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COLORED ALEXANDER INVARIANTS AND CONE-MANIFOLDS

Dedicated to Professor Noriaki Kawanaka on his sixtieth birthday

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Abstract

In this paper, we reconstruct the link invariant of framed links introduced in [1] by the universal R-matrix of $\mathcal{U}_q(sl_2)$ and name it the colored Alexander invariant. We check that the optimistic limit o-lim of this invariant is determined by the volume of the knot and link cone-manifold for figure eight knot, Whitehead link and Borromean rings. We also propose the A-polynomials of these examples obtained from the colored Alexander invariant.

1. Introduction

New link invariants are introduced in [1] for colored links. They are defined for each positive integer N and considered as a generalization of the multivariable Alexander polynomial [12], which corresponds to the case $N = 2$. Here we redefine these invariants by using the universal R-matrix of $\mathcal{U}_q(sl_2)$.

Let $q = \exp(\pi\sqrt{-1}/N)$ be a $2N$-th root of unity. Let $\mathcal{U}_q(sl_2)$ be the quantum enveloping algebra corresponding to the Lie algebra sl_2 defined by the following generators and relations:

$$\mathcal{U}_q(sl_2) = \left\langle K,\ K^{-1},\ E,\ F \;\middle|\; KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\right.$$
$$\left. [E,\ F] = \frac{K - K^{-1}}{q - q^{-1}} \right\rangle.$$

$\mathcal{U}_q(sl_2)$ also has a hopf algebra structure with the following coproduct Δ.

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F.$$

The N-dimensional irreducible representation of $\mathcal{U}_q(sl_2)$ at $q = \exp(\pi\sqrt{-1}/N)$ admits central deformation parametrized by the scalar α correponding to the central element K^{2N}.

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Instead of α, we use parameter λ satisfying $\alpha = q^{N\lambda}$. Let $(\rho_\lambda, V_\lambda)$ be the corresponding representation

$$\rho_\lambda : \mathcal{U}_q(sl_2) \to \mathrm{End}(V_\lambda).$$

Then ρ_λ is isomorphic to the irreducible representation of the highest weight λ if λ is not an integer.

The R-matrix G in [1] is compatible with the action of $\mathcal{U}_q(sl_2)$ on $V_\lambda \otimes V_\mu$ and $V_\mu \otimes V_\lambda$, i.e.

$$\Delta(x)G = G\Delta(x) \colon V_\lambda \otimes V_\mu \to V_\mu \otimes V_\lambda.$$

We modify the above R-matrix G so that it coincides with the representation of the universal R-matrix when λ is an integer. By using this modified R-matrix, we construct a framed link invariant which we call the *colored Alexander invariant*, and investigate its relation to the hyperbolic volume of the knot and link cone-manifolds. Let $M_{\theta_1,\theta_2,\dots,\theta_k}$ be the cone manifold obtained from the link L with the cone angle θ_i along the i-th component of L. Then we expect the following.

Naive expectation (Volume conjecture for the colored Alexander invariant). If $M_{\theta_1,\theta_2,\cdots,\theta_k}$ is a hyperbolic cone manifold,

$$\mathrm{Vol}(M_{\theta_1,\theta_2,\cdots,\theta_k}) = \lim_{N\to\infty} \frac{2\pi \log|\Phi_L^N(N\theta_1/(2\pi), N\theta_2/(2\pi), \dots, N\theta_k/(2\pi))|}{N}.$$

If $M_{\theta_1,\theta_2,\dots,\theta_k}$ is a spherical cone manifold,

$$\Phi_L^N\left(\frac{N\theta_1}{2\pi}, \frac{N\theta_2}{2\pi}, \dots, \frac{N\theta_k}{2\pi}\right) \underset{N\to\infty}{\sim} \exp \frac{N\sqrt{-1}}{2\pi} \mathrm{Vol}(M_{\theta_1,\theta_2,\dots,\theta_k}).$$

It is known that the above does not hold if one of θ_i's is rational with respect to π. But if we use the notion of optimistic limit o-lim introduced in [9], we would declare the following conjecture.

Conjecture. *If $M_{\theta_1,\theta_2,\dots,\theta_k}$ is a hyperbolic cone manifold,*

$$\mathrm{Vol}(M_{\theta_1,\theta_2,\dots,\theta_k}) = \underset{N\to\infty}{\text{o-lim}}\, 2\pi \frac{\log|\Phi_L^N(N\theta_1/(2\pi), N\theta_2/(2\pi), \dots, N\theta_k/(2\pi))|}{N}.$$

If $M_{\theta_1,\theta_2,\dots,\theta_k}$ is a spherical cone manifold,

$$\mathrm{Vol}(M_{\theta_1,\theta_2,\cdots,\theta_k}) = \underset{N\to\infty}{\text{o-lim}}\, 2\pi \frac{\mathrm{Im}(\log \Phi_L^N(N\theta_1/(2\pi), N\theta_2/(2\pi), \dots, N\theta_k/(2\pi)))}{N}.$$

The optimistic limit o-lim means to apply the saddle point method formally as in [5], [9] and [10]. This conjecture would be proved by applying Yokota's theory developed in [13] to the colored Alexander invariant. However, in this paper, we show some examples instead of proving the conjecture. Such relation is first observed in [11] by considering the deformation of the parameter q. Here, instead of q, we deform the highest weights corresponding to the components of a link, and this deformation can be accomplished independently for each component while the perturbation of q gives a simultaneous deformation for the all components.

As an application, a system of A-polynomials for a multi-component link is proposed at the end of the paper.

2. *R*-matrix

2.1. Representation of $\mathcal{U}_q(sl_2)$. We first describe the highest weight representation $(\rho_\lambda, V_\lambda)$ of $\mathcal{U}_q(sl_2)$. Let $\{v_0^\lambda, v_1^\lambda, \dots, v_{N-1}^\lambda\}$ be the basis of V_λ, and the actions of K, E, F are given by the following:

$$\begin{aligned} E v_i^\lambda &= \frac{\{\lambda - i + 1\}}{\{1\}} v_{i-1}^\lambda, \ (i \geq 1), \quad E v_0^\lambda = 0, \\ F v_i^\lambda &= \frac{\{i+1\}}{\{1\}} v_{i+1}^\lambda, \ (i \leq N-1), \quad F v_{N-1}^\lambda = 0, \\ K v_i^\lambda &= q^{\lambda - 2i} v_i^\lambda, \end{aligned}$$

where $\{a\} = q^a - q^{-a}$. We also use the notations $\{x; n\} = \prod_{i=0}^{n-1}\{x - i\}$ for positive integer n and $\{x; 0\} = 1$.

2.2. Universal *R*-matrix. Let R_u be the universal R-matrix R_u of $\mathcal{U}_q(sl_2)$ given as follows:

$$R_u = q^{\frac{1}{2}H \otimes H} \sum_{n=0}^{\infty} \frac{\{1\}^{2n}}{\{n; n\}} q^{n(n-1)/2} (E^n \otimes F^n),$$

where H is an element such that $q^H = K$. H is not an element of $\mathcal{U}_q(sl_2)$, but we define the action of H toV_λ so that $q^H = K$, i.e.

$$H v_i^\lambda = (\lambda - 2i) v_i^\lambda.$$

Similarly,

$$q^{\frac{1}{2}H \otimes H} v_i^\lambda \otimes v_j^\mu = q^{\frac{1}{2}(\lambda - 2i)(\mu - 2j)} v_i^\lambda \otimes v_j^\mu.$$

Let R be the R-matrix corresponding to the braid generator

$$R = R_u P,$$

where P is the permutation

$$P(x \otimes y) = y \otimes x.$$

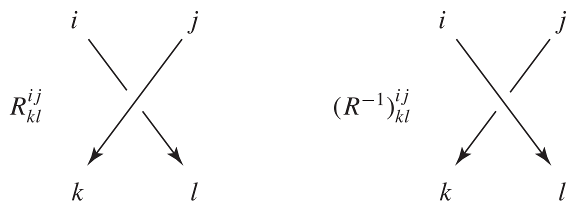


Fig. 1. R-matrices at positive and negative crossings.

Since R_u is the universal R-matrix, R satisfies the braid relation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \tag{1}$$

2.3. Representation of the *R*-matrix. The R-matrix R defined in the previous section gives a mapping $V_\lambda \otimes V_\mu \to V_\mu \otimes V_\lambda$ as follows.

$$\begin{aligned} &R \cdot (v_i^\lambda \otimes v_j^\mu) \\ &= \sum_n q^{\frac{1}{2}(\lambda-2i-2n)(\mu-2j+2n)+n(n-1)/2} \frac{\{i+n;n\}\{\mu-j+n;n\}}{\{n;n\}} (v_{j-n}^\mu \otimes v_{i+n}^\lambda), \\ &R^{-1} \cdot (v_i^\lambda \otimes v_j^\mu) \\ &= \sum_n (-1)^n q^{-\frac{1}{2}(\lambda-2i)(\mu-2j)-n(n-1)/2} \frac{\{j+n;n\}\{\lambda-i+n;n\}}{\{n;n\}} (v_{j+n}^\mu \otimes v_{i-n}^\lambda). \end{aligned} \tag{2}$$

In the following, R_{kl}^{ij} means the matrix element of R, i.e.

$$R \cdot (v_i^\lambda \otimes v_j^\mu) = \sum_{k,l} R_{kl}^{ij} (v_k^\mu \otimes v_l^\lambda),$$

and R, R^{-1} correspond to the crossings as in Fig. 1.

