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SOME PROPERTIES OF MARKOV PROCESSES ASSOCIATED WITH TIME DEPENDENT DIRICHLET FORMS

Dedicated to Professor Takesi Watanabe on his 60th birthday

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1. Introduction

In the previous paper [8], we showed the existence of the space-time Markov processes associated with certain time dependent Dirichlet forms. The situation treated there can not be covered by either the theory of symmetric Dirichlet forms or the theory of coercive non-symmetric Dirichlet foms. Nevertheless, as we saw in that paper, the methods of symmetric Dirichlet forms had been effective for the construction of space-time Hunt processes associated with time dependent Dirichlet forms. The purpose of this paper is to show that many of the properties similar to the case of symmetric Dirichlet forms given by Fukushima [4] still hold in the time dependent cases. To give the more precise statements of the results, we shall introduce the notations which will be used in this paper. Let X be a locally compact separable metric space and m be an positive Radon measure on X such that Supp $[m]=X$. We shall suppose that we are given a family $E^{(\tau)}$ ($\tau\in R^1$) of Dirichlet forms on $H\equiv L^2(X;m)$ with common domain V, that is, we are given a Hilbert space $(V, \|\cdot\|_V)$ which is densely and continuously embedded in H and a family of bilinear forms $E^{(\tau)}(\varphi, \psi)$ on $V\times V$ satisfying the following conditions:

(E.1) For all $\varphi, \psi\in V$, $E^{(\tau)}(\varphi, \psi)$ is a measurable function of $\tau\in R^1$.
(E.2) For any $p>0$, there exists a positive constant $M=M(p)$ such that

$$|E_p^{(\tau)}(\varphi, \psi)|\leq M\,\|\varphi\|_V\,\|\psi\|_V\,,$$

for all $\varphi, \psi\in V$ and $\tau\in R^1$, where by using the inner product $(\cdot, \cdot)_H$ in H, $E_p^{(\tau)}$ is given by

$$E_p^{(\tau)}(\varphi, \psi)=E^{(\tau)}(\varphi, \psi)+p(\varphi, \psi)_H\,.$$

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(E.3) For any $p>0$, there exists a constant $\alpha=\alpha(p)$ such that

$$E_p^{(\tau)}(\varphi, \varphi)\geq\alpha\ ||\varphi||_V^2 ,$$

for all $\varphi\in V$.

(E.4) For any $\varphi\in V$ and $c\geq 0$, $\varphi\wedge c$ belongs to V and satisfies

$$E^{(\tau)}(\varphi\wedge c, \varphi-\varphi\wedge c)\geq 0\ , \quad E^{(\tau)}(\varphi-\varphi\wedge c, \varphi\wedge c)\geq 0\ ,$$

for all $\tau\in R^1$.

(E.5) There exists a subset D of the space $C_0(X)$ of continuous functions on X with compaat support such that $D\cap V$ is $||\cdot||_V$-dense in V and uniformly dense in $C_0(X)$.

We shall identify H and its dual space H'. Under this identification, H is continuously and densely embedded in the dual space V' of V. Let $\mathcal{X}=R^1\times X$ and $d\nu=d\tau\times dm$, where $d\tau$ is the Lebesgue measure on R^1. Define the spaces $\mathcal{H}, \mathcal{V}$ and $\mathcal{V}'$ by $\mathcal{H}=L^2(R^1; H)$, $\mathcal{V}=L^2(R^1; V)$ and $\mathcal{V}'=L^2(R^1; V')$, respectively. The norm of $\mathcal{H}$ is defined by

$$(1.1) \qquad ||u||_{\mathcal{H}}^2 = \int_{R^1} ||u(\tau, \cdot)||_H^2\, d\tau\ .$$

Define $||\cdot||_{\mathcal{V}}$ and $||\cdot||_{\mathcal{V}'}$ similarly by using $||\cdot||_V$ and $||\cdot||_{V'}$ instead of $||\cdot||_H$ in (1.1), respectively. We shall define the space $(\mathcal{F}, ||\cdot||_{\mathcal{F}})$ by

$$(1.2) \qquad \mathcal{F} = \{u\in\mathcal{V}; \frac{\partial u}{\partial\tau}\in\mathcal{V}'\}\ , \quad ||u||_{\mathcal{F}}^2 = ||u||_{\mathcal{V}}^2+||\frac{\partial u}{\partial\tau}||_{\mathcal{V}'}^2\ .$$

