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# ON THE COEFFICIENTS OF CERTAIN FAMILY OF MODULAR EQUATIONS

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**Abstract**

The $n$-th modular equation for the elliptic modular function $j(z)$ has large coefficients even for small $n$, and those coefficients grow rapidly as $n \to \infty$. The growth of these coefficients was first obtained by Cohen ([5]). And, recently Cais and Conrad ([1], §7) considered this problem for the Hauptmodul $j_5(z)$ of the principal congruence group $\Gamma(5)$. They found that the ratio of logarithmic heights of $n$-th modular equations for $j(z)$ and $j_5(z)$ converges to 60 as $n \to \infty$, and observed that 60 is the group index $[\overline{\Gamma(1)} : \overline{\Gamma(5)}]$. In this paper we prove that their observation is true for Hauptmoduln of somewhat general Fuchsian groups of the first kind with genus zero.

## 1. Introduction

Let $\mathfrak{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the complex upper half plane and $j(z) = q^{-1} + 744 + 196884q + \cdots$ be the elliptic modular function on $SL_2(\mathbb{Z})$ with $z \in \mathfrak{H}$ and $q = e^{2\pi i z}$. Further, let $\Phi_n^j(X, Y) = 0$ be the $n$-th modular equation for $j(z)$ (see [6, 10, 11]). Then $\Phi_n^j(X, Y)$ is a polynomial with integral coefficients satisfying $\Phi_n^j(j(z), j(nz)) = 0$, and is irreducible as a polynomial in $X$ over $\mathbb{C}(Y)$. Moreover it is known that $\Phi_p^j(X, Y)$ satisfies the Kronecker congruences, and $\Phi_n^j(X, Y)$ has large coefficients even for small $n$. For example,

$$\begin{aligned}
\Phi_3^j(X, Y) = {} & X(X + 2^{15} \cdot 3 \cdot 5^3)^3 + Y(Y + 2^{15} \cdot 3 \cdot 5^3)^3 - X^3Y^3 \\
& + 2^3 \cdot 3^2 \cdot 31X^2Y^2(X + Y) - 2^2 \cdot 3^3 \cdot 9907XY(X^2 + Y^2) \\
& + 2 \cdot 3^4 \cdot 13 \cdot 193 \cdot 6367X^2Y^2 + 2^{16} \cdot 3^5 \cdot 5^3 \cdot 17 \cdot 263XY(X + Y) \\
& - 2^{31} \cdot 5^6 \cdot 22973XY.
\end{aligned}$$

Note that the coefficients of $\Phi_n^j(X, Y)$ grow quite rapidly as $n \to \infty$, which was first estimated by Cohen ([5]) as follows.

For a nonzero polynomial $P(X_1, \dots, X_r) \in \mathbb{C}[X_1, \dots, X_r]$, let $h(P(X_1, \dots, X_r))$ be the *logarithmic height* of $P(X_1, \dots, X_r)$ defined by the logarithm of the maximum of

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the absolute values of its coefficients. And, throughout this article we use $\mathcal{O}$-notation which has the following meaning; let $f$ and $g$ be complex valued functions defined on some set $S$ and $h$ be a real valued positive function defined on $S$. Then $f = g + \mathcal{O}(h)$ means that there exists an absolute positive constant $A$ such that $|f - g| \leq A \cdot h$ on $S$. With the aid of height and $\mathcal{O}$-notation Cohen showed that how rapidly $h(\Phi_n^j(X, Y))$ grows as $n \to \infty$, that is, for any positive integer $n$ we have

$$h(\Phi_n^j(X, Y)) = 6\psi(n)\left\{\log n - 2\sum_{p|n}\frac{\log p}{p} + \mathcal{O}(1)\right\}, \tag{1.1}$$

where $\psi(n) = n\prod_{p|n}(1 + 1/p)$.

On the other hand Cais and Conrad recently considered the modular equations of the Hauptmodul $j_5(z) = q^{-1/5}(1 + q - q^3 + q^5 + \cdots)$ of $\Gamma(5)$. For a positive integer $n$ with $(n, 5) = 1$ we let $\Phi_n^{j_5}(X, Y) = 0$ be the $n$-th modular equation for $j_5(z)$ defined as in [1, Definition 6.4]. Then $\Phi_n^{j_5}(X, Y)$ is a polynomial with integral coefficients satisfying $\Phi_n^{j_5}(j_5(z), j_5(nz)) = 0$, and is irreducible as a polynomial in $X$ over $\mathbb{C}(Y)$. In addition, $\Phi_p^{j_5}(X, Y)$ also satisfies the Kronecker congruences ([1, Theorem 6.8]). But unlike the case of $\Phi_n^j(X, Y)$, $\Phi_n^{j_5}(X, Y)$ has much smaller coefficients, for example,

$$\Phi_3^{j_5}(X, Y) = X^4Y^3 + X^3 - 3X^2Y^2 - XY^4 - Y.$$

They indeed estimated the logarithmic height of $\Phi_n^{j_5}(X, Y)$, precisely, for any positive integer $n$ with $(n, 5) = 1$

$$h(\Phi_n^{j_5}(X, Y)) = \frac{1}{10}\psi(n)\left\{\log n - 2\sum_{p|n}\frac{\log p}{p} + \mathcal{O}(1)\right\},$$

from which they derived by comparing with $h(\Phi_n^j(X, Y))$ that

$$\lim_{\substack{n\to\infty \\ (n,5)=1}} \frac{h(\Phi_n^j(X, Y))}{h(\Phi_n^{j_5}(X, Y))} = 60 = [\overline{\Gamma(1)} : \overline{\Gamma(5)}]$$

where $\overline{\Gamma(1)}$ and $\overline{\Gamma(5)}$ denote the images of $\Gamma(1)$ and $\Gamma(5)$ in $PSL_2(\mathbb{R})$. But Cais and Conrad did not explain why the ratio of logarithmic heights converges to the group index.

So it is natural and worthwhile to ask whether

$$\frac{h(\Phi_n^j(X, Y))}{h(\Phi_n^f(X, Y))} \to [\overline{\Gamma(1)} : \overline{\Gamma}]$$

as $n \to \infty$ with some conditions on $n$ for a Hauptmodul $f(z)$ of arbitrary congruence subgroup $\Gamma$. In Theorem 2.1 (1) we shall prove that the answer is affirmative for classical congruence subgroups. We further consider a similar question about subgroups $\langle \Gamma_0(N), W_e\rangle_{e\in S}$ of $SL_2(\mathbb{R})$ which appear in "Monstrous Moonshine" phenomenon. And we will prove in Theorem 2.1 (2) that the ratio of logarithmic heights in this case is also related to a certain summand of group indices.

In what follows we fix an integer $N$, and define necessary congruence subgroups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\},$$
$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \bmod N \right\},$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\},$$
$$\Gamma^1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \bmod N \right\},$$
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

## 2. Preliminaries and statements of the results

In this section we recall the definition of modular equations for Hauptmoduln of various subgroups of $SL_2(\mathbb{R})$.

For a Fuchsian group $\Gamma$ of the first kind with genus zero, we define a Hauptmodul of $\Gamma$ by an automorphic function $f(z)$ for $\Gamma$ satisfying $A_0(\Gamma) = \mathbb{C}(f(z))$. Here by $A_0(\Gamma)$ we mean the field of all automorphic functions for $\Gamma$ (see [11]). In this paper we fix that $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$ for a positive integer $m$, and $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ is a Hauptmodul of $\Gamma$ with $a_n \in \mathbb{R}$ for all $n \geq 0$. While considering this Hauptmodul $f(z)$ of $\Gamma$, it is a necessary condition that the genus of $\Gamma$ is zero, and as for the genus formula of $\Gamma$ we refer to [9, Theorem 1.1].

For a positive integer $n$ with $(n, mN) = 1$ we have the following disjoint coset decomposition

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma = \bigcup_{\substack{a>0 \\ ad=n}} \bigcup_{\substack{0\leq b<d \\ (a,b,d)=1}} \Gamma\sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where $\sigma_a \in SL_2(\mathbb{Z})$ satisfies $\sigma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \bmod mN$. This can be proved by observing

$$\left| \Gamma \backslash \Gamma \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma \right| = n \prod_{p|n} \left(1 + \frac{1}{p}\right) = \psi(n)$$

and using [11, Proposition 3.36].

REMARK. Since $\sigma_a\Gamma\sigma_a^{-1} = \Gamma$ for any positive divisor $a$ of $n$, we have $\mathbb{C}(f) = A_0(\Gamma) = A_0(\sigma_a^{-1}\Gamma\sigma_a) = \mathbb{C}(f\circ\sigma_a)$, and hence for given $a$ we can define a rational function $P_a(T) \in \mathbb{C}(T)$ such that $f\circ\sigma_a = P_a(f)$. For positive divisors $a$, $b$ of $n$ we easily see that

(1) $a \equiv \pm 1 \bmod N \Leftrightarrow P_a(T) = T$, and $\overline{a} = \overline{b} \in (\mathbb{Z}/N\mathbb{Z})^\times/\{\pm 1\} \Leftrightarrow P_a(T) = P_b(T)$,

(2) $P_a(P_b(T)) = P_{ab}(T) = P_b(P_a(T))$.