2.4. *R*-matrix in [1]. We introduce some symbols for q-analogues.

$$(z;n)_q = \prod_{j=0}^{n-1} (1 - zq^j),$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \begin{cases} \dfrac{(q;m)_q}{(q;m-n)_q (q;n)_q} & \text{for } m-n \geq 0, \\ 0 & \text{for } m-n < 0, \end{cases}$$

$$\begin{pmatrix} m \\ n \end{pmatrix}_{z,q} = \frac{(z;m)_q}{(z;n)_q} \quad \text{for } m,\ n \geq 0.$$

Let $G^{ab}_{cd}(\alpha, \beta; \pm)$ be the R-matrix given in [1],

$$G^{ab}_{cd}(\alpha, \beta; +) = \begin{bmatrix} a \\ d \end{bmatrix}_{q^2} \begin{pmatrix} c \\ b \end{pmatrix}_{\beta, q^2} \beta^d q^{2bd} \alpha^{sb+tc} \beta^{-sd-ta} \times f(\alpha, \beta, q^2) \frac{F(\alpha, a)F(\beta, b)}{F(\alpha, d)F(\beta, c)} q^{u(a+d-b-c)+v(ab-cd)},$$

$$G^{ab}_{cd}(\alpha, \beta; -) = \begin{bmatrix} b \\ c \end{bmatrix}_{1/q^2} \begin{pmatrix} d \\ a \end{pmatrix}_{1/\beta, 1/q^2} \beta^{-c} q^{-2ac} \beta^{sb+tc} \alpha^{-sd-ta} \times f(\alpha, \beta, q^2)^{-1} \frac{F(\alpha, b)F(\beta, a)}{F(\alpha, c)F(\beta, d)} q^{u(a+d-b-c)+v(ab-cd)},$$

where f, F are arbitrary functions, s, t, u, v are arbitrary numbers, and α, β are the parameters for the under and the over paths respectively. Note that

$$\begin{bmatrix} i+n \\ i \end{bmatrix}_{q^2} = q^{in} \frac{\{i+n; n\}}{\{n; n\}},$$

$$\begin{pmatrix} j \\ j-n \end{pmatrix}_{q^{-2\mu}, q^2} = (-1)^n q^{jn-(\mu+1)n-\frac{1}{2}n(n-1)} \{j-1-\mu; n\} = q^{jn-(\mu+1)n-\frac{1}{2}n(n-1)} \{\mu - j + n; n\}.$$

Theorem 2. *The above R-matrix is equal to $G^{lk}_{ji}(\alpha, \beta; \pm)$ as follows*:

$$R^{ij}_{kl} = G^{lk}_{ji}(q^{-2\lambda}, q^{-2\mu}; +), \quad (R^{-1})^{ij}_{kl} = G^{lk}_{ji}(q^{-2\mu}, q^{-2\lambda}; -),$$

where the arbitrary parameters and functions are fixed as follows.

$$s = \frac{1}{2}, \quad t = 0, \quad u = 0, \quad v = 1, \quad F(\alpha, i) = 1, \quad f(q^{-2\lambda}, q^{-2\mu}, q^2) = q^{(1/2)\lambda\mu}.$$

Proof. Compare R^{ij}_{kl} in (2) and $G^{lk}_{ji}(q^{-2\lambda}, q^{-2\mu}; +)$ by putting $n = l - i = j - k$.

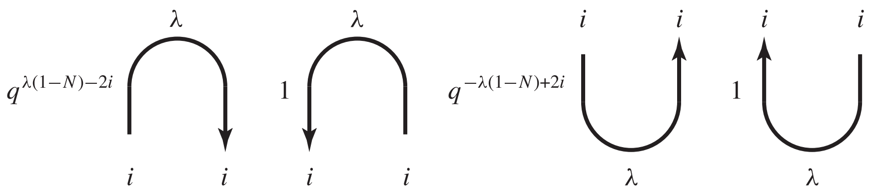
$$\begin{aligned} &G^{lk}_{ji}(q^{-2\lambda}, q^{-2\mu}; +) \\ &= q^{\frac{1}{2}\lambda\mu+in+jn-(\mu+1)n-(1/2)n(n-1)-2\mu i+2i(j-n)-\lambda(j-n)+\mu i+(i+n)(j-n)-ij} \frac{\{i+n; n\}\{\mu - j + n; n\}}{\{n; n\}} \\ &= q^{\frac{1}{2}(\lambda-2i-2n)(\mu-2j+2n)+n(n-1)/2} \frac{\{i+n; n\}\{\mu - j + n; n\}}{\{n; n\}} \\ &= R^{ij}_{kl}. \end{aligned}$$

Proof for $(R^{ij}_{kl})^{-1}$ is similar. $\square$

REMARK 3. The action of $\mathcal{U}_q(sl_2)$ on V_λ and $V_{\lambda+2N}$ are the same one. However, our R-matrices concerning to λ and $\lambda+2N$ are different. This is the reason why we use the parameter λ instead of $\alpha = q^{N\lambda}$.

3. Colored Alexander invariants

3.1. Modified invariant. Let T be a (1, 1) tangle with parametrized components. Let λ_1 be the parameter corresponding to the open component and $\lambda_2, \dots, \lambda_k$ be the parameters for other components. These parameters are called the *colors* of the components. Let $O_T^N(\lambda_1, \dots, \lambda_k)$ be the operator in $\mathrm{End}(V_{\lambda_1})$ constructed by the state sum as in [12] obtained by assigning the R-matrices for crossings of T and the following scalars for maximal and minimal points of T.



Note that

$$\rho_\lambda(K^{-(N-1)})v_i^\lambda = q^{\lambda(1-N)-2i}v_i^\lambda, \quad \rho_\lambda(K^{(N-1)})v_i^\lambda = q^{-\lambda(1-N)+2i}v_i^\lambda.$$

Let

$$\Phi_T^N(\lambda_1, \lambda_2, \dots, \lambda_k) = \{\lambda_1 + N; N-1\}^{-1} O_T^N(\lambda_1, \lambda_2, \dots, \lambda_k),$$

where λ_1 is the color of the open component of T.

Theorem 4. $\Phi_T^N(\lambda_1, \lambda_2, \dots, \lambda_k)$ *is an invariant of a colored framed link L with colors $\lambda_1, \lambda_2, \dots, \lambda_k$ obtained by closing the tangle T.*

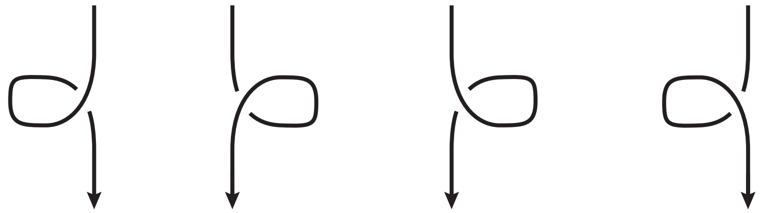
DEFINITION 5. We write $\Phi_L^N(\lambda_1, \lambda_2, \dots, \lambda_k)$ instead of $\Phi_T^N(\lambda_1, \lambda_2, \dots, \lambda_k)$ and call it the *colored Alexander invariant* of colored links.

Proof of Theorem 4. Let $\tilde{\Phi}_L^N(\lambda_1, \lambda_2, \dots, \lambda_k)$ be the function defined as above from $f^{-1}R$ instead of R, where f is the function in Theorem 2. Since the product of the f terms in $\tilde{\Phi}_L^N$ only depends on the framing and linking numbers of L, if $\tilde{\Phi}$ is a link invariant, then Φ is also a framed link invariant. By the way,

$$\alpha^{(N-1)/2}(\alpha; N-1)_{q^2}^{-1} = (-1)^{N-1} q^{-(N-1)(N-2)/2}\{\lambda_1 + N; N-1\}^{-1}$$

for $\alpha = q^{-2\lambda_1}$, where $(\alpha; k)_\omega = \prod_{j=0}^{k-1}(1-\alpha\omega^j)$. Therefore

$$\tilde{\Phi}_L^N(\lambda_1, \lambda_2, \dots, \lambda_k) = (-1)^{N-1} q^{(N-1)(N-2)/2}\hat{\Phi}_L(\lambda_1, \lambda_2, \dots, \lambda_k),$$



Positive twist of type I, type II, negative twist of type I, type II.

