$

where $f \geqq 0$, $f_u' \geqq 0$ and f_u' is bounded.

EXAMPLE. $\Delta u=\dfrac{c(x, y)}{1+e^{-u}}+\varphi(x, y)$, $c(x, y) \geqq 0$, $\varphi(x, y) \geqq 0$.