If we let $P_a(T) = A(T)/B(T) \in \mathbb{C}(T)$ with $A(T), B(T) \in \mathbb{C}[T]$ and $(A(T), B(T)) = 1$, then $\deg A(T)$, $\deg B(T) \le 1$ except when $\deg A(T) = \deg B(T) = 0$ because $\mathbb{C}(f\circ\sigma_a) = \mathbb{C}(f)$.

We now consider the following polynomial $\Psi_n^f(X, z)$ with the indeterminate $X$

$$\Psi_n^f(X, z) = \prod_{\substack{a>0\\ ad=n}}\ \prod_{\substack{0\le b<d\\ (a,b,d)=1}} \left(X - f\circ\sigma_a\circ\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}(z)\right).$$

Note that $\deg_X \Psi_n^f(X, z) = \psi(n)$. Since all the coefficients of $\Psi_n^f(X, z)$ are the elementary symmetric functions of the $f\circ\sigma_a\circ\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}$, they are invariant under $\Gamma$, i.e., $\Psi_n^f(X, z) \in \mathbb{C}(f(z))[X]$ and we may write $\Psi_n^f(X, f(z))$ instead of $\Psi_n^f(X, z)$. Then as in the usual argument of modular equations, we see that $\Psi_n^f(X, f(z))$ is irreducible over $\mathbb{C}(f(z))$. And we see from [8] that $f(z)^{r_n}\Psi_n^f(X, f(z)) \in \mathbb{C}[X, f(z)]$ for $r_n = -\sum_{s\in S_{1,\infty}\cap S_{2,0}} \mathrm{ord}_s\, f(z)$, where $S_{1,\infty}$ (respectively, $S_{2,0}$) is the set of all points of $\left(\Gamma \cap \begin{pmatrix} n & 0\\ 0 & 1\end{pmatrix}^{-1}\Gamma\begin{pmatrix} n & 0\\ 0 & 1\end{pmatrix}\right) \setminus \mathfrak{H}^*$ such that $f(z)$ (respectively, $f(nz)$) has poles (respectively, zeros) (see also [3, Theorem 3.3] or the proof of [4, Theorem 10]). Here we note that $r_n \le -\sum_{s\in S_{1,\infty}} \mathrm{ord}_s\, f(z) = [\mathbb{C}(f(z), f(nz)) : \mathbb{C}(f(z))] \le n\prod_{p|n}(1 + 1/p)$, because $\Psi_n^f(P_n(f(nz)), f(z)) = 0$.

Therefore for those Hauptmoduln $f(z)$ of $\Gamma$ and integer $n$ with $(n, mN) = 1$ we define the $n$-th modular equation $\Phi_n^f(X, Y) = Y^{r_n}\Psi_n^f(X, Y)$, namely

$$\Phi_n^f(X, f(z)) = f(z)^{r_n}\cdot \prod_{\substack{a>0\\ ad=n}}\ \prod_{\substack{0\le b<d\\ (a,b,d)=1}} \left(X - f\circ\sigma_a\circ\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}(z)\right).$$

Here we remark that if we confine ourselves to a Hauptmodul $f(z) = q^{-1} + \sum_{n=0}^\infty a_n q^n$ with $a_n \in \mathbb{Z}$, we could justify that $\Phi_n^f(X, Y) \in \mathbb{Z}[X, Y]$ and $\Phi_p^f(X, Y)$ satisfies the Kronecker congruences depending on $P_p(T)$ in the above remark. But we will not go further into this direction.

Next, unlike the case $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$ we further consider a subgroup $\langle \Gamma_0(N), W_e\rangle_{e\in S}$ of $SL_2(\mathbb{R})$ which appears in "Monstrous Moonshine" phenomenon. For the details, we recommend the readers to refer [2].

Let $N > 1$ be an integer and $e$ be a Hall divisor of $N$, that is, $e$ is a positive divisor of $N$ such that $(e, N/e) = 1$. For a Hall divisor $e$ of $N$ we define an Atkin-Lehner involution of $\Gamma_0(N)$ as a matrix with determinant 1 of the form

$$\begin{pmatrix} a\sqrt{e} & \dfrac{b}{\sqrt{e}} \\ c\dfrac{N}{\sqrt{e}} & d\sqrt{e} \end{pmatrix} \quad \text{where} \quad a, b, c, d \in \mathbb{Z}.$$

Let $W_e$ be the set of all Atkin-Lehner involutions with a fixed Hall divisor $e$ of $N$. Then these sets satisfy the following multiplication rule:

$$W_e W_f = W_f W_e = W_k \quad \text{where} \quad k = \frac{e}{(e,\, f)} \cdot \frac{f}{(e,\, f)}. \tag{2.1}$$

Notice that $k$ is a Hall divisor of $N$ if $e$ and $f$ are Hall divisors of $N$. Assume that $S$ is a subset of the Hall divisors of $N$ closed under the above multiplication rule. By $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ we mean the subgroup of $SL_2(\mathbb{R})$ generated by all elements of $\Gamma_0(N)$ and $W_e$ for all $e \in S$. If $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ is of genus zero, then we can choose a Hauptmodul $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ with $a_n \in \mathbb{Z}$. In [2] Chen and Yui defined, for a positive integer $n$ prime to $N$, the $n$-th modular equation $\Phi_n^f(X, Y) = 0$ for which

$$\Phi_n^f(X,\, f(z)) = \prod_{\substack{a>0 \\ ad=n}} \prod_{\substack{0 \le b < d \\ (a,b,d)=1}} \left( X - f \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} (z) \right).$$

And they proved that $\Phi_n^f(X, Y)$ is a polynomial with integral coefficients satisfying $\Phi_n^f(f(z),\, f(nz)) = 0$ and it is irreducible as a polynomial in $X$ over $\mathbb{C}(Y)$. But, for the purpose of this article, it is enough to assume that $f(z)$ has only real Fourier coefficients, i.e., $a_n \in \mathbb{R}$ for all $n \geq 0$.

Now we are ready to state our main theorem.

**Theorem 2.1.** (1) *Let $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ be a Hauptmodul of $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$ with $a_n \in \mathbb{R}$. For a positive integer $n$ with $(n, mN) = 1$, we get*

$$h(\Phi_n^f(X, Y)) = \frac{6\psi(n)}{[\overline{\Gamma(1)} : \overline{\Gamma}]} \left\{ \log n - 2 \sum_{p \mid n} \frac{\log p}{p} + \mathcal{O}(1) \right\}.$$

(2) *Let $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ be a Hauptmodul of $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ with $a_n \in \mathbb{R}$. For a positive integer $n$ with $(n, N) = 1$, we have*

$$h(\Phi_n^f(X, Y)) = \sum_{e \in S} \frac{6\psi(n)}{[\overline{\Gamma(1)} : \overline{\Gamma_0(N/e)}]} \left\{ \log n - 2 \sum_{p \mid n} \frac{\log p}{p} + \mathcal{O}(1) \right\}.$$

Combining (1.1) and Theorem 2.1, we can readily achieve the following corollary.

**Corollary 2.2.** (1) *With the notations as in* Theorem 2.1 (1), *we obtain*

$$\lim_{\substack{n\to\infty\\(n,mN)=1}} \frac{h(\Phi_n^f(X, Y))}{h(\Phi_n^j(X, Y))} = \frac{1}{[\overline{\Gamma(1)} : \overline{\Gamma}]}.$$

(2) *With the notations as in* Theorem 2.1 (2), *we get*

$$\lim_{\substack{n\to\infty\\(n,N)=1}} \frac{h(\Phi_n^f(X, Y))}{h(\Phi_n^j(X, Y))} = \sum_{e\in S} \frac{1}{[\overline{\Gamma(1)} : \overline{\Gamma_0(N/e)}]}.$$

We conclude this section with some remarks. For an arbitrary intersection of classical congruence subgroups

$$\Gamma' = \Gamma_0(N_1) \cap \Gamma^0(N_2) \cap \Gamma_1(N_3) \cap \Gamma^1(N_4) \cap \Gamma(N_5),$$

we have $\alpha^{-1}\Gamma'\alpha = \Gamma_1(N) \cap \Gamma_0(mN)$ where $N = lcm(N_3, N_4, N_5)$ and

$$\alpha = \begin{pmatrix} lcm(N_2, N_4, N_5) & 0 \\ 0 & 1 \end{pmatrix}, \quad m = \frac{lcm(N_1, N_3, N_5)\, lcm(N_2, N_4, N_5)}{N}.$$

If $g(z) = q_h^{-1} + \sum_{n=0}^{\infty} a_n q_h^n$ is a Hauptmodul of $\Gamma'$ with $h = lcm(N_2, N_4, N_5)$ and $q_h = e^{2\pi iz/h}$, then $f(z) := g \circ \alpha(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ is a Hauptmodul of $\Gamma_1(N) \cap \Gamma_0(mN)$. Since the $n$-th modular equation $\Phi_n^g(X, Y)$ for $g(z)$ is, essentially, irreducible as a polynomial in $X$ over $\mathbb{C}(Y)$ satisfying $\Phi_n^g(g(z), g(nz)) = 0$, we obtain $\Phi_n^f(X, Y) = \Phi_n^g(X, Y)$ by observing $\Phi_n^g(g(hz), g(hnz)) = 0$ and $f(z) = g(hz)$. Thus Theorem 2.1 (1) holds for any congruence subgroup of $\Gamma_0(N_1)$, $\Gamma^0(N_2)$, $\Gamma_1(N_3)$, $\Gamma^1(N_4)$, $\Gamma(N_5)$ or arbitrary intersection of them. For example, since

$$\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma(5) \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} = \Gamma_1(5) \cap \Gamma_0(25)$$

and $f(z) := j_5(5z)$ is a Hauptmodul of $\Gamma_1(5) \cap \Gamma_0(25)$ with the same $n$-th modular equation when $(n, 5) = 1$, we can recover the result of Cais and Conrad from Theorem 2.1 (1).