Fig. 2. Positive twists and negative twists.

where $\hat{\Phi}$ is the isotopy link invariant in Definition 5.4 of [1]. In the definition of $\hat{\Phi}$, the function f is normalized as $f(\alpha, \alpha; q^2) = \alpha^{-(N-1)/2}$ so that formulas (3.11) and (3.12) in [1] hold. In the definition of $\tilde{\Phi}$, we add the factor $\alpha^{-(N-1)/2} = q^{\lambda_1(N-1)}$ to the contribution of minimal and maximal point instead of the R-matrix so that the similar formulas hold. Therefore, Φ is a framed link invariant. □

3.2. Framing contribution. The only difference of our Φ and $\hat{\Phi}$ in [1] is that Φ depends on the framing of the link. We check for the four types of twist given in Fig. 2. We assume that the string is colored by λ. For the positive twist of type I, the corresponding scalar is obtained by computing the action to v_0^λ.

$$q^{\lambda(1-N)}q^{\frac{1}{2}\lambda^2} = q^{\lambda(\lambda+2-2N)/2}.$$

For the positive twist of type II, the corresponding scalar is obtained by computing the action to v_{N-1}^λ.

$$q^{-\lambda(1-N)+2(N-1)}q^{(\lambda-2N+2)(\lambda/2-N+1)} = q^{(\lambda^2+2\lambda-2\lambda N+4N^2-4N)/2} = q^{\lambda(\lambda+2-2N)/2}.$$

For the negative twist of type I, the corresponding scalar is obtained by computing the action to v_0^λ.

$$q^{-\lambda(1-N)}q^{-(1/2)\lambda^2} = q^{-\lambda(\lambda+2-2N)/2}.$$

For the negative twist of type II, the corresponding scalar is obtained by computing the action to v_{N-1}^λ.

$$q^{\lambda(1-N)-2(N-1)}q^{-(\lambda+2)(\lambda/2+1)} = q^{-\lambda(\lambda+2-2N)/2}.$$

Therefore, we have the following.

Proposition 6. *Let K_f and K_0 be ambient isotopic knots with framing f and* 0, *we have*

$$\Phi_{K_f}^N(\lambda) = q^{\frac{\lambda(\lambda+2-2N)}{2}f}\,\Phi_{K_0}^N(\lambda). \tag{3}$$

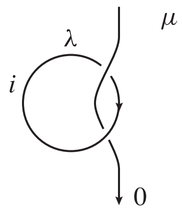


Fig. 3. Tangle for the Hopf link.

4. Examples

4.1. Hopf link. For the positive Hopf link H_+, we compute the colored Alexander invariant by using the tangle in Fig. 3.

$$\begin{aligned}
\Phi^N_{H_+}(\lambda, \mu) &= \{\mu + N; N-1\}^{-1} \sum_{i=0}^{N-1} q^{\lambda(1-N)-2i} q^{(\lambda/2-i)\mu} q^{\mu(\lambda/2-i)} \\
&= \{\mu + N; N-1\}^{-1} \sum_{i=0}^{N-1} q^{\lambda(\mu+1)-N\lambda-2(\mu+1)i} \\
&= \{\mu + N; N-1\}^{-1} q^{\lambda(\mu+1)-N\lambda} \frac{1-q^{-2N(\mu+1)}}{1-q^{-2(\mu+1)}} \\
&= \{\mu + N; N-1\}^{-1} q^{(\lambda+1)(\mu+1)-N(\lambda+\mu)} \frac{q^{N\mu}-q^{-N\mu}}{q^{\mu+1}-q^{-(\mu+1)}} \\
&= \{\mu + N; N\}^{-1} q^{(\lambda+1-N)(\mu+1-N)-N^2} 2\sqrt{-1}\sin \pi\mu \\
&= -\frac{\sqrt{-1}^N}{2\sin \pi\mu} q^{(\lambda+1-N)(\mu+1-N)} 2\sqrt{-1}\sin \pi\mu = \sqrt{-1}^{N-1} q^{(\lambda+1-N)(\mu+1-N)}.
\end{aligned}$$

Here we use the relation ([1], 1.392, 1)

$$\{\lambda + N; N\} = 2^N \sqrt{-1}^N \prod_{i=1}^{N} \sin\frac{(\lambda+i)\pi}{N} = 2\sqrt{-1}^N \sin(\lambda+1)\pi = -2\sqrt{-1}^N \sin\lambda\pi. \tag{4}$$

Therefore, we get

$$\Phi^N_{H_+}(\lambda, \mu) = \sqrt{-1}^{N-1} q^{(\lambda+1-N)(\mu+1-N)}. \tag{5}$$

Similarly, for negative Hopf link H_-, we have

$$\Phi^N_{H_-}(\lambda, \mu) = \sqrt{-1}^{-N+1} q^{-(\lambda+1-N)(\mu+1-N)}. \tag{6}$$

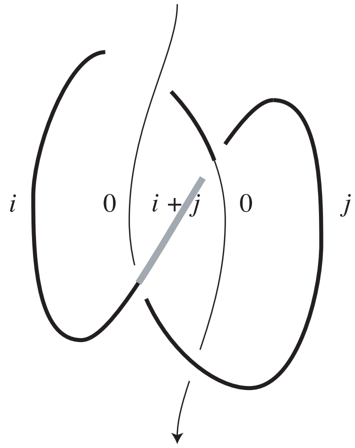


Fig. 4. Tangle for the figure eight knot.

4.2. Figure eight knot. For the figure eight knot 4_1, $\Phi^N_{4_1}(\lambda)$ is given as follows.

$$\begin{aligned}
\Phi^N_{4_1}(\lambda) &= \{\lambda+N; N-1\}^{-1} \sum_{i,j} q^{\lambda(1-N)-2i} q^{-\lambda(1-N)+2j} R^{i,0}_{0,i} \overline{R}^{i,j}_{i+j,0} R^{0,i+j}_{i,j} \overline{R}^{j,0}_{0,j} \\
&= \{\lambda+N; N-1\}^{-1} \sum_{i,j} q^{2(j-i)} q^{\lambda(\lambda/2-i)} \\
&\qquad \times (-1)^i q^{-\frac{1}{2}(\lambda-2i)(\lambda-2j)-i(i-1)/2} \frac{\{i+j; i\}\{\lambda; i\}}{\{i; i\}} \\
&\qquad \times q^{\frac{1}{2}(\lambda-2j)(\lambda-2i)+j(j-1)/2} \{\lambda-i; j\} \times q^{-\lambda(\lambda/2-j)} \\
&= \{\lambda+N; N-1\}^{-1} \sum_{i,j} (-1)^i q^{(j-i)(2\lambda+i+j+3)/2} \frac{\{i+j; i\}\{\lambda; i+j\}}{\{i; i\}} \\
&= \{\lambda+N; N-1\}^{-1} \sum_{0 \le j \le k \le N-1} (-1)^{k-j} q^{-3k/2-k^2/2-k\lambda+j(3+k+2\lambda)} \frac{\{k; i\}\{\lambda; k\}}{\{i; i\}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k = i+j).
\end{aligned}$$

By using Lemma A in Appendix, the above is equal to

$$\begin{aligned}
&\{\lambda+N; N-1\}^{-1} \sum_k (-1)^k q^{-3k/2-k^2/2-k\lambda} \prod_{j=1}^{k} (1-q^{4+2k+2\lambda-2j}) \{\lambda; k\} \\
&= \{\lambda+N; N-1\}^{-1} \sum_k \{\lambda+k+1; k\}\{\lambda; k\} = \{\lambda+N; N\}^{-1} \sum_k \{\lambda+k+1; 2k+1\} \\
&= -\frac{1}{2\sqrt{-1}^N \sin \lambda\pi} \sum_k \{\lambda+k+1; 2k+1\} \quad \text{(by (4))}.
\end{aligned}$$

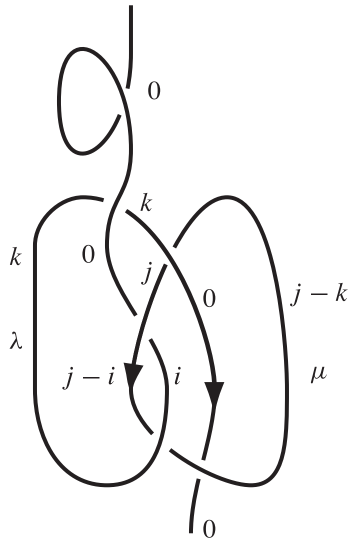


Fig. 5. Tangle of Whitehead link.