The condition (E.5) guarantees that $C_0(\mathcal{X})\cap\mathcal{F}$ is uniformly dense in the space $C_0(\mathcal{X})$ of continuous functions on $\mathcal{X}$ with compact support, and $||\cdot||_{\mathcal{F}}$-dense in $\mathcal{F}$ (see [9]). Define the bilinear form $\mathcal{E}$ by

$$(1.3) \qquad \mathcal{E}(u, v) = \begin{cases} \mathcal{A}(u, v)-\left(\dfrac{\partial u}{\partial\tau}, v\right), & u\in\mathcal{F}, v\in\mathcal{V} \\[2ex] \mathcal{A}(u, v)+\left(\dfrac{\partial v}{\partial\tau}, u\right), & u\in\mathcal{V}, v\in\mathcal{F}\ , \end{cases}$$

where

$$(1.4) \qquad \mathcal{A}(u, v) = \int_{R^1} E^{(\tau)}(u(\tau, \cdot), v(\tau, \cdot))\, d\tau$$

and $(\cdot, \cdot)$ is the coupling between the elements of $\mathcal{V}'$ and $\mathcal{V}$. As usual, set

$$\mathcal{E}_p(u, v) = \mathcal{E}(u, v)+p\,(u, v)\ .$$

Then, for any $p>0$ and $f\in\mathcal{V}'$, there exist resolvents $G_pf\in\mathcal{F}$ and $\hat{G}_pf\in\mathcal{F}$ such that

$$(1.5) \qquad \mathcal{E}_p(G_p f, u) = \mathcal{E}_p(u, \hat{G}_p f) = (f, u) \qquad \text{for all} \quad u \in \mathcal{V},$$

(see [5]). In [8] we have seen that there exists a Hunt process $\boldsymbol{M}=(Y(t), P_y)$ on the extended state space $\mathcal{X}$ whose resolvent $V_p f$ is a quasi-continuous (q.c. in abbreviation) modification of $G_p f$ for all $f \in \mathcal{H}$, where the quasi-continuity is defined with respect to the capacity defined by $(\mathcal{E}, \mathcal{F})$. Similarly there exists the (weak) dual Hunt process $\hat{\boldsymbol{M}}=(\hat{Y}(t), \hat{P}_y)$ associated with $\hat{G}_p$ in a similar sense.

In Section 2 in this paper, we shall give the notations and preliminary results which will be used in this paper. In Section 4, we shall discuss the $\mathcal{E}_p$-orthogonality property of the resolvent $V_p^{\mathcal{X}-B} f$ of the part process on $\mathcal{X}-B$ and $\hat{H}_B^p \tilde{V}_p g$ as well as its consequences, where $\hat{H}_B^p$ is the p-th order hitting distribution of the set B with respect to the dual process $\hat{\boldsymbol{M}}$. To show it, we need to extend the domain of the form $\mathcal{E}$ to a class of pairs of functions containing $\{\mathcal{F} \oplus (\mathcal{P} \ominus \mathcal{P})\} \times \{\mathcal{F} \oplus (\hat{\mathcal{P}} \ominus \hat{\mathcal{P}})\}$, where $\mathcal{P}$ and $\hat{\mathcal{P}}$ are the set of all 1-excessive and 1-coexcessive functions, respectively, $\mathcal{F} \oplus (\mathcal{P} \ominus \mathcal{P}) = \{u = u_1 + u_2 - u_3;\ u_1 \in \mathcal{F},\ u_2, u_3 \in \mathcal{P}\}$ and $\mathcal{F} \oplus (\hat{\mathcal{P}} \ominus \hat{\mathcal{P}})$ is defined similarly. Such an extension will be given in Section 3. In Section 5, we shall introduce the notions of measures of finite energy integral as well as smooth measures and show a correspondence between such measures and positive natural additive functionals (PNAF's in abbreviation) (cf. [3], [4], [11], [12]). As in [4], the *energy* of an additive functional (AF in abbreviation) $A(t)$ of $\boldsymbol{M}$ is defined by

$$(1.6) \qquad e(A) = \frac{1}{2} \lim_{p \to \infty} p^2 E_\nu\left(\int_0^\infty e^{-pt} A^2(t)\, dt\right),$$

(see also [7], [13]). An AF $M(t)$ is called a *martingale additive functional* (abbreviated to MAF) if $M(t)$ is an AF such that $E_y(M^2(t)) < \infty$ q.e. and $E_y(M(t)) = 0$ q.e. The set of all MAF's with finite energy is denoted by $\mathring{\mathcal{M}}$. Also, let $\mathcal{N}$ be the set of all NAF's of zero energy. Note that an additive functional $A(t)$ is called a NAF if $A(t)$ and $Y(t)$ do not jump simultaneously a.s. P_y for q.e.y. In Section 6, we shall show the following decomposition theorem which have been given by Fukushima in the case of symmetric Dirichlet forms; if $u \in \mathcal{F} \oplus (\mathcal{P} \ominus \mathcal{P})$, then there exist uniquely $M^{[u]} \in \mathring{\mathcal{M}}$ and $N^{[u]} \in \mathcal{N}$ such that