If $S$ contains all the Hall divisors of $N$, we write $\Gamma_0(N)+$ as the group $\langle \Gamma_0(N), W_e\rangle_{e\in S}$. In [2, Appendix 2] Chen and Yui calculated some modular equations for Hauptmoduln of $\Gamma_0(N)$ and $\Gamma_0(N)+$. For instance,

$$\begin{aligned}\Phi_2^{\Gamma_0(3)}(X, Y) = X^3 &+ (-Y^2 + 108)X^2 + (-153Y + 2268)X \\ &+ (Y^3 + 108Y^2 + 2268Y - 46224),\end{aligned}$$

$$\Phi_2^{\Gamma_0(3)+}(X, Y) = X^3 + (-Y^2 + 1566)X^2 + (17343Y + 741474)X \\ + (Y^3 + 1566Y^2 + 7417474Y - 28166076),$$

where $\Phi_2^{\Gamma_0(3)}$ and $\Phi_2^{\Gamma_0(3)+}$ stand for the second modular equations of the (normalized) Hauptmoduln of $\Gamma_0(3)$ and $\Gamma_0(3)+$, respectively. We remark that Theorem 2.1 (2) also gives a reason why the logarithmic height of $\Phi_n^{\Gamma_0(3)}$ is smaller than that of $\Phi_n^{\Gamma_0(3)+}$ for not only $n = 2$ but also sufficiently large $n$.

## 3. Proof of Theorem 2.1

To prove Theorem 2.1 it is necessary to study the behavior of Hauptmodul at each cusp of $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$ or $\langle \Gamma_0(N), W_e \rangle_{e \in S}$. In this section we recall some lemmas which give us useful informations about these cusps.

First lemma provides us a criterion to determine whether or not given two cusps are equivalent under $\Gamma$.

**Lemma 3.1.** *Let* $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$ *and*

$$\Delta = \{\overline{\pm(1 + Nk)} \in (\mathbb{Z}/mN\mathbb{Z})^\times \mid k = 0, 1, \ldots, m - 1\}.$$

*We assume that* $a, c, a'$ *and* $c'$ *are integers such that* $(a, c) = (a', c') = 1$. *By* $\pm 1/0$ *we mean* $\infty$. *Then the cusp* $a/c$ *is equivalent to* $a'/c'$ *under* $\Gamma$ *if and only if there exist* $x \in \Delta$ *and* $n \in \mathbb{Z}$ *such that*

$$\begin{pmatrix} a' \\ c' \end{pmatrix} \equiv \begin{pmatrix} xa + nc \\ x^{-1}c \end{pmatrix} \mod mN.$$

Proof. Suppose that $a/c$ is equivalent to $a'/c'$ under $\Gamma$, i.e., there exists $\gamma \in \Gamma$ such that $a'/c' = \gamma(a/c)$. Since $a, c, a', c'$ are integers satisfying $(a, c) = (a', c') = 1$, we have $\begin{pmatrix} a' \\ c' \end{pmatrix} = \pm\gamma\begin{pmatrix} a \\ c \end{pmatrix}$. By putting $\gamma = \begin{pmatrix} x & n \\ z & w \end{pmatrix} \in \Gamma$ we have the desired assertion. Conversely suppose that there exist $x \in \Delta$ and $n \in \mathbb{Z}$ satisfying the above congruence in the hypothesis. Since the natural reduction map of $SL_2(\mathbb{Z})$ into $SL_2(\mathbb{Z}/mN\mathbb{Z})$ is surjective, let $\gamma \in SL_2(\mathbb{Z})$ be a preimage of $\begin{pmatrix} x & n \\ 0 & x^{-1} \end{pmatrix} \in SL_2(\mathbb{Z}/mN\mathbb{Z})$. Note that $\gamma \in \{\pm 1\} \cdot \Gamma$ and $\begin{pmatrix} a' \\ c' \end{pmatrix} \equiv \gamma\begin{pmatrix} a \\ c \end{pmatrix} \bmod mN$. Now it is an elementary fact that if $u, v, z, w$ are integers such that $(u, v) = (z, w) = 1$ and $\begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} z \\ w \end{pmatrix} \bmod N$, then $u/v$ and $z/w$ are equivalent under $\Gamma(N)$ ([11, Lemma 1.42]). So in our case there exists $\gamma' \in \Gamma(mN)$ such that $a'/c' = \gamma'(\gamma(a/c))$. This completes the proof since $\Gamma(mN) \subset \Gamma$. □

Let $\phi(x)$ be the Euler function. Then it is worthy of remarking that

$$[\overline{\Gamma(1)} : \overline{\Gamma}] = [\overline{\Gamma(1)} : \overline{\Gamma_0(mN)}][\overline{\Gamma_0(mN)} : \overline{\Gamma}] = \frac{[\overline{\Gamma(1)} : \overline{\Gamma_0(mN)}]\phi(mN)}{|\Delta|}, \tag{3.1}$$

which will be used in the proof of Lemma 3.10 and Lemma 3.11. From the next two lemmas we can determine whether a given cusp is equivalent to the cusp infinity under $\langle \Gamma_0(N), W_e \rangle_{e \in S}$.

**Lemma 3.2.** *Let $S_{\Gamma_0(N)}$ be the set of pairs $(c, a)$ satisfying*
(1) $(1, 0) \in S_{\Gamma_0(N)}$.
(2) $c > 1,\ c \mid N,\ 1 \le a < c,\ (c, a) = 1$.
(3) *If* $(c, a), (c, a_1) \in S_{\Gamma_0(N)}$ *and* $a_1 \equiv a \bmod (c, N/c)$ *then* $a = a_1$.
*Then the set $\{a/c \mid (c, a) \in S_{\Gamma_0(N)}\}$ is a set of complete representatives of all inequivalent cusps of $\Gamma_0(N)$.*

Proof. This lemma is indeed well-known ([7, Proposition 1.23]). For the reader's convenience we give an alternative proof. We first observe that the cardinality of $S_{\Gamma_0(N)}$ is $1 + \sum_{c>1, c \mid N} \varphi((c, N/c))$ because the natural map $(\mathbb{Z}/c\mathbb{Z})^\times \to (\mathbb{Z}/(c, N/c)\mathbb{Z})^\times$ is surjective. Since the number of inequivalent cusps of $\Gamma_0(N)$ is $\sum_{d \mid N} \varphi((d, N/d))$ (see [11, Proposition 1.43]), it is enough to prove that arbitrary two distinct pairs $(c, a), (c', a') \in S_{\Gamma_0(N)}$ are inequivalent to each other. Suppose that they are equivalent under $\Gamma_0(N)$. By substituting $N = 1$, $m = N$, and $\Delta = (\mathbb{Z}/N\mathbb{Z})^\times$ in Lemma 3.1, we must have that $c = c'$ and $x \equiv 1 \bmod N/c$. Thus $a' \equiv xa + nc \bmod N$ with $x \equiv 1 \bmod N/c$ implies that $a' \equiv a \bmod (c, N/c)$. By hypothesis (3) we have $a' = a$. □

**Lemma 3.3.** *Let $S$ be a subset of Hall divisors of $N$ closed under the multiplication rule* (2.1). *Then the cusps*

$$\left\{ \frac{1}{N/e} \,\middle|\, e \in S \right\}$$

*are all those equivalent under $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ to $\infty$ among the set of representatives $\{a/c \mid (c, a) \in S_{\Gamma_0(N)}\}$ described in* Lemma 3.2.

Proof. For given $e \in S$ there exist $b, d \in \mathbb{Z}$ satisfying $de - b(N/e) = 1$. Thus we have $W_e = \Gamma_0(N) \begin{pmatrix} \sqrt{e} & b/\sqrt{e} \\ N/\sqrt{e} & d\sqrt{e} \end{pmatrix}$. Since $\begin{pmatrix} \sqrt{e} & b/\sqrt{e} \\ N/\sqrt{e} & d\sqrt{e} \end{pmatrix}(\infty) = 1/(N/e)$, we have the assertion. □

Using the above lemmas we are able to prove Theorem 2.1 by adopting the idea of Cais and Conrad ([1]). For convenience, if $f(z) = q^{-1} + \sum_{n=0}^{\infty} a_n q^n$ with $a_n \in \mathbb{R}$ is a Hauptmodul of $\Gamma$ (respectively, $\langle \Gamma_0(N), W_e \rangle_{e \in S}$), then we simply write "$f(z)$ is on $\Gamma$" (respectively, on $\langle \Gamma_0(N), W_e \rangle_{e \in S}$).

**Lemma 3.4.** $\Gamma$ *and* $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ *have no elliptic points on* $i\mathbb{R}_{>1}$.