Therefore

$$\Phi^N_{4_1}(\lambda) = -\frac{1}{2\sqrt{-1}^N \sin \lambda\pi} \sum_{k=0}^{N-1} \{\lambda + k + 1; 2k + 1\}. \tag{7}$$

4.3. Whitehead link. Let K_W be the Whitehead link of framings 0 for both components. The colored Alexander invariant of K_W is given by

$$\begin{aligned}
&\Phi^N_{K_W}(\lambda, \mu) \\
&= q^{-((\lambda+1-N)^2-(1-N)^2)/2}\{\lambda + N; N - 1\}^{-1} \\
&\quad \times \sum_{0\le k\le i\le j\le N-1} q^{\lambda(1-N)-2k} q^{-\mu(1-N)+2(j-k)} q^{\lambda(\lambda/2-k)} q^{-\frac{1}{2}(\mu-2j+2k)\lambda} \\
&\qquad\qquad \times q^{((\lambda-2i)(\mu-2j+2i)+i(i-1))/2}\{\mu - j + i; i\} \\
&\qquad\qquad \times (-1)^k q^{-((\lambda-2k)(\mu-2j+2k)+k(k-1))/2} \frac{\{j; k\}\{\lambda; k\}}{\{k; k\}} \\
&\qquad\qquad \times q^{\frac{1}{2}(\mu-2j+2k)(\lambda-2k)+(i-k)(i-k-1)/2} \frac{\{j - k; i - k\}\{\lambda - k; i - k\}}{\{i - k; i - k\}} \\
&= \{\lambda + N; N - 1\}^{-1} \sum_{0\le i\le j\le N-1} q^{-(i+1)(i-2j)+\lambda i+\mu(N-i-1)}\{\lambda; i\}\{j; i\}\{\mu - j + i; i\} \\
&\qquad\qquad \times \sum_{0\le k\le i} (-1)^k q^{k(-i-2\lambda-3)} \frac{1}{\{i - k; i - k\}\{k; k\}}.
\end{aligned}$$

By using Lemma A in Appendix, the above is equal to

$$\sum_{0\le i\le j\le N-1} q^{-(i+1)(i-2j)+\lambda i+\mu(N-i-1)} \frac{\{\lambda;i\}\{j;i\}\{\mu-j+i;i\}}{\{i;i\}\{\lambda+N;N-1\}} \prod_{k=1}^{i}(1-q^{-2\lambda-2-2k})$$
$$= \sum_{0\le i\le j\le N-1} q^{-(i+1)(i-2j)+\lambda i+\mu(N-i-1)} \frac{\{\lambda;i\}\{j;i\}\{\mu-j+i;i\}}{\{i;i\}\{\lambda+N;N-1\}} \prod_{k=1}^{i} q^{-\lambda-1-k}(q^{\lambda+1+k}-q^{-(\lambda+1+k)})$$
$$= \sum_{0\le i\le j\le N-1} q^{-3i^2/2+2ji-\mu i-5i/2+2j-\mu+\mu N} \frac{\{\lambda;i\}\{j;i\}\{\mu-j+i;i\}\{\lambda+1+i;i\}}{\{i;i\}\{\lambda+N;N-1\}}.$$

Now, replacing j by $i+l$ and using Lemma B in Appendix, we have

$$\sum_{0\le i\le N-1} q^{i^2/2-\mu i-2Ni-i/2-\mu+\mu N} \frac{\{\lambda+i+1;2i+1\}}{\{i;i\}\{\lambda+N;N\}} \sum_{0\le l\le N-1-i} q^{2l(i+1)}\{l+i;i\}\{\mu-l;i\}$$
$$= \sum_{0\le i\le N-1} q^{i^2/2-\mu i-i/2-\mu+\mu N} \frac{\{\lambda+i+1;2i+1\}}{\{i;i\}\{\lambda+N;N\}}$$
$$\times\, q^{(i+1)(\mu-i)-N\mu} \frac{\{i;i\}^2\{\mu+i+1;2i+1\}}{\{\mu+N;N\}\{2i+1;2i+1-N\}}$$
$$= \frac{(-1)^N}{4\sin\pi\lambda\sin\pi\mu} \sum_{[N/2]\le i\le N-1} q^{-i^2/2-3i/2} \frac{\{i;i\}\{\lambda+1+i;2i+1\}\{\mu+i+1;2i+1\}}{\{2i+1;2i+1-N\}}.$$

Therefore

(8)
$$\Phi^N_{K_W}(\lambda,\mu) = \frac{(-1)^N}{4\sin\pi\lambda\sin\pi\mu} \sum_{[N/2]\le i\le N-1} q^{-i^2/2-3i/2} \frac{\{i;i\}\{\lambda+1+i;2i+1\}\{\mu+i+1;2i+1\}}{\{2i+1;2i+1-N\}}.$$

4.4. Borromean rings. Let K_B be the Borromean rings. Then

$$\Phi^N_{K_B}(\lambda,\mu,\nu)$$
$$= \{\mu+N;N-1\}^{-1}$$
$$\times \sum_{i+j\ge l\ge i,\, i+j\ge k\ge j} q^{\lambda(1-N)-2i-\nu(1-N)+2j}$$
$$\times\, R^{i,0}_{0,i}\, \overline{R}^{i,j}_{k,i+j-k}\, R^{0,k}_{i+j-l,k+l-i-j}\, \overline{R}^{k+l-i-j,i+j-k}_{l,0}\, R^{i+j-l,l}_{i,j}\, \overline{R}^{j,0}_{0,j}$$

$$
\begin{aligned}
&= \{\mu+N;N-1\}^{-1} \\
&\quad\times \sum_{i+j\ge l\ge i,\, i+j\ge k\ge j} q^{(N-1)(\nu-\lambda)-2i+2j} q^{\mu(\lambda/2-i)} q^{-\mu(\nu/2-j)} \\
&\qquad\qquad \times (-1)^{k-j} q^{-\frac{1}{2}(\lambda-2i)(\nu-2j)-(k-j)(k-j-1)/2} \\
&\qquad\qquad \times \frac{\{k;k-j\}\{\lambda-i-j+k;k-j\}}{\{k-j;k-j\}} \\
&\qquad\qquad \times q^{\frac{1}{2}(\mu+2i+2j-2k-2l)(\nu-2i-2j+2l)+(k+l-i-j)(k+l-i-j-1)/2} \\
&\qquad\qquad \times \{\nu-i-j+l;k+l-i-j\} \\
&\qquad\qquad \times (-1)^{k+l-i-j} q^{-\frac{1}{2}(\lambda-2i-2j+2k)(\mu+2i+2j-2k-2l)-(k+l-i-j)(k+l-i-j-1)/2} \\
&\qquad\qquad \times \frac{\{l;k+l-i-j\}\{\mu;k+l-i-j\}}{\{k+l-i-j;k+l-i-j\}} \\
&\qquad\qquad \times q^{\frac{1}{2}(\lambda-2i)(\nu-2j)+(l-i)(l-i-1)/2} \frac{\{j;l-i\}\{\lambda-i;l-i\}}{\{l-i;l-i\}} \\
&= \{\mu+N;N-1\}^{-1} \\
&\quad\times \sum_{i+j\ge l\ge i,\, i+j\ge k\ge j} (-1)^{l-i} q^{(N-1)(\nu-\lambda)} \\
&\qquad\qquad \times q^{i(i-3)/2-j(j-3)/2+k(3k+1)/2-l(3l+1)/2-2ik+il-jk+2jl+\lambda(-i-j+k+l)+\mu(-i+j-k+l)+\nu(i+j-k-l)} \\
&\qquad\qquad \times \frac{\{k;j\}\{\lambda-i-j+k;k+l-i-j\}\{\nu-i-j+l;k+l-i-j\}\{l;i\}\{\mu;k+l-i-j\}}{\{k+l-i-j;k+l-i-j\}\{i+j-k;i+j-k\}\{i+j-l;i+j-l\}}.
\end{aligned}
$$

By putting $i=l+r-s$ and $j=k-r$, we get

$$
\begin{aligned}
&\{\mu+N;N-1\}^{-1} \sum_{l\ge s,k\ge s} (-1)^s q^{N(\nu-\lambda)+(\lambda-\nu+2k-2l)(s+1)+\mu s+s(3+s)/2} \\
&\qquad\qquad \times \frac{\{k;s\}\{\lambda-l+s;s\}\{\nu-k+s;s\}\{l;s\}\{\mu;s\}}{\{s;s\}^2} \\
&\qquad\qquad \times \sum_{s\ge r} (-1)^r q^{-r(2\mu+s+3)} \frac{\{s;r\}}{\{r;r\}} \\
&= \{\mu+N;N-1\}^{-1} \sum_{k,l,s} (-1)^s q^{N(\nu-\lambda)+(\lambda-\nu+2k-2l)(s+1)+\mu s+s(3+s)/2} \\
&\qquad\qquad \times \frac{\{k;s\}\{\lambda-l+s;s\}\{\nu-k+s;s\}\{l;s\}\{\mu;s\}}{\{s;s\}^2} \\
&\qquad\qquad \times (1-q^{-2\mu-4})(1-q^{-2\mu-6})\cdots(1-q^{-2\mu-2s-2}) \\
&= \{\mu+N;N-1\}^{-1} \sum_{l\ge s,k\ge s} (-1)^s q^{(N-1)(\nu-\lambda)+(\lambda-\nu+2k-2l)(s+1)+\mu s+s(s+3)/2-s(2\mu+s+3)/2} \\
&\qquad\qquad \times \frac{\{k;s\}\{\lambda-l+s;s\}\{\nu-k+s;s\}\{l;s\}\{\mu;s\}}{\{s;s\}^2} \times \{\mu+s+1;s\}
\end{aligned}
$$