$$(1.7) \qquad \tilde{u}(Y(t)) - \tilde{u}(Y(0)) = M^{[u]}(t) + N^{[u]}(t),$$

where $\tilde{u}$ is a q.c. modification of u if $u \in \mathcal{F}$ and the difference of 1-excessive regularizations if u is the difference of 1-excessive functions. In particular, if $u \in \mathcal{F}$, then the NAF $N^{[u]}$ in (1.7) becomes continuous. In the decomposition (1.7), the energy of $M^{[u]}$ is given by

$$(1.8) \qquad e(M^{[u]}) = \mathcal{A}(u, u) - \int_{\mathcal{X}} \tilde{u}^2(y)\, k(dy),$$

for a positive Radon measure k on $\mathcal{X}$. From this representation, most of the stochastic calculi discussed in [4: Chapter 5] follow in our setting but we shall not trace it because the proofs are similar. We shall only conider the case

$$(1.9) \qquad E^{(\tau)}(\varphi, \psi) = \sum_{i,j=1}^{d} \int_{R^d} a_{ij}(\tau, x) \frac{\partial \varphi(x)}{\partial x_i} \frac{\partial \psi(x)}{\partial x_j} dx, \ \varphi, \psi \in C_0^\infty(R^d),$$

in which the explicit form of the quadratic variation of $M^{[u]}$, $N^{[u]}$ will be given and an equivalence of the $E^{(\tau)}$-polarity of a subset B of R^d and the $\mathcal{E}$-polaiity of a subset $(a, b)\times B$ of $R^1\times R^d$ will be established.

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2. Notations and preliminary results

Let $(\mathcal{E}, \mathcal{F})$ be the form defined by (1.2) and (1.3). As we mentioned in Section 1, for any $f\in\mathcal{V}'$, there exist resolvents $G_p f\in\mathcal{F}$ and $\hat{G}_p f\in\mathcal{F}$ satisfying $\mathcal{E}_p(G_p f, v)=\mathcal{E}_p(v, \hat{G}_p f)=(f, v)$, where $\mathcal{E}_p(u, v)=\mathcal{E}(u, v)+p(u, v)$ and (u, v) is the coupling between $\mathcal{V}'$ and $\mathcal{V}$. In what follows we shall also use the notation (u, v) to represent the inner product of functions $u, v\in\mathcal{H}$ because it will not cause any confusion. The resolvent G_p satisfies the following conditions (see [8], [9], [10]):

(i) for any $f\in\mathcal{H}$ such that $0\leq f\leq 1$ ν—a.e., $0\leq pG_p f\leq 1$ ν—a.e.,

(ii) if $f\in\mathcal{H}$ then $pG_p f$ converges to f in $\mathcal{H}$ as $p\to\infty$, furthermore, if $f\in\mathcal{V}$ [resp. $f\in\mathcal{F}$] then $pG_p f$ converges weakly to f in $\mathcal{V}$ [resp. in $\mathcal{F}$].

A function $u\in\mathcal{V}$ is called *p-excessive* if it satisfies $\mathcal{E}_p(u, v)\geq 0$ for all $v\in \mathcal{F}^+\equiv\{w\in\mathcal{F}; w\geq 0\ \nu\text{—a.e.}\}$. Then the following statements are equivalent to each other:

(i) u is 1-excessive.

(ii) u is non-negative and satisfies $qG_{q+1}u\leq u$, ν—a.e. for any $q\geq 0$.

(iii) There exists a positive Radon measure μ_u on $\mathcal{X}$ such that $\mathcal{E}_1(u, v)= \int v(y)\, \mu_u(dy)$ for any $v\in\mathcal{F}\cap C_0(\mathcal{X})$.

In this case u is called the *1-potential of* μ_u and denote by $u=V_1\,\mu_u$.

The family of all 1-excessive functions is denoted by $\mathcal{P}$. Similarly, the notions of *p-coexcessive functions*, $\hat{V}_1\,\hat{\mu}_u$ and the family $\hat{\mathcal{P}}$ of all 1-coexcessive functions are defined by using $\mathcal{E}_p(v, u)$ instead of $\mathcal{E}_p(u, v)$ in the definition of p-excessive functions.