Proof. If $it$ ($t > 1$) is a fixed point of an elliptic element $\sigma \in SL_2(\mathbb{R})$, then the absolute value of the trace of $\sigma$, $|\mathrm{tr}(\sigma)|$, is less than 2. Moreover, if $\sigma \in SL_2(\mathbb{Z})$, we have $\sigma = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which gives rise to a contradiction. If $\sigma \in SL_2(\mathbb{R}) \setminus SL_2(\mathbb{Z})$, then we may assume $\sigma = \pm\begin{pmatrix} a\sqrt{e} & b/\sqrt{e} \\ cN/\sqrt{e} & d\sqrt{e} \end{pmatrix}$ for $a, b, c, d \in \mathbb{Z}$ and a Hall divisor $e$ of $N$. Since $\sigma$ fixes $it$ and $|\mathrm{tr}(\sigma)| < 2$, we have $a = d$; hence $a = 0$ and $\sigma = \pm\begin{pmatrix} 0 & b/\sqrt{e} \\ cN/\sqrt{e} & 0 \end{pmatrix}$. Since $\sigma$ has determinant 1, we obtain $-bcN = e$ and so $bc(N/e) = -1$, that is, $b/c = -1$. On the other hand $\sigma$ fixes $it$, so we have $b = -cNt$. Thus $Nt^2 = 1$, which is a contradiction. □

Since $f(z) = q^{-1} + \cdots$ has real Fourier coefficients, $f(it)$ is real and $|f(it)| \to \infty$ as $t \to \infty$. Moreover $f'(z)$ is nonvanishing on $i\mathbb{R}_{>1}$ by Lemma 3.4, so we see that $f(it)$ is strictly increasing for $t \geq 1$. Thus we can choose real numbers $s > 1$ and $1 \leq t_0 \leq t_1$ such that $f(it_0) = s$, $f(it_1) = 2s$.

**Lemma 3.5.** *For $t_0 \leq t \leq t_1$, we have*

$$h(\Phi_n^f(X,\ f(it))) = \sum_{\substack{a>0\\ ad=n}} S_d(t) + \mathcal{O}(\psi(n)),$$

*where*

$$S_d(t) = \begin{cases} \displaystyle\sum_{\substack{0\leq b<d\\ (a,b,d)=1}} \log\max\left\{1,\ \left|f\circ\sigma_a\left(\frac{ait+b}{d}\right)\right|\right\} & \text{if} \quad f(z) \quad \text{is on} \quad \Gamma, \\ \displaystyle\sum_{\substack{0\leq b<d\\ (a,b,d)=1}} \log\max\left\{1,\ \left|f\left(\frac{ait+b}{d}\right)\right|\right\} & \text{if} \quad f(z) \quad \text{is on} \quad \langle\Gamma_0(N), W_e\rangle_{e\in S}. \end{cases}$$

*Here the implicit $\mathcal{O}$-constant depends only on $f$, $t_0$ and $t_1$.*

Proof. It is well-known that the coefficients of a monic polynomial $P(x) = (x - w_1)\cdots(x - w_d)$ are laid in between $2^{-d}M$ and $2^dM$ where $M = \prod_{j=1}^d \max\{1, |w_j|\}$. Taking logarithm we see that

$$h(P) = \sum_{j=1}^{d} \log\max\{1, |w_j|\} + \mathcal{O}(d) \tag{3.2}$$

with an implicit absolute $\mathcal{O}$-constant which is independent of $d$ and $P$.

If $f(z)$ is on $\Gamma$, then for $t_0 \leq t \leq t_1$

$$\Phi_n^f(X,\ f(it)) = f(z)^{r_n} \prod_{\substack{a>0\\ ad=n}} \prod_{\substack{0\leq b<d\\ (a,b,d)=1}} \left(X - (f\circ\sigma_a)\left(\frac{ait+b}{d}\right)\right).$$

Applying (3.2) we have

$$\begin{aligned} &h(\Phi_n^f(X,\ f(it)))\\ &= r_n \log f(it) + \sum_{\substack{a>0\\ ad=n}} \sum_{\substack{0\leq b<d\\ (a,b,d)=1}} \log\max\left\{1,\ \left| f\circ\sigma_a\left(\frac{ait+b}{d}\right)\right|\right\} + \mathcal{O}(\psi(n)). \end{aligned}$$

Since $0 \leq r_n \leq \psi(n)$ and $s = f(it_0) \leq f(it) \leq f(it_1) = 2s$, we get $r_n \log f(it) = \mathcal{O}(\psi(n))$ where the implicit $\mathcal{O}$-constant depends only on $f$, $t_0$ and $t_1$.

As for the case where $f(z)$ is on $\langle \Gamma_0(N),\ W_e\rangle_{e\in S}$, the same argument can be applied, and hence we omit the detailed proof. $\square$

Next goal is to calculate each term in the summation $S_d(t)$. For this purpose we are in need of the following lemma.

**Lemma 3.6.** *For $z = \xi + i\eta \in \mathfrak{H}$, let $g(z) = a_{-1}q_h^{-1} + \sum_{n=0}^{\infty} a_n {q_h}^n$ with $q_h = e^{2\pi i z/h}$ for a positive integer $h$. We assume that if $a_{-1} = 0$ (respectively, $a_{-1} \neq 0$), then $g(z)$ (respectively, $q_h g(z)$) is absolutely convergent for $\eta > 0$. Then for $\eta \geq 1/2$, we have*

$$\log\max\{1,\ |g(z)|\} = \begin{cases} \mathcal{O}(1) & \text{if} \quad a_{-1} = 0,\\ \dfrac{2\pi i\eta}{h} + \mathcal{O}(1) & \text{if} \quad a_{-1} \neq 0. \end{cases}$$

*Here the implicit $\mathcal{O}$-constants depend only $g(z)$.*

Proof. Since $g(z+h) = g(z)$, we may assume that $-h/2 \leq \xi \leq h/2$. Suppose first that $a_{-1} = 0$. Since $|g(z)| \to |a_0|$ as $\eta \to \infty$, there is a real number $\eta_0 \geq 1/2$ such that for $\eta > \eta_0$, $|a_0|/2 \leq |g(z)| \leq |a_0| + 1$. Hence, for $\eta > \eta_0$ we derive $\log\max\{1, |g(z)|\} = \mathcal{O}(1)$. Here the implicit $\mathcal{O}$-constant depends only on $a_0$, that is $g$. For $1/2 \leq \eta \leq \eta_0$ we note that $\log\max\{1, |g(z)|\}$ is a continuous function on the set

$$\left\{\xi + i\eta \in \mathfrak{H}\ \middle|\ -\frac{h}{2} \leq \xi \leq \frac{h}{2} \text{ and } \frac{1}{2} \leq \eta \leq \eta_0\right\}$$

and hence is bounded on this set. Note that the upper bound depends only on $g$ and is independent of the choice of $\eta_0$.

If $a_{-1} \neq 0$, $|q_h g(z)| \to |a_{-1}|$ as $\eta \to \infty$ so that we obtain the assertion by the same argument as above. $\square$

Let $M$ be a positive integer. Then it is more convenient to consider the displaced interval $I_M = [1/(M+1), (M+2)/(M+1))$ rather than the usual interval $[0, 1)$. Cohen proved in [5] that $I_M$ can be expressed as

$$I_M = \bigcup_{k=1}^{M} \bigcup_{\substack{h=1\\(h,k)=1}}^{k} I_M\left(\frac{h}{k}\right),$$

which is a disjoint union of sets $I_M(h/k)$. Here each $I_M(h/k)$ is an interval of the form $[\rho_1^{(h/k)}, \rho_2^{(h/k)})$ containing $h/k$ and

$$\frac{1}{2Mk} \le \frac{h}{k} - \rho_1^{(h/k)} \le \frac{1}{(M+1)k},$$
$$\frac{1}{2Mk} \le \rho_2^{(h/k)} - \frac{h}{k} \le \frac{1}{(M+1)k}.$$

For real numbers $h$, $k$ and $x$, we put

$$g_{h,k}(x) = \frac{2\pi nt/d^2k^2}{(at/d)^2 + (x - h/k)^2},$$

which will be used for estimating the sum $S_d(t)$. Thus $a$, $d$ and $t$ are related to $S_d(t)$. Note that the width of the cusp $\sigma_a(\infty)$ is 1, because $f \circ \sigma_a = P_a(f)$ as remarked in §2. Also observe that $\langle \Gamma_0(N), W_e\rangle_{e\in S}$ contains $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Hence in any case we may reindex the sum in $S_d(t)$ via

$$b \mapsto \begin{cases} b & \text{if} \quad \dfrac{b}{d} \in \left[\dfrac{1}{N+1}, 1\right), \\[2ex] b + d & \text{if} \quad \dfrac{b}{d} \in \left[0, \dfrac{1}{N+1}\right). \end{cases}$$

**Lemma 3.7.** *Let $f$ be on $\Gamma$.*

(1) *If $at/d \ge 1/2$, then we have*

$$\log\max\left\{1, \left|f \circ \sigma_a\left(\frac{ait+b}{d}\right)\right|\right\} = \begin{cases} \dfrac{2\pi nt}{d^2} + \mathcal{O}(1) & \textit{if} \quad \overline{a} \in \Delta, \\ \mathcal{O}(1) & \textit{otherwise.} \end{cases}$$

(2) *Put $M = [d/\sqrt{nt}]$. If $at/d \le 1$, then $M \ge 1$ and, for $b/d \in I_M(h/k)$, we get*

$$\log\max\left\{1, \left|f \circ \sigma_a\left(\frac{ait+b}{d}\right)\right|\right\}$$
$$= \begin{cases} g_{h,k}(b/d) + \mathcal{O}(1) & \textit{if} \quad k \equiv 0 \bmod mN \quad \textit{and} \quad \overline{h} \in \overline{a}\Delta, \\ \mathcal{O}(1) & \textit{otherwise.} \end{cases}$$

*In both cases the implicit $\mathcal{O}$-constants depend only on $f$.*

Proof. (1) By Lemma 3.1, $\sigma_a(\infty)$ is equivalent to $\infty$ under $\Gamma$ if and only if $\overline{a} \in \Delta$. Using this, Lemma 3.6 gives us the assertion.