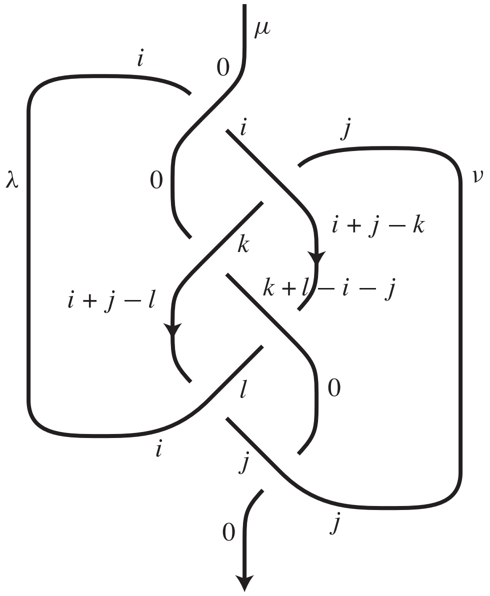


Fig. 6. Borromean rings.

$$
\begin{aligned}
&= \sum_{l \ge s, k \ge s} (-1)^s q^{N(\nu-\lambda)+(\lambda-\nu+2k-2l)(s+1)} \\
&\qquad \times \frac{\{k; s\}\{\lambda - l + s; s\}\{\nu - k + s; s\}\{l; s\}\{\mu + s + 1; 2s + 1\}}{\{\mu + N; N\}\{s; s\}^2} \\
&= \sum_s (-1)^s q^{N(\nu-\lambda)+(\lambda-\nu)(s+1)} \frac{\{\mu + s + 1; 2s + 1\}}{\{\mu + N; N\}\{s; s\}^2} \\
&\qquad \times \sum_{k \ge s} q^{2k(s+1)}\{k; s\}\{\nu - k + s; s\} \sum_{l \ge s} q^{-2l(s+1)}\{l; s\}\{\lambda - l + s; s\} \\
&= q^{N(\nu-\lambda)} \sum_s (-1)^s q^{(\lambda-\nu)(s+1)} \frac{\{\mu + s + 1; 2s + 1\}}{\{\mu + N; N\}\{s; s\}^2} \\
&\qquad \times \sum_k q^{2k(s+1)}\{k + s; s\}\{\nu - k; s\} \sum_l q^{-2l(s+1)}\{l + s; s\}\{\lambda - l; s\}.
\end{aligned}
$$

By using Lemma B in Appendix, we have

$$
\sum_k q^{2k(s+1)}\{k + s; s\}\{\nu - k; s\} = q^{(s+1)(\nu-s)-N\nu} \frac{\{s; s\}^2\{\nu + s + 1; 2s + 1\}}{\{\nu + N; N\}\{2s + 1; 2s + 1 - N\}}
$$

and

$$
\sum_l q^{-2l(s+1)}\{l + s; s\}\{\lambda - l; s\} = q^{-(s+1)(\lambda-s)+N\lambda} \frac{\{s; s\}^2\{\lambda + s + 1; 2s + 1\}}{\{\lambda + N; N\}\{2s + 1; 2s + 1 - N\}}.
$$

Hence

$$\begin{aligned}
&\Phi^N_{K_B}(\lambda,\mu,\nu)\\
&=\sum_s(-1)^s\frac{\{s;s\}^2\{\lambda+s+1;2s+1\}\{\mu+s+1;2s+1\}\{\nu+s+1;2s+1\}}{\{\lambda+N;N\}\{\mu+N;N\}\{\nu+N;N\}\{2s+1;2s+1-N\}^2}\\
&=\frac{-\sqrt{-1}^{\,-3N}}{8\sin\pi\lambda\sin\pi\mu\sin\pi\nu}\\
&\quad\times\sum_{[N/2]\le s\le N-1}(-1)^s\frac{\{s;s\}^2\{\lambda+s+1;2s+1\}\{\mu+s+1;2s+1\}\{\nu+s+1;2s+1\}}{\{2s+1;2s+1-N\}^2}.
\end{aligned}$$

Therefore

(9)

$$\begin{aligned}
&\Phi^N_{K_B}(\lambda,\mu,\nu)\\
&=\frac{\sqrt{-1}^{\,N}}{8\sin\pi\lambda\sin\pi\mu\sin\pi\nu}\\
&\quad\times\sum_{[N/2]\le s\le N-1}(-1)^{s+1}\frac{\{s;s\}^2\{\lambda+s+1;2s+1\}\{\mu+s+1;2s+1\}\{\nu+s+1;2s+1\}}{\{2s+1;2s+1-N\}^2}.
\end{aligned}$$

4.5. Colored Jones invariant. For integer λ, μ, ν, the colored Jones invariants $V_L(\lambda,\dots)$ of 4_1, K_W, K_B are given in [3].

$$\begin{aligned}
V_{4_1}(\lambda)&=\sum_{i=0}^{\lambda}\frac{\{\lambda+i+1;2i+1\}}{\{1\}},\\
V_{K_W}(\lambda)&=\sum_{i=0}^{\min(\lambda,\mu)}q^{-i^2/2-3i/2}\frac{\{\lambda+i+1;2i+1\}\{\mu+i+1;2i+1\}}{\{1\}\{2i+1;i+1\}},\\
V_{K_B}(\lambda)&=\sum_{i=0}^{\min(\lambda,\mu,\nu)}\frac{\{\lambda+i+1;2i+1\}\{\mu+i+1;2i+1\}\{\nu+i+1;2i+1\}}{\{1\}\{2i+1;i+1\}^2}.
\end{aligned}$$

Comparing with the colored Alexander invariants, the terms in the summations are similar, but the range of the summations are quite different, especially for the link case.

5. Volume of cone manifolds

5.1. Figure eight knot. Let 4_1 be the figure eight knot and let $(a;x)_n=\prod_{j=0}^{n-1}(1-ax^j)$. Then, from (7), we have

$$\Phi^N_{4_1}(\lambda)=\frac{-\sqrt{-1}^{\,-N}}{2\sin\lambda\pi}\sum_k q^{(2k+1)(\lambda+1)}(q^{-2\lambda-2k+2};q^2)_{2k+1}.$$

Let $\mathrm{Li}_2(z)$ be the analytic continuation of the following function:

$$\mathrm{Li}_2(x) = \int_0^x -\frac{\log(1-x)}{x}\,dx = \sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$

Then $\mathrm{Li}_2(z)$ is a multivalued function and its branches are given by

$$\mathrm{Li}_2(z) + 2\pi\sqrt{-1}p\log z + 4q\pi^2 \quad (p, q \in \mathbb{Z}).$$

In the rest of this paper, $\mathrm{Li}_2(z)$ means an appropriate branch of it. Following Kashaev's way in [7] (see also [10] for more examples) to consider the relation to the hyperbolic volume, let $V(a, z)$ be the following function.

$$V(a, z) = \pi\sqrt{-1}\log a + \log a \log z + \mathrm{Li}_2\left(\frac{1}{az}\right) - \mathrm{Li}_2\left(\frac{z}{a}\right). \tag{10}$$

Here a, z correspond to $q^{2\lambda}$, q^{2k} respectively, and the term $\pi\sqrt{-1}\log a$ is added so that the value of $V(a, z_0)$ in the following coincide with the volume not only for hyperbolic case but also for spherical case. Adding $\pi\sqrt{-1}\log a$ to V corresponds to multiplying $q^{N\lambda}$ to Φ. The optimistic limit introduced in [9] of the colored Alexander invariants of 4_1 with fixed a is given by $V(a, z_0)$ where z_0 is the solution of the following equation.

$$\frac{\partial V}{\partial z} = 0, \quad \text{i.e.} \quad z^2 - (a - 1 + a^{-1})z + 1 = 0.$$

The solution of this equation is given as follows. Let $a = \exp\sqrt{-1}\alpha$ and K_α be the figure eight knot cone-manifold with the cone angle α along the knot.

Hyperbolic case ($0 \le \alpha < 2\pi/3$)	$\lvert z_0\rvert = 1$,	$\cos(\arg z_0) = \cos\alpha - \dfrac{1}{2}$,
Euclidean case ($\alpha = 2\pi/3$)	$z_0 = -1$,	
Spherical case ($2\pi/3 < \alpha < 4\pi/3$)	$z_0 =$ real number.	