For any function $h\in\mathcal{H}$, set

$$(2.1) \qquad e_h = \inf\{u\in\mathcal{P};\ u\geq h\ \nu\text{—a.e.}\}.$$

Then $e_h\in\mathcal{P}$ and $e_h=e_{h^+}$ ν—a.e. for all $h\in\mathcal{H}$ (see [10]). Mignot and Puel [6] proved that the solution $h_\varepsilon \in \mathcal{F}$ of

$$(2.2) \qquad \mathcal{E}_1(h_\varepsilon, v) = \frac{1}{\varepsilon}((h_\varepsilon - h)^-, v), \quad \forall v \in \mathcal{V},$$

converges increasingly a.e., strongly in $\mathcal{H}$ and weakly in $\mathcal{V}$ to e_h as $\varepsilon \downarrow 0$. When $h=I_B$ for a Borel set B we shall denote e_h by e_B. Since e_h is a 1-excessive function, there exists a measure μ_{e_h} such that $e_h = V_1 \mu_{e_h}$. For any open set O of $\mathcal{X}$, define the *capacity* Cap(O) of O by Cap$(O)=\mu_O(\bar{O})$, where $\mu_O=\mu_{e_h}$ for $h=I_O$. The capacity can be extended to all Borel sets as a Choquet capacity. A function u on $\mathcal{X}$ is called *quasi-continuous* (q.c. in abbreviation) if there exists a decreasing sequence of open sets $\{O_n\}$ such that Cap$(O_n) \downarrow 0$ and u is continuous on each $\mathcal{X}-O_n$. It is known that any function of $\mathcal{F}$ has a q.c. modification ([9], [10]). In the previous paper [8], we proved that there exist a Hunt process $\boldsymbol{M}=(Y(t), P_y)$ and its dual Hunt process $\hat{\boldsymbol{M}}=(\hat{Y}(t), \hat{P}_y)$ on $\mathcal{X}$, whose resolvents $V_p f$ and $\hat{V}_p f$ are q.c.modifications of $G_p f$ and $\hat{G}_p f$, respeatively. For any 1-excessive [resp. 1-coexcessive] function u, we shall define its 1-*excessive* [resp. 1-*coexcessive*] *regularization* $\tilde{u}$ by

$$(2.3) \qquad \tilde{u}(y) = \lim_{n\to\infty} nV_{n+1}u(y) \text{ [resp. } \tilde{u}(y) = \lim_{n\to\infty} n\hat{V}_{n+1}u(y)] .$$

We shall start with the following lemma which will be used in many places.

Lemma 2.1. *For any $u \in \mathcal{H}$, if $p(u-pV_p u, u)$ remains bounded as $p\to\infty$, then $u \in \mathcal{V}$ and*

$$(2.4) \qquad \liminf_{p\to\infty} p(u-pV_p u, u) \geq \mathcal{A}(u, u) .$$

In particular, if $u \in \mathcal{F}$ then, for any $v \in \mathcal{V}$, $\lim_{p\to\infty} p(u-pV_p u, v)$ exists and equals to $\mathcal{E}(u, v)$.

Proof. For any $p>0$, since

$$\begin{aligned} \mathcal{A}(pV_p u, pV_p u) &= \mathcal{E}(pV_p u, pV_p u) = p(u-pV_p u, pV_p u) \\ &= p(u-pV_p u, u) - p\|u-pV_p u\|_{\mathcal{H}}^2 \leq p(u-pV_p u, u), \end{aligned}$$

$\{pV_p u\}$ is uniformly bounded in $\mathcal{V}$ from (E.3). Hence there exist a sequence $\{p_n\}$ increasing to infinity and a function $u_0 \in \mathcal{V}$ such that $p_n V_{p_n} u$ converges to u_0 weakly in $\mathcal{V}$ as $n\to\infty$. Then, for any $f \in \mathcal{H}$,

$$\begin{aligned} (f, u_0)_{\mathcal{H}} &= \mathcal{E}_1(V_1 f, u_0) = \lim_{n\to\infty} \mathcal{E}_1(V_1 f, p_n V_{p_n} u) \\ &= \lim_{n\to\infty} (f, p_n V_{p_n} u) = (f, u) . \end{aligned}$$

This implies that $u=u_0 \in \mathcal{V}$.

(2.4) follows from the inequality in the fiist paragraph of the proof.

The last assertion is obvious from

$$\mathcal{E}(pV_p u, v) = p(u - pV_p u, v), \quad \forall v \in \mathcal{V}.$$

For any open set O of $\mathcal{X}$, the 1-excessive function e_O was defined by (2.1) with $h = I_O$. For any compact set F, the following result holds.

Lemma 2.2. *If F is a compact subset of $\mathcal{X}$, then*

$$e_F = \inf\{u \in \mathcal{P};\, u \geq 1 \ \nu\text{—a.e. on a neighbourhood of } F\},\ \nu\text{—a.e.}$$

Proof. Denote