(2) Since $(h,k)=1$, we can find $\gamma_{h,k} := \begin{pmatrix} v & u \\ -k & h \end{pmatrix} \in SL_2(\mathbb{Z})$. By routine calculation we see that

$$\mathrm{Im}\left(\gamma_{h,k}\left(\frac{ait+b}{d}\right)\right) = \frac{nt/d^2k^2}{(at/d)^2+(b/d-h/k)^2} = \frac{1}{2\pi} g_{h,k}\left(\frac{b}{d}\right).$$

Since $b/d \in I_M(h/k) = [\rho_1^{(h/k)}, \rho_2^{(h/k)})$, we obtain

$$\left|\frac{b}{d}-\frac{h}{k}\right| \leq \frac{1}{(M+1)k} \leq \frac{\sqrt{nt}}{dk}.$$

Moreover, we achieve

$$\frac{at}{d} = \frac{nt}{d^2} \leq \frac{\sqrt{nt}}{dk}$$

which implies that

$$\mathrm{Im}\left(\gamma_{h,k}\left(\frac{ait+b}{d}\right)\right) \geq \frac{1}{2}.$$

By Lemma 3.1, $\sigma_a(\gamma_{h,k}^{-1}(\infty))$ is equivalent to $\infty$ under $\Gamma$ if and only if $k \equiv 0 \bmod mN$ and $\overline{h} \in \overline{a}\Delta$. Taking $g(z) = f \circ \sigma_a \circ \gamma_{h,k}^{-1}(z)$ in Lemma 3.6, we have the assertion. More precisely, if $k \equiv 0 \bmod mN$ and $\overline{h} \in \overline{a}\Delta$, then

$$\begin{aligned}\left|f \circ \sigma_a\left(\frac{ait+b}{d}\right)\right| &= \left|f \circ \sigma_a \circ \gamma_{h,k}^{-1}\left(\gamma_{h,k}\left(\frac{ait+b}{d}\right)\right)\right| \\ &= 2\pi\ \mathrm{Im}\left(\gamma_{h,k}\left(\frac{ait+b}{d}\right)\right) = g_{h,k}\left(\frac{b}{d}\right).\end{aligned}$$

Other case corresponds to the holomorphic one in Lemma 3.6. Therefore we prove the lemma. □

**Lemma 3.8.** *Let $f$ be on $\langle \Gamma_0(N), W_e\rangle_{e \in S}$.*
(1) *If $at/d \geq 1/2$, then we have*

$$\log\max\left\{1, \left|f\left(\frac{ait+b}{d}\right)\right|\right\} = \frac{2\pi at}{d} + \mathcal{O}(1).$$

(2) *Put* $M = [d/\sqrt{nt}]$. *If* $at/d \leq 1$, *then* $M \geq 1$ *and, for* $b/d \in I_M(h/k)$, *we establish*

$$\log\max\left\{1, \left|f\left(\frac{ait+b}{d}\right)\right|\right\}$$
$$= \begin{cases} g_{h,k}\left(\dfrac{b}{d}\right) + \mathcal{O}(1) & \text{for} \quad e \in S, \quad k \equiv 0 \bmod N/e \quad \text{and} \quad \overline{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^{\times}, \\ \mathcal{O}(1) & \text{otherwise.} \end{cases}$$

*In both cases the implicit* $\mathcal{O}$*-constants depend only on* $f$.

Proof. Since the first assertion can be proved in a similar way to Lemma 3.7, we only prove (2). The fact that $\gamma_{h,k}^{-1}(\infty)$ is equivalent to $\infty$ under $\langle \Gamma_0(N), W_e \rangle_{e \in S}$ yields by Lemma 3.7 that $h/k$ is equivalent to $1/(N/e)$ under $\Gamma_0(N)$ for some Hall divisor $e \in S$ exactly. In other words, by Lemma 3.1 there are $\overline{x} \in (\mathbb{Z}/N\mathbb{Z})^{\times}$, $n \in \mathbb{Z}$ such that $h \equiv x^{-1} + n \cdot (N/e) \bmod N$ and $k \equiv x \cdot (N/e) \bmod N$. This is equivalent to $h \equiv x^{-1} \bmod (N/e)$ and $k \equiv 0 \bmod (N/e)$, because $N/e$ is also a Hall divisor. Thus we have the conclusion. □

Now, we calculate $S_d(t)$ more precisely in Lemma 3.10 and 3.10. To this end we need the following lemma in advance.

**Lemma 3.9.** *Let* $k$, $j$ *and* $a$ *be positive integers satisfying* $j \mid k$ *and* $(j, a) = 1$. *We further let* $\zeta$ *be a primitive* $k$*-th root of unity and let*

$$c'_k(l) = \sum_{\substack{h \in (\mathbb{Z}/k\mathbb{Z})^{\times} \\ h \equiv a \bmod j}} \zeta^{hl} \quad \text{for} \quad l \in \mathbb{Z}.$$

*Then*

$$|c'_k(l)| \leq j \cdot (k, l) \quad \text{for any} \quad l \in \mathbb{Z}. \tag{3.3}$$

Proof. Using a primitive $j$-th root of unity $\zeta^{k/j}$ we may rewrite the sum as

$$c'_k(l) = \frac{1}{j} \sum_{i \in \mathbb{Z}/j\mathbb{Z}} \zeta^{-kia/j} \sum_{h \in (\mathbb{Z}/k\mathbb{Z})^{\times}} \zeta^{(l+ik/j)h}.$$

Let $\mu(x)$ be the Möbius function. Since the Ramanujan's sum satisfies

$$\sum_{h \in (\mathbb{Z}/k\mathbb{Z})^{\times}} \zeta^{hx} = \mu\left(\frac{k}{(k, x)}\right) \cdot \phi(k) \Big/ \phi\left(\frac{k}{(k, x)}\right)$$

for $x \in \mathbb{Z}$ and $\phi(xy) \le x\phi(y)$ for any $x, y \in \mathbb{Z}_{>0}$, we have

$$\left|\sum_{h\in(\mathbb{Z}/k\mathbb{Z})^\times} \zeta^{(l+ik/j)h}\right| \le \frac{\phi(k)}{\phi(k/(k, l+ik/j))} \le \left(k, l+\frac{ik}{j}\right) = \left(k, l+\frac{ik}{j}, jl\right)$$
$$\le (k, jl) \le j\cdot\left(\frac{k}{j}, l\right) \le j\cdot(k, l),$$

which implies $|c_k'(l)| \le j\cdot(k, l)$. □

Here we remark that Cais and Conrad dealt with the case of a rational prime $j$ dividing $k$ in [1, Lemma D.3], but it seems to be not true. Indeed, we can find a counterexample when $k = p = 3$, $a = m = 1$ with the notations as in there. So we correct it and prove the expanded version. It doesn't crucially matter, however, to the results because we need just its boundedness.

**Lemma 3.10.** *Let $f$ be on $\Gamma$.*
(1) *If $d < \sqrt{nt}$, then $S_d(t) = \mathcal{O}(n/d)$. Here the implicit $\mathcal{O}$-constant depends only upon $f$, $t_0$ and $t_1$.*
(2) *If $d \ge \sqrt{nt}$, then*

$$S_d(t) = \frac{1}{[\overline{\Gamma(1)} : \overline{\Gamma}]}\cdot\frac{6d}{(a, d)}\phi((a, d))\log\left(\frac{d^2}{n}\right) + \mathcal{O}\left(\sigma_1\left(\frac{d}{(a, d)}\right)\right) + \mathcal{O}\left(\frac{d\sigma_1((a, d))}{(a, d)}\right),$$

*where $\phi(x)$ is the Euler function and $\sigma_1(x)$ is the sum of positive divisors of $x$. Here the implicit $\mathcal{O}$-constant depends only upon $\Gamma$, $f$, $t_0$ and $t_1$.*

Proof. (1) Since the number of elements in $\{b \mid 0 \le b < d,\ (a, b, d) = 1\}$ is $d\phi((a, d))/(a, d)$, by Lemma 3.6 and the fact that $\phi((a, d))/(a, d) \le 1$ we have

$$\begin{aligned}|S_d(t)| &\le \sum_{\substack{0\le b<d\\(a,b,d)=1}} \log\max\left\{1, \left|f\circ\sigma_a\left(\frac{ait+b}{d}\right)\right|\right\}\\
&= \begin{cases}\dfrac{d\phi((a, d))}{(a, d)}\dfrac{2\pi nt}{d^2} + \mathcal{O}\left(\dfrac{d\phi((a, d))}{(a, d)}\right) & \text{if } \ \overline{a}\in\Delta,\\ \mathcal{O}\left(\dfrac{d\phi((a, d))}{(a, d)}\right) & \text{otherwise}\end{cases}\\
&\le \begin{cases}\dfrac{2\pi nt}{d} + C\cdot d & \text{if } \ \overline{a}\in\Delta\\ C'd & \text{otherwise.}\end{cases}\end{aligned}$$

Using the fact that $d < nt/d \le nt_1/d$ we conclude the first assertion.