Comparing with the results of [8], we have

$$-\sqrt{-1}\frac{\partial V(a, z)}{\partial\alpha} = \operatorname{arccosh}(1 + \cos\alpha - \cos 2\alpha) = \pm\frac{\partial\,\mathrm{Vol}(K_\alpha)}{\partial\alpha},$$

and we get the following.

Theorem 7. *For hyperbolic case, i.e.* $0 \le \alpha < (2/3)\pi$,

$$\pm\sqrt{-1}V(a, z_0) = \mathrm{Vol}(K_\alpha).$$

For spherical case,

$$\pm V(a, z_0) = \mathrm{Vol}(K_\alpha).$$

5.2. Whitehead link. Let K_W be the Whitehead link. Then, from (8), we have

$$\begin{aligned}
&\Phi^N_{K_W}(\lambda, \mu) \\
&= \frac{\sqrt{-1}^{\,-N-1} q^{-(\lambda+\mu+1)}}{4\sin\pi\lambda\sin\pi\mu} \\
&\quad\times\sum_i (-1)^{i+1} q^{(-2\lambda-2\mu+i-3)i}\frac{(q^2;q^2)_N(q^2;q^2)_i(q^{2\lambda-2i+2};q^2)_{2i+1}(q^{2\mu-2i+2};q^2)_{2i+1}}{(q^2;q^2)_{2i+1}}.
\end{aligned}$$

Now we compute the optimistic limit of $\Phi^N_{K_W}(\lambda, \mu)$. Let $V(a, b, z)$ be the following function.

$$\begin{aligned}
Va(a, b, z) = \pi\sqrt{-1}(\log a + \log b + \log z) - \mathrm{Li}_2(z) - \mathrm{Li}_2(az) - \mathrm{Li}_2(bz) + \mathrm{Li}_2(z^2) \\
+ \mathrm{Li}_2\left(\frac{a}{z}\right) + \mathrm{Li}_2\left(\frac{b}{z}\right) - \left(\log a + \log b - \frac{\log z}{2}\right)\log z,
\end{aligned}$$

where a, b, z correspond to $q^{2\lambda}$, $q^{2\mu}$, q^{2i} respectively, and the term $\pi\sqrt{-1}(\log a + \log b)$ is added so that the value of $V(a, b, z_0)$ in the following coincide with the volume not only for hyperbolic case but also for spherical case. By using

$$\mathrm{Li}_2(z^{-1}) = -\mathrm{Li}_2(z) - \frac{1}{2}(\log z)^2 + \pi\sqrt{-1}\log z + \frac{\pi^2}{3},$$

we have

$$\begin{aligned}
V(a, b, z) = \pi\sqrt{-1}(\log a + \log b + \log z) - \mathrm{Li}_2(z) + \mathrm{Li}_2(z^2) - \mathrm{Li}_2(az) - \mathrm{Li}_2\left(\frac{z}{a}\right) \\
- \mathrm{Li}_2(bz) - \mathrm{Li}_2\left(\frac{z}{b}\right) - \frac{1}{2}(\log a)^2 - \frac{1}{2}(\log b)^2 - \frac{1}{2}(\log z)^2 + \frac{2\pi^2}{3}.
\end{aligned} \tag{11}$$

Here $2\pi\sqrt{-1}\log z$ is absorbed by $\mathrm{Li}_2(z)$ as a branch of it. The optimistic limit of the colored Alexander invariants of K_B with fixed a, b and c is given by $V(a, b, c, z_0)$ where z_0 is a solution of the following equation.

$$\frac{\partial V(a, b, z)}{\partial z} = 0.$$

By putting $z = (x-1)/(x+1)$, this equation becomes to

$$x^3 + \frac{1}{2}(A^2B^2 + A^2 + B^2 - 1)x^2 - A^2B^2x + A^2B^2 = 0, \tag{12}$$

where

$$A = \frac{a+1}{a-1}\sqrt{-1}, \quad B = \frac{b+1}{b-1}\sqrt{-1}.$$

Let $a = \exp\sqrt{-1}\alpha$ and $b = \exp\sqrt{-1}\beta$, then $A = \cot(\alpha/2)$, $B = \cot(\beta/2)$ and we have

$$-\sqrt{-1}\frac{\partial V(a,b,z)}{\partial\alpha} = -2\arctan\frac{A}{x}, \quad -\sqrt{-1}\frac{\partial V(a,b,z)}{\partial\beta} = -2\arctan\frac{B}{x}. \tag{13}$$

Comparing with the results of [8], (13) shows that the partial derivatives of V coinside with the lengths of the singular geodesics of $W_{\alpha,\beta}$, where $W_{\alpha,\beta}$ is the Whitehead link cone-manifold with cone angles α and β. Hence we get the following.

Theorem 8. *If $W_{\alpha,\beta}$ is hyperbolic, then the volume of $W_{\alpha,\beta}$ is given by*

$$\mathrm{Vol}(W_{\alpha,\beta}) = \pm\frac{V(a,b,z_1) - V(a,b,z_2)}{2\sqrt{-1}},$$

where $z_1 = \bar{z}$, $z_2 = z$, $z = (x-1)/(x+1)$, $\mathrm{Im}(x) \neq 0$ and x is a root of the cubic equation (12).

If $W_{\alpha,\beta}$ is spherical, then the volume of $W_{\alpha,\beta}$ is given by

$$\mathrm{Vol}(W_{\alpha,\beta}) = \pm\frac{V(a,b,z_1) - V(a,b,z_2)}{2},$$

where $z_1 = (x_1-1)/(x_1+1)$, $z_2 = (x_2-1)/(x_2+1)$, and x_1, x_2 are nonnegative roots of the cubic equation (12).

REMARK 9. (1) The above gives a new formula for the volume of $W_{\alpha,\beta}$ without using integral expression.
(2) For hyperbolic case, $\pm\mathrm{Im}\,V(a,b,z_1)$ and $\mp\mathrm{Im}\,V(a,b,z_2)$ are equal to the volume of $W_{\alpha,\beta}$ since $\bar{a} = a^{-1}$, $\bar{b} = b^{-1}$ and $V(a,b,z_2) = \overline{V(a,b,z_1)}$.
(3) For spherical case, $V(a,b,z_1)$ and $V(a,b,z_2)$ are real numbers.

5.3. Borromean rings. Let K_B be the Borromean rings. Then, from (9), we have

$$\begin{aligned}
&\Phi_N^{\lambda,\mu,\nu}(K_B)\\
&= \frac{\sqrt{-1}^N q^{-\lambda-\mu-\nu-1}}{8\sin\pi\lambda\,\sin\pi\mu\,\sin\pi\nu}\\
&\quad\times\sum_s (-1)^{s+1} q^{-(2\lambda+2\mu+2\nu+1-3s)s}\\
&\qquad\quad\times\frac{((q^2;q^2)_s)^2 (q^{2\lambda-2s+2};q^2)_{2s+1}(q^{2\mu-2s+2};q^2)_{2s+1}(q^{2\nu-2s+2};q^2)_{2s+1}}{((q^2;q^2)_{2s+1-N})^2}.
\end{aligned}$$

Now we compute the optimistic limit of $\Phi_{K_B}^N(\lambda,\mu,\nu)$. Let $V(a,b,c,z)$ be the following function.

$$\begin{aligned}
V(a,b,c,z) &= \pi\sqrt{-1}(\log a+\log b+\log c+\log z) - 2\mathrm{Li}_2(z) + 2\mathrm{Li}_2(z^2) - \mathrm{Li}_2(az)\\
&\quad - \mathrm{Li}_2(bz) - \mathrm{Li}_2(cz) + \mathrm{Li}_2\left(\frac{a}{z}\right) + \mathrm{Li}_2\left(\frac{b}{z}\right) + \mathrm{Li}_2\left(\frac{c}{z}\right)\\
&\quad + \frac{3}{2}\log^2 z - (\log a+\log b+\log c)\log z,
\end{aligned}\tag{14}$$

where a, b, c, z correspond to $q^{2\lambda}$, $q^{2\mu}$, $q^{2\nu}$, q^{2s} respectively, and the term $\pi\sqrt{-1}(\log a+\log b+\log c)$ is added so that the value of $V(a,b,c,z_0)$ in the following coincide with the volume not only for hyperbolic case but also for spherical case. By using

$$\begin{aligned}
&\mathrm{Li}_2(z^2) = 2\mathrm{Li}_2(z) + 2\mathrm{Li}_2(-z),\\
&\mathrm{Li}_2(z^{-1}) = -\mathrm{Li}_2(z) - \frac{1}{2}(\log z)^2 + \pi\sqrt{-1}\log z + \frac{\pi^2}{3},
\end{aligned}$$

we have

$$\begin{aligned}
V(a,b,c,z) = &-\mathrm{Li}_2(az) - \mathrm{Li}_2(bz) - \mathrm{Li}_2(cz) + \mathrm{Li}_2\left(\frac{a}{z}\right) + \mathrm{Li}_2\left(\frac{b}{z}\right) + \mathrm{Li}_2\left(\frac{c}{z}\right) + \mathrm{Li}_2(z)\\
&- \mathrm{Li}_2(z^{-1}) + 2\mathrm{Li}_2(-z) - 2\mathrm{Li}_2(-z^{-1})\\
&- (\log a+\log b+\log c)(\log z - \pi\sqrt{-1}).
\end{aligned}$$