(2) Note that the assumption $d \geq \sqrt{nt}$ implies $at/d \leq 1$. Put $M = [d/\sqrt{nt}] \geq 1$. Then we have by Lemma 3.7

$$\begin{aligned}
S_d(t) &= \sum_{k=1}^{M} \sum_{\substack{h=1\\(h,k)=1}}^{k} \sum_{\substack{b/d \in I_M(h/k)\\ 0\leq b<d\\ (a,b,d)=1}} \log\max\left\{1, \left| f\circ\sigma_a\left(\frac{ait+b}{d}\right)\right|\right\}\\
&= \sum_{\substack{1\leq h\leq k\leq M\\(h,k)=1\\ k\equiv 0 \bmod mN\\ \overline{h}\in\overline{a}\Delta}} \sum_{\substack{b/d\in I_M(h/k)\\0\leq b<d\\(a,b,d)=1}} \left(g_{h,k}\left(\frac{b}{d}\right) + \mathcal{O}(1)\right) + \sum_{\substack{1\leq h\leq k\leq M\\(h,k)=1\\ \text{otherwise}}} \sum_{\substack{b/d\in I_M(h/k)\\0\leq b<d\\(a,b,d)=1}} \mathcal{O}(1)\\
&= \sum_{\substack{1\leq h\leq k\leq M\\(h,k)=1\\ k\equiv 0 \bmod mN\\ \overline{h}\in\overline{a}\Delta}} \sum_{\substack{b/d\in I_M(h/k)\\0\leq b<d\\(a,b,d)=1}} g_{h,k}\left(\frac{b}{d}\right) + \mathcal{O}(d).
\end{aligned}$$

Since the total number for error terms $\mathcal{O}(1)$ is less than $d$ and so $\mathcal{O}(d)$ lies outside of the summation, we can get the last expression in the above summation.

Meanwhile, we see from [5, Lemma 6] that

$$\sum_{\substack{b/d\in I_M(h/k)\\ 0\leq b<d,\ (a,b,d)=1}} g_{h,k}\left(\frac{b}{d}\right) = k^{-2} \sum_{f|(a,d)} \mu(f) F_f\left(\frac{dh}{fk}\right) + \mathcal{O}\left(\frac{\sqrt{n}\sigma_1((a,d))}{k(a,d)}\right),$$

where $F_f(\theta) = (2\pi^2 d/f)\sum_{v\in\mathbb{Z}} e^{-2\pi|v|nt/df} e^{2\pi i v\theta}$ and $\mu(x)$ is the Möbius function.

Since we have as in [1]

$$\begin{aligned}
&C\frac{\sqrt{n}\sigma_1((a,d))}{(a,d)} \sum_{\substack{1\leq h\leq k\leq M\\(h,k)=1}} \frac{1}{k} = C\frac{\sqrt{n}\sigma_1((a,d))}{(a,d)} \sum_{1\leq k\leq M} \frac{\phi(k)}{k} \leq C\cdot M\frac{\sqrt{n}\sigma_1((a,d))}{(q,d)}\\
&\qquad \leq C\frac{\sqrt{n}\sigma_1((a,d))}{(a,d)}\frac{d}{\sqrt{nt}} \leq C\frac{d\sigma_1((a,d))}{(a,d)},
\end{aligned}\tag{3.4}$$

we establish that

$$\begin{aligned}
S_d(t) &= \sum_{\substack{1\leq h\leq k\leq M\\(h,k)=1\\ k\equiv 0 \bmod mN\\ \overline{h}\in\overline{a}\Delta}} \left\{k^{-2}\sum_{f|(a,d)} \mu(f)F_f\left(\frac{dh}{fk}\right) + \mathcal{O}\left(\frac{\sqrt{n}\sigma_1((a,d))}{k(a,d)}\right)\right\} + \mathcal{O}(d)\\
&= \sum_{\substack{1\leq h\leq k\leq M\\(h,k)=1\\ k\equiv 0 \bmod mN\\ \overline{h}\in\overline{a}\Delta}} \left\{k^{-2}\sum_{f|(a,d)} \mu(f)F_f\left(\frac{dh}{fk}\right)\right\} + \mathcal{O}\left(\frac{d\sigma_1((a,d))}{(a,d)}\right)
\end{aligned}$$

$$= \sum_{f|(a,d)} \mu(f) \sum_{\substack{1\le h\le k\le M \\ (h,k)=1 \\ k\equiv 0 \bmod mN \\ \overline{h}\in\overline{a}\Delta}} k^{-2} F_f\left(\frac{dh}{fk}\right) + \mathcal{O}\left(\frac{d\sigma_1((a,d))}{(a,d)}\right).$$

We now consider the sum

$$\sum_{\substack{1\le h\le k\le M \\ (h,k)=1 \\ k\equiv 0 \bmod mN \\ \overline{h}\in\overline{a}\Delta}} k^{-2} F_f\left(\frac{dh}{fk}\right) = \frac{2\pi^2 d}{f} \sum_{v\in\mathbb{Z}} C_M\left(\frac{dv}{f}\right) e^{-2\pi|v|nt/df}, \tag{3.5}$$

where

$$C_M(l) = \sum_{\substack{1\le k\le M \\ k\equiv 0 \bmod mN}} k^{-2} c_k(l)$$

and

$$c_k(l) = \sum_{\substack{1\le h\le k \\ (h,k)=1 \\ \overline{h}\in\overline{a}\Delta}} e^{2\pi i hl/k} \quad \text{for any} \quad l\in\mathbb{Z}.$$

We have to calculate $C_M(l)$ and $c_k(l)$ to know the upper bound of the sum of (3.5). By Lemma 3.9 we know that $|c_k(l)| \le |\Delta| mN(k,l)$ for $l\in\mathbb{Z}-\{0\}$. So when $l\ne 0$, we have

$$\begin{aligned} |C_M(l)| &\le |\Delta| mN \sum_{k=1}^{\infty} k^{-2}(k,l) \le |\Delta| mN \sum_{d|l} d \sum_{j=1}^{\infty} \frac{1}{j^2 d^2} \\ &= |\Delta| mN \frac{\pi^2}{6} \frac{1}{|l|} \sum_{d|l} \frac{|l|}{d} = |\Delta| mN \frac{\pi^2}{6} \frac{\sigma_1(|l|)}{|l|}; \end{aligned}$$

hence

$$|C_M(l)| = \mathcal{O}\left(\frac{\sigma_1(|l|)}{|l|}\right)$$

for $l\ne 0$, where the implicit $\mathcal{O}$-constant depends only on $\Gamma$. In case of $l=0$ we consider the natural surjective homomorphism $\pi:(\mathbb{Z}/k\mathbb{Z})^\times \to (\mathbb{Z}/mN\mathbb{Z})^\times$ which gives us

$$c_k(0) = |\pi^{-1}(\Delta)| = |\Delta|\,|\ker\pi| = |\Delta| \frac{\phi(k)}{\phi(mN)}.$$

Hence by [1, Lemma D.1] and (3.1) we obtain

$$\begin{aligned} C_M(0) &= \sum_{\substack{1\le k\le M \\ k\equiv 0 \bmod mN}} k^{-2}\frac{|\Delta|}{\phi(mN)}\phi(k) \\ &= \frac{6}{\pi^2}\frac{|\Delta|}{\phi(mN)[\Gamma(1):\Gamma_0(mN)]}\log M + \mathcal{O}(1) \\ &= \frac{6}{\pi^2[\overline{\Gamma(1)}:\overline{\Gamma}]}\log M + \mathcal{O}(1), \end{aligned}$$

where the implicit $\mathcal{O}$-constant is absolute, i.e., it is independent of $\Gamma$ and $M$.

Therefore we get

$$\begin{aligned} \sum_{\substack{1\le h\le k\le M \\ (h,k)=1 \\ h\equiv 0 \bmod mN \\ \overline{h}\in\overline{a}\Delta}} k^{-2}F_f\left(\frac{dh}{fk}\right) &= \frac{12d}{f[\overline{\Gamma(1)}:\overline{\Gamma}]}\log M + \mathcal{O}\left(\frac{d}{f}\right) \\ &\quad + \mathcal{O}\left(\sum_{v\in\mathbb{Z}-\{0\}}\frac{\sigma_1(d|v|/f)}{|v|}e^{-2\pi|v|nt/df}\right), \end{aligned}$$

where the implicit $\mathcal{O}$-constants depend only on $\Gamma$ and $t$. Since $f \mid (a,d)$ and $(a,d)\mid a = n/d$, we have $df \le n$; hence $1\le t_0\le t$ implies that

$$e^{-2\pi(|v|-1)nt/df} \le e^{-2\pi(|v|-1)t} \le e^{-2\pi(|v|-1)}.$$

By putting

$$C_1 = \sum_{v\in\mathbb{Z}-\{0\}}\frac{\sigma_1(|v|)}{|v|}e^{-2\pi(|v|-1)}$$

and using the fact

$$\sigma_1\left(\frac{d}{f}|v|\right) \le \sigma_1\left(\frac{d}{f}\right)\sigma_1(|v|),$$

we obtain

$$\sum_{v\in\mathbb{Z}-\{0\}}\frac{\sigma_1(d|v|/f)}{|v|}e^{-2\pi|v|nt/df} \le C_1\sigma_1\left(\frac{d}{f}\right)e^{-2\pi nt/df} \le C_1\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}.$$

Thus we deduce

$$\mathcal{O}\left(\sum_{v\in\mathbb{Z}-\{0\}}\frac{\sigma_1(d|v|/f)}{|v|}e^{-2\pi|v|nt/df}\right) = \mathcal{O}\left(\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}\right),$$

where the implicit $\mathcal{O}$-constant depends only on $\Gamma$.

Since $M = [\sqrt{d^2/(nt)}]$ and $1 \le t_0 \le t \le t_1$, we see that $\log M = (1/2)\log(d^2/n) + \mathcal{O}(1)$ where the implicit $\mathcal{O}$-constant depends only on $t_0$ and $t_1$.

Consequently, we have

$$\sum_{\substack{1\le h\le k\le M \\ (h,k)=1 \\ k\equiv 0 \bmod mN \\ \overline{h}\in\overline{a}\Delta}} k^{-2} F_f\left(\frac{dh}{fk}\right) = \frac{6d}{f[\overline{\Gamma(1)} : \overline{\Gamma}]}\log\left(\frac{d^2}{n}\right) + \mathcal{O}\left(\frac{d}{f}\right) + \mathcal{O}\left(\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}\right)$$

where the implicit $\mathcal{O}$-constants depend only on $\Gamma$, $t_0$ and $t_1$. By substituting this for the sum of $S_d(t)$ we obtain

$$S_d(t) = \sum_{f|(a,d)} \mu(f)\left(\frac{6d}{f[\overline{\Gamma(1)} : \overline{\Gamma}]}\log\left(\frac{d^2}{n}\right) + \mathcal{O}\left(\frac{d}{f}\right) + \mathcal{O}\left(\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}\right)\right) + \mathcal{O}\left(\frac{d\sigma_1((a,d))}{(a,d)}\right),$$

where the implicit $\mathcal{O}$-constants depend only on $\Gamma$, $f$, $t_0$ and $t_1$. Since

$$\sum_{f|(a,d)}\left|\mu(f)\frac{d}{f}\right| \le \sum_{f|(a,d)}\frac{d}{f} = \frac{d\sigma_1((a,d))}{(a,d)},$$

the first error term contributes $\mathcal{O}(d\sigma_1((a,d))/(a,d))$.

Similarly, since $\sigma_1(df/(a,d)) \le \sigma_1(d/(a,d))\sigma_1(f)$ and $e^{-2\pi nf/d(a,d)} \le e^{-2\pi f}$, we derive

$$\begin{aligned}\sum_{f|(a,d)}\left|\mu(f)\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}\right| &\le \sum_{f|(a,d)}\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df} = \sum_{f|(a,d)}\sigma_1\left(\frac{df}{(a,d)}\right)e^{-2\pi nf/d(a,d)} \\ &\le \sigma_1\left(\frac{d}{(a,d)}\right)\sum_{f|(a,d)}\sigma_1(f)e^{-2\pi f},\end{aligned}$$

and so the second error term contributes $\mathcal{O}(\sigma_1(d/(a,d)))$. From the fact $\phi((a,d)) = \sum_{f|(a,d)}\mu(f)(a,d)/f$ we finally obtain

$$S_d(t) = \frac{6}{[\overline{\Gamma(1)} : \overline{\Gamma}]}\frac{d}{(a,d)}\phi((a,d))\log\left(\frac{d^2}{n}\right) + \mathcal{O}\left(\sigma_1\left(\frac{d}{(a,d)}\right)\right) + \mathcal{O}\left(\frac{d\sigma_1((a,d))}{(a,d)}\right),$$

where the implicit $\mathcal{O}$-constants depend only on $\Gamma$, $f$, $t_0$ and $t_1$. This completes the proof. □

**Lemma 3.11.** *Let $f$ be on $\langle \Gamma_0(N), W_e \rangle_{e \in S}$.*
(1) *If $d < \sqrt{nt}$, then $S_d = \mathcal{O}(n/d)$. Here the implicit $\mathcal{O}$-constant depends only upon $f$, $t_0$ and $t_1$.*
(2) *If $d \geq \sqrt{nt}$, then*

$$S_d = \sum_{e \in S} \frac{1}{[\Gamma(1) : \Gamma_0(N/e)]} \frac{6d}{(a, d)} \phi((a, d)) \log\left(\frac{d^2}{n}\right) + \mathcal{O}\left(\sigma_1\left(\frac{d}{(a, d)}\right)\right) + \mathcal{O}\left(\frac{d\sigma_1((a, d))}{(a, d)}\right).$$

*Here the implicit $\mathcal{O}$-constant depends only upon $\langle \Gamma_0(N), W_e \rangle_{e \in S}$, $f$, $t_0$ and $t_1$.*

Proof. It is possible for us to prove (1) with similar arguments as in Lemma 3.10, so we omit the detail. We only give a proof of (2). By using Lemma 3.8 we have

$$\begin{aligned} S_d(t) &= \sum_{k=1}^{M} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} \log \max\left\{1, \left| f\left(\frac{ait + b}{d}\right) \right| \right\} \\ &= \sum_{e \in S} \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \bmod N/e \\ \overline{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} \sum_{\substack{b/d \in I_M(h/k) \\ 0 \leq b < d \\ (a,b,d)=1}} g_{h,k}\left(\frac{b}{d}\right) + \mathcal{O}(d \cdot s), \end{aligned}$$

where $M = [d/\sqrt{nt}]$ and $s$ is the number of Hall divisors in $S$. From [5, Lemma 6] and (3.4) we can see that

$$\begin{aligned} S_d(t) &= \sum_{e \in S} \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \bmod N/e \\ \overline{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} \left\{ k^{-2} \sum_{f | (a,d)} \mu(f) F_f\left(\frac{dh}{fk}\right) + \mathcal{O}\left(\frac{\sqrt{n}\sigma_1((a, d))}{k(a, d)}\right) \right\} + \mathcal{O}(d \cdot s) \\ &= \sum_{e \in S} \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \bmod N/e \\ \overline{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} \left\{ k^{-2} \sum_{f | (a,d)} \mu(f) F_f\left(\frac{dh}{fk}\right) + \mathcal{O}\left(\frac{\sqrt{n}\sigma_1((a, d))}{k(a, d)}\right) \right\} + \mathcal{O}(d \cdot s) \\ &= \sum_{e \in S} \sum_{f | (a,d)} \mu(f) \sum_{\substack{1 \leq h \leq k \leq M \\ (h,k)=1 \\ k \equiv 0 \bmod N/e \\ \overline{h} \in (\mathbb{Z}/(N/e)\mathbb{Z})^\times}} k^{-2} F_f\left(\frac{dh}{fk}\right) + \mathcal{O}\left(\frac{d\sigma_1((a, d))}{(a, d)} \cdot s\right). \end{aligned}$$

As we did in (3.5) we change the inner summand as

$$\sum_{\substack{1\le h\le k\le M\\(h,k)=1\\k\equiv 0 \bmod N/e\\ \overline{h}\in(\mathbb{Z}/(N/e)\mathbb{Z})^\times}} k^{-2}F_f\left(\frac{dh}{fk}\right)=\frac{2\pi^2 d}{f}\sum_{v\in\mathbb{Z}} C_M\left(\frac{dv}{f}\right)e^{-2\pi|v|nt/df},$$

where

$$C_M(l)=\sum_{\substack{1\le k\le M\\k\equiv 0 \bmod N/e}} k^{-2}c_k(l)$$

and

$$c_k(l)=\sum_{\substack{1\le h\le k\le M\\(h,k)=1\\ \overline{h}\in(\mathbb{Z}/(N/e)\mathbb{Z})^\times}} e^{2\pi i hl/k} \quad \text{for any} \quad l\in\mathbb{Z}.$$

Then, by Lemma 3.9 we know that $|c_k(l)|\le\phi(N/e)\cdot(N/e)\cdot(k,l)$ for $l\in\mathbb{Z}-\{0\}$. So when $l\neq 0$, we have

$$|C_M(l)|=\mathcal{O}\left(\frac{\sigma_1(|l|)}{|l|}\right),$$

where the implicit $\mathcal{O}$-constant depends only on $\langle\Gamma_0(N),W_e\rangle_{e\in S}$. When $l=0$, we obtain $c_k(0)=\phi(k)$. Hence it follows from [1, Lemma D.1] that

$$\begin{aligned}C_M(0)&=\sum_{\substack{1\le k\le M\\k\equiv 0 \bmod N/e}} k^{-2}\phi(k)\\&=\frac{6}{\pi^2}\frac{1}{[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}\log M+\mathcal{O}(1),\end{aligned}$$

where the implicit $\mathcal{O}$-constant is absolute, namely it is independent of $\langle\Gamma_0(N),W_e\rangle_{e\in S}$ and $M$.