The optimistic limit of the colored Alexander invariants of K_B with fixed a, b and c is given by $V(a,b,c,z_0)$ where z_0 is a solution of the following equation.

$$\frac{dV}{dz} = 0. \tag{15}$$

Let $a = \exp\sqrt{-1}\alpha$, $b = \exp\sqrt{-1}\beta$, $c = \exp\sqrt{-1}\gamma$, $z = \exp\sqrt{-1}\zeta$, $A = \tan(\alpha/2)$, $B = \tan(\beta/2)$, $C = \tan(\gamma/2)$ and $T = \tan(\zeta/2)$. Then (15) is transformed to

$$T^4 - (A^2 + B^2 + C^2 + 1)T^2 - A^2B^2C^2 = 0. \tag{16}$$

For real parameters α and ζ, let $\Delta(\alpha, \zeta) = \Lambda(\alpha + \zeta) - \Lambda(\alpha - \zeta)$, where $\Lambda(x) = -\int_0^x \log|2\sin t|\, dt$ is the Lobachevski function. Then

$$2\Delta\left(\frac{\alpha}{2}, \frac{\zeta}{2}\right) = \operatorname{Im}\left(\mathrm{Li}_2(az) - \mathrm{Li}_2\left(\frac{a}{z}\right) + \log a \log z\right).$$

Therefore, for real parameters α, β, γ, and ζ,

$$\begin{aligned}&\operatorname{Im} V(a, b, c, z)\\ &= -2\left(\Delta\left(\frac{\alpha}{2}, \frac{\zeta}{2}\right) + \Delta\left(\frac{\beta}{2}, \frac{\zeta}{2}\right) + \Delta\left(\frac{\gamma}{2}, \frac{\zeta}{2}\right) - 2\Delta\left(\frac{\pi}{2}, \frac{\zeta}{2}\right) - \Delta\left(0, \frac{\zeta}{2}\right)\right).\end{aligned}$$

Now, consider the case that the parameter z is a real number. In this case, we use parameter T satisfying $T = (z-1)/(z+1)$. Then the equation (15) is transformed to

$$T^4 + (A^2 + B^2 + C^2 + 1)T^2 - A^2B^2C^2 = 0. \tag{17}$$

For $\theta > \pi/2$, let $\delta(\xi, \theta)$ be the function in [8],

$$\delta(\xi, \theta) = \int_\theta^{\pi/2} \log(1 - \cos 2\xi \cos 2\tau)\frac{d\tau}{\cos 2\tau}.$$

By putting $t = \tan\tau$,

$$\delta(\xi, \theta) = \int_{-\infty}^{\tan\theta} \log\left(\frac{2(t^2 + \tan^2\xi)}{(1+t^2)(1+\tan^2\xi)}\right)\frac{dt}{t^2 - 1}.$$

Now replace t by $(z-1)/(z+1)$, we get

$$\delta\left(\frac{\alpha}{2}, \theta\right) = \frac{1}{2}\int_{-1}^{(1+\tan\theta)/(1-\tan\theta)} \log\left(\frac{(1-az)(1-az^{-1})}{(1+z^2)}\right)\frac{dz}{z}.$$

Let T_0 be the negative root of (17) and z_0, θ_0 be the real numbers satisfying $z_0 = (1+T_0)/(1-T_0)$ and $\tan\theta_0 = T_0$. Then

$$2\left(\delta\left(\frac{\alpha}{2}, \theta_0\right) + \delta\left(\frac{\beta}{2}, \theta_0\right) + \delta\left(\frac{\gamma}{2}, \theta_0\right) - 2\delta\left(\frac{\pi}{2}, \theta_0\right) - \delta(0, \theta_0)\right) = \operatorname{Re} V(a, b, c, z_0).$$

Comparing with the results of [6] and [8], we get the following.

Theorem 10. *For* $0 < \alpha, \beta, \gamma < 2\pi$, *let* $M_{\alpha,\beta,\gamma}$ *be the cone-manifold with singlar set* K_B *whose cone angles of three components of* K_B *are* α, β, γ. *If* $M_{\alpha,\beta,\gamma}$ *is a hyperbolic cone-manifold with* $0 < \alpha, \beta, \gamma < \pi$, *then*

$$\begin{aligned}\mathrm{Vol}(M_{\alpha,\beta,\gamma}) &= \pm\,\mathrm{Im}(V(a, b, c, z_1)) = \mp\,\mathrm{Im}(V(a, b, c, z_2))\\ &= \pm\frac{1}{2\sqrt{-1}}(V(a, b, c, z_1) - V(a, b, c, z_2)),\end{aligned}$$

where z_1, z_2 *correspond to the real solutions of the equation* (16).

If $M_{\alpha,\beta,\gamma}$ *is a spherical cone-manifold with* $\pi < \alpha, \beta, \gamma < 2\pi$, *then*

$$\begin{aligned}\mathrm{Vol}(M_{\alpha,\beta,\gamma}) &= \pm\,\mathrm{Re}\,V(a, b, c, z_3) = \mp\,\mathrm{Re}\,V(a, b, c, z_4)\\ &= \pm\frac{1}{2}(V(a, b, c, z_3) - V(a, b, c, z_4)),\end{aligned}$$

where z_3, z_4 *correspond to the real solutions of the equation* (17).

6. *A*-polynomials

In this section, we introduce a polynomial or a system of polynomials of the above three examples by using the method in [13]. For figure eight knot, it coinsides with the A-polynomial, and for link cases, these are generalization of the A-polynomial for links.

6.1. Figure eight knot. The A-polynomial of 4_1 is obtained by eliminating the parameter x from the following system of equations.

$$\frac{\partial V(a, x)}{\partial x} = 0, \quad a\frac{\partial V(a, x)}{\partial a} = \log L.$$

Here $V(a, x)$ is given by two parameters a and L correspond to the meridian and longitude. The resulting polynomial is

$$a^2 - L + aL + 2a^2L + a^3L - a^4L + a^2L^2 = 0. \tag{18}$$

In fact, this polynomial coincides with $A_{J(2,-2)}(L, M)$ in Theorem 7 of [4] with $a = M^2$.

6.2. Whitehead link. After the above construction of the A-polynomial of 4_1, the system of A-polynomials of the link K_W may be obtained by eliminating the parameter z from the following system of equations.

$$\frac{\partial V(a, b, z)}{\partial z} = 0, \quad a\frac{\partial V(a, b, z)}{\partial a} = \log L_a, \quad b\frac{\partial V(a, b, z)}{\partial b} = \log L_b,$$

where $V(a, b, z)$ is the function given by (11). The parameters a, b correspond to the meridians of the first and second components, and L_a, L_b correspond to the longitudes of them respectively. The result is

(19)

$$a^2bL_a{}^3 + (a^2b^2 - ab^2 + a^2 - 2ab - a + b)L_a{}^2 + (a^2b - ab^2 - 2ab + b^2 - a + 1)L_a + b = 0,$$

$$ab^2L_b{}^3 + (a^2b^2 - a^2b - 2ab + b^2 + a - b)L_b{}^2 + (ab^2 - a^2b + a^2 - 2ab - b + 1)L_b + a = 0.$$

6.3. Borromean rings. The system of A-polynomials of the link K_W may be obtained by eliminating the parameter z from the following system of equations.

$$\frac{\partial V}{\partial z} = 0, \quad a\frac{\partial V}{\partial a} = \log L_a, \quad b\frac{\partial V}{\partial b} = \log L_b, \quad c\frac{\partial V}{\partial c} = \log L_c,$$

where $V = V(a, b, c, z)$ is the function given by (14). The parameters a, b, c correspond to the meridians of the first, second and third components, and L_a, L_b, L_c correspond to the longitudes of them respectively. By eliminating z from the first two equations, we get the following. Let $a = \exp\sqrt{-1}\alpha$, $b = \exp\sqrt{-1}\beta$, $c = \exp\sqrt{-1}\gamma$, $A = \cot(\alpha/2)$, $B = \cot(\beta/2)$, $C = \cot(\gamma/2)$, $D = A^2(B^2C^2 + B^2 + C^2 + 1)$, and $E = A^2 + B^2C^2$, then

$$DL_a{}^4 - 4EL_a{}^3 - 2(D - 4A^2 + 4B^2C^2)L_a{}^2 - 4EL_a + D = 0. \tag{20}$$

For L_b and L_c, we get the similar equation corresponding to the symmetry of a, b and c.

Appendix

Here we show some formulas we used in the computation.

Lemma A. *The following formulas hold.*

$$\sum_{i=0}^{\alpha}(-1)^i q^{\beta i}\begin{bmatrix}\alpha\\ i\end{bmatrix} = \prod_{j=1}^{\alpha}(1 - q^{\beta+\alpha+1-2j}).$$

This formula comes from the following quantized Pascal relation and an induction.

$$\begin{bmatrix}n\\ s\end{bmatrix} = q^{\pm s}\begin{bmatrix}n-1\\ s\end{bmatrix} + q^{\mp(n-s)}\begin{bmatrix}n-1\\ s-1\end{bmatrix}. \tag{A.1}$$

Lemma B. *For any a, b, c such that b, $a-c$ are nonnegative integers and a is not an integer, we have the following identity*:

$$\begin{aligned}&\sum_{k=0}^{N-1-b} q^{\pm k(a+b-c+2)}\{a-k;a-c\}\{b+k;b\}\\ &= q^{\pm((b+1)c-Na)}\frac{\{N;N\}\{b;b\}\{a+b+1;a+b-c+1\}}{\{a+N;N\}\{a+b-c+1;b+1\}}.\end{aligned}$$

Proof. Downword induction on b.

STEP 1. If $b=N-1$,

$$\sum_{k=0}^{N-1-b} q^{\pm k(a+b-c+2)}\{a-k;a-c\}\{b+k;b\} = \{a;a-c\}\{N-1;N-1\}.$$

On the other hand,

$$\begin{aligned}&q^{\pm((b+1)c-Na)}\frac{\{N;N\}\{b;b\}\{a+b+1;a+b-c+1\}}{\{a+N;N\}\{a+b-c+1;b+1\}}\\ &= (-1)^{a-c}\frac{\{N;N\}\{N-1;N-1\}\{a+N;a-c+N\}}{\{a+N;N\}\{a-c+N;N\}}\\ &= \{N-1;N-1\}\{a;a-c\}.\end{aligned}$$

Hence the equality holds for $b=N-1$.

STEP 2. Assume that Lemma holds for b and prove for $b-1$. First, we use the relation $\{n\}=\{N-n\}$ and $\{n;n\}=\{N-1;n\}$. Apply this to $\{b+k-1;b+k-1\}$, we have

$$\begin{aligned}&\sum_{k=0}^{N-b} q^{\pm k(a+b-c+1)}\{a-k;a-c\}\{b+k-1;b-1\}\\ &= \sum_{k=0}^{N-b} q^{\pm k(a+b-c+1)}\frac{\{a-k;a-c\}\{N-1;b+k-1\}}{\{k;k\}}\\ &= \{N-1;b-1\}\sum_{k=0}^{N-b} q^{\pm k(a+b-c+1)}\frac{\{a-k;a-c\}\{N-b;k\}}{\{k;k\}}.\end{aligned}$$

Now we use Pascal retation (A.1).

$$\{N-1;b-1\}\sum_{k=0}^{N-b} q^{\pm k(a+b-c+1)}\frac{\{a-k;a-c\}\{N-b;k\}}{\{k;k\}}$$

$$
\begin{aligned}
&= \{N-1; b-1\} \sum_{k=0}^{N-b} q^{\pm k(a+b-c+1)} \{a-k; a-c\} \\
&\qquad \times \left(q^{\pm k} \frac{\{N-b-1; k\}}{\{k;k\}} + q^{\mp(N-b-k)} \frac{\{N-b-1; k-1\}}{\{k-1; k-1\}} \right) \\
&= \frac{1}{\{N-b\}} \left(\sum_{k=0}^{N-b-1} q^{\pm k(a+b-c+2)} \frac{\{a-k; a-c\}\{N-1; b+k\}}{\{k;k\}} \right. \\
&\qquad \left. - q^{\pm b} \sum_{k=0}^{N-b-1} q^{\pm(k+1)(a+b-c+2)} \frac{\{a-k; a-c\}\{N-1; b+k-1\}}{\{k-1; k-1\}} \right) \\
&= \frac{1}{\{b\}} \sum_{k=0}^{N-b-1} \left(q^{\pm k(a+b-c+2)} \frac{\{a-k; a-c\}\{b+k; b+k\}}{\{k;k\}} \right. \\
&\qquad \left. - q^{\pm(a+2b-c+2)} q^{\pm k(a+b-c+2)} \frac{\{a-k-1; a-c\}\{b+k; b+k\}}{\{k;k\}} \right) \\
&= \sum_{k=0}^{N-b-1} \frac{q^{\pm k(a+b-c+2)}}{\{b\}} (\{a-k; a-c\}\{b+k; b\} \\
&\qquad - q^{\pm(a+2b-c+2)} \{a-k-1; a-c\}\{b+k; b\}) \\
&= \frac{1}{\{b\}} \left(q^{\pm((b+1)c-Na)} \frac{\{N;N\}\{b;b\}\{a+b+1; a+b-c+1\}}{\{a+N; N\}\{a+b-c+1; b+1\}} \right. \\
&\qquad \left. - q^{\pm(a+2b-c+2)} q^{\pm((b+1)(c-1)-N(a-1))} \frac{\{N;N\}\{b;b\}\{a+b; a+b-c+1\}}{\{a+N-1; N\}\{a+b-c+1; b+1\}} \right) \\
&= \frac{q^{\pm((b+1)c-Na)}}{\{b\}} \frac{\{N;N\}\{b;b\}\{a+b; a+b-c\}}{\{a+N; N\}\{a+b-c+1; b+1\}} (\{a+b+1\} - q^{\pm(a+b-c+1)}\{c\}) \\
&= \frac{q^{\pm((b+1)c-Na)}}{\{b\}} q^{\pm((b+1)c-Na)} \frac{\{N;N\}\{b;b\}\{a+b; a+b-c\}}{\{a+N; N\}\{a+b-c+1; b+1\}} q^{\mp c} \{a+b-c+1\} \\
&= q^{\pm(bc-Na)} \frac{\{N;N\}\{b-1; b-1\}\{a+b; a+b-c\}}{\{a+N; N\}\{a+b-c; b\}}
\end{aligned}
$$

Hence the formula also holds for $b-1$. □

References

[1] Y. Akutsu, T. Deguchi and T. Ohtsuki: *Invariants of colored links*, J. Knot Theory Ramifications **1** (1992), 161–184.

[2] I.S. Gradshteyn and I.M. Ryzhik: Table of Integrals, Series, and Products, Fifth edition, Academic Press, San Diego, CA, 1996.

[3] K. Habiro: *On the colored Jones polynomials of some simple links*; in Recent Progress Towards the Volume Conjecture (Japanese) (Kyoto, 2000), Sūrikaisekikenkyūsho Kōkyūroku **1172**, 2000, 34–43.
[4] J. Hoste and P.D. Shanahan: *A formula for the A-polynomial of twist knots*, J. Knot Theory Ramifications **13** (2004), 193–209.
[5] R.M. Kashaev: *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), 269–275.
[6] R. Kellerhals: *On the volume of hyperbolic polyhedra*, Math. Ann. **285** (1989), 541–569.
[7] A.N. Kirillov: *Clebsh-Gordan quantum coefficients*, (Russian), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **168** (1988), Anal. Teor. Chisel i Teor. Funktsii. **9**, 67–84, 188; translation in J. Soviet Math. **53** (1991), 264–276.
[8] A.D. Mednykh: *On hyperbolic and spherical volumes for knot and link cone-manifolds*; in Kleinian Groups and Hyperbolic 3-Manifolds (Warwick, 2001), London Math. Soc. Lecture Note Ser. **299**, Cambridge Univ. Press, Cambridge, 2003, 145–163.
[9] H. Murakami: *Some limits of the colored Jones polynomials of the figure-eight knot*, Kyungpook Math. J. **44** (2004), 369–383.
[10] H. Murakami, J. Murakami, M. Okamoto, T. Takata and Y. Yokota: *Kashaev's conjecture and the Chern-Simons invariants of knots and links*, Experiment. Math. **11** (2002), 427–435.
[11] H. Murakami and Y. Yokota: *The colored Jones polynomials of the figure-eight knot and its Dehn surgery spaces*, J. Reine Angew. Math. **607** (2007), 47–68.
[12] J. Murakami: *A state model for the multivariable Alexander polynomial*, Pacific J. Math. **157** (1993), 109–135.
[13] Y. Yokota, *On the volume conjecture for hyperbolic knots*, arXiv:`math.QA/0009165` (2000).
[14] Y. Yokota: *From the Jones polynomial to the A-polynomial of hyperbolic knots*; Proceedings of the Winter Workshop of Topology/Workshop of Topology and Computer (Sendai, 2002/Nara, 2001), Interdiscip. Inform. Sci. **9**, 2003, 11–21.

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