Therefore we get

$$\begin{aligned}&\sum_{\substack{1\le h\le k\le M\\(h,k)=1\\k\equiv 0 \bmod N/e\\ \overline{h}\in(\mathbb{Z}/(N/e)\mathbb{Z})^\times}} k^{-2}F_f\left(\frac{dh}{fk}\right)\\&=\frac{12d}{f[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}\log M+\mathcal{O}\left(\frac{d}{f}\right)+\mathcal{O}\left(\sum_{v\in\mathbb{Z}-\{0\}}\frac{\sigma_1(d|v|/f)}{|v|}e^{-2\pi|v|nt/df}\right),\end{aligned}$$

where the implicit $\mathcal{O}$-constants depend only on $\langle\Gamma_0(N), W_e\rangle_{e\in S}$ and $t$. Applying the same estimates as in the proof of Lemma 3.10 we have

$$\begin{aligned}&\sum_{\substack{1\le h\le k\le M\\ (h,k)=1\\ k\equiv 0 \bmod N/e\\ \overline{h}\in(\mathbb{Z}/(N/e)\mathbb{Z})^\times}} k^{-2}F_f\left(\frac{dh}{fk}\right)\\ &= \frac{6d}{f[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}\log\left(\frac{d^2}{n}\right)+\mathcal{O}\left(\frac{d}{f}\right)+\mathcal{O}\left(\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}\right),\end{aligned}$$

where the implicit $\mathcal{O}$-constants depend only on $\langle\Gamma_0(N), W_e\rangle_{e\in S}$, $t_0$ and $t_1$. By plugging this into the sum of $S_d(t)$ we achieve

$$\begin{aligned}S_d(t) &= \sum_{e\in S}\sum_{f|(a,d)}\mu(f)\left(\frac{6d}{f[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}\log\left(\frac{d^2}{n}\right)+\mathcal{O}\left(\frac{d}{f}\right)+\mathcal{O}\left(\sigma_1\left(\frac{d}{f}\right)e^{-2\pi n/df}\right)\right)\\ &\quad+\mathcal{O}\left(\frac{d\sigma_1((a,d))}{(a,d)}\cdot s\right),\end{aligned}$$

where the implicit $\mathcal{O}$-constants depend only on $\langle\Gamma_0(N), W_e\rangle_{e\in S}$, $f$, $t_0$ and $t_1$. Thus, in like manner as in the proof of Lemma 3.10 we finally conclude

$$\begin{aligned}S_d(t) &= \sum_{e\in S}\frac{6}{[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}\frac{d}{(a,\,d)}\phi((a,\,d))\log\left(\frac{d^2}{n}\right)\\ &\quad+\mathcal{O}\left(\sigma_1\left(\frac{d}{(a,\,d)}\right)\cdot s\right)+\mathcal{O}\left(\frac{d\sigma_1((a,\,d))}{(a,\,d)}\cdot s\right),\end{aligned}$$

where the implicit $\mathcal{O}$-constants depend only on $\langle\Gamma_0(N), W_e\rangle_{e\in S}$, $f$, $t_0$ and $t_1$. The number $s$ depends only on the group $\langle\Gamma_0(N), W_e\rangle_{e\in S}$, and so we have the assertion. □

**Lemma 3.12.** *For* $1\le t_0\le t\le t_1$,
(1) *if* $f(z)$ *is on* $\Gamma$, *then we have*

$$h(\Phi_n^f(X,\ f(it))) = \frac{6\psi(n)}{[\overline{\Gamma(1)}:\overline{\Gamma}]}\left(\log n - 2\sum_{p|n}\frac{\log p}{p}+\mathcal{O}(1)\right),$$

(2) *if* $f(z)$ *is on* $\langle\Gamma_0(N), W_e\rangle_{e\in S}$, *then we achieve*

$$h(\Phi_n^f(X,\ f(it))) = \sum_{e\in S}\frac{6\psi(n)}{[\overline{\Gamma(1)}:\overline{\Gamma_0(N/e)}]}\left(\log n - 2\sum_{p|n}\frac{\log p}{p}+\mathcal{O}(1)\right).$$

*In case* (1) (*resp.,* (2)) *the implicit* $\mathcal{O}$*-constant depends only on* $\Gamma$ (*resp.,* $\langle\Gamma_0(N), W_e\rangle_{e\in S}$), $f$, $t_0$ *and* $t_1$.

Proof. In each case we are able to use the same method. Thus we put down only the first case. From Lemma 3.10 we know that

$$h(\Phi_n^f(X,\ f(it))) = \sum_{\substack{a>0\\ ad=n}} S_d(t) + \mathcal{O}(\psi(n)) = H_1 + H_2 + \mathcal{O}(\psi(n)),$$

where

$$H_1 = \sum_{\substack{a>0,\ ad=n\\ d<\sqrt{nt}}} S_d(t) = \mathcal{O}(\psi(n))$$

and

$$H_2 = \sum_{\substack{a>0,\ ad=n\\ d\geq\sqrt{nt}}} S_d(t) = \frac{6\psi(n)}{[\overline{\Gamma(1)} : \overline{\Gamma}]}\left(\log n - 2\sum_{p|n}\frac{\log p}{p} + \mathcal{O}(1)\right)$$

by means of Cohen's results in [5, §4]. □

**Lemma 3.13.** *Let $P(X) \in \mathbb{C}[X]$ be any nonzero polynomial of degree $\leq D$. Then for any $\theta > 0$, there exists an absolute constant $c_\theta > 0$, depending only on $\theta$, such that*

$$\left|(h(P(X)) - \log \sup_{\theta\leq x\leq 2\theta} |P(x)|\right| \leq c_\theta D.$$

Proof. We refer to [1] or [5]. □

Now we are ready to prove our main theorem.

Proof of Theorem 2.1. To avoid troublesome, we define $h(0) = -\infty$. Let $D = \psi(n)$ and we write

$$\Phi_n^f(X, Y) = P_0(Y)X^D + P_1(Y)X^{D-1} + \cdots + P_D(Y)$$

with $P_j(Y) \in \mathbb{C}[Y]$ and $P_0(Y) \neq 0$. Certainly, $h(\Phi_n^f(X,Y)) = \max_{0\leq j\leq D} h(P_j(Y))$. Since $\deg P_j(Y) \leq D$, Lemma 3.13 yields that

$$\begin{aligned} h(\Phi_n^f(X, Y)) &= \max_{0\leq j\leq D} \log \sup_{s\leq y\leq 2s} |P_j(y)| + \mathcal{O}(D) \\ &= \sup_{s\leq y\leq 2s} \max_{0\leq j\leq D} \log |P_j(y)| + \mathcal{O}(D) \end{aligned}$$

where the implicit $\mathcal{O}$-constant depends only on $s$. Since $\max_{0\leq j\leq D} \log|P_j(y)| = h(\Phi_n^f(X, y))$, we obtain

$$h(\Phi_n^f(X, Y)) = \sup_{s\leq y\leq 2s} h(\Phi_n^f(X, y)) + \mathcal{O}(D).$$

Here we note that the interval $[t_0, t_1]$ corresponds bijectively to the interval $[s, 2s]$, and so we have

$$h(\Phi_n^f(X, Y)) = \sup_{t_0 \le t \le t_1} h(\Phi_n^f(X, f(it))) + \mathcal{O}(D).$$

Therefore, we get the conclusion by Lemma 3.12. □

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## References

[1] B. Cais and B. Conrad: *Modular curves and Ramanujan's continued fraction*, J. Reine Angew. Math. **597** (2006), 27–104.
[2] I. Chen and N. Yui: *Singular values of Thompson series*; in Groups, Difference Sets, and the Monster (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ. **4**, de Gruyter, Berlin, 1996, 255–326.
[3] B. Cho, N.M. Kim and J.K. Koo: *Affine models of the modular curves $X(p)$ and its application*, submitted.
[4] B. Cho, J.K. Koo and Y.K. Park: *Arithmetic of the Ramanujan-Göllnitz-Gordon continued fraction*, submitted.
[5] P. Cohen: *On the coefficients of the transformation polynomials for the elliptic modular function*, Math. Proc. Cambridge Philos. Soc. **95** (1984), 389–402.
[6] D.A. Cox: Primes of the form $x^2 + ny^2$, Wiley, New York, 1989.
[7] K. Harada: Moonshine of Finite Groups, Lecture Notes of The Ohio State University, 1987.
[8] N. Ishida and N. Ishii: *The equations for modular function fields of principal congruence subgroups of prime level*, Manuscripta Math. **90** (1996), 271–285.
[9] D. Jeon and C.H. Kim: *On the arithmetic of certain modular curves*, preprint.
[10] S. Lang: Elliptic Functions, second edition, Springer, New York, 1987.
[11] G. Shimura: Introduction to the Arithmetic Theory of Automorphic Functions, Princeton University Press, 1971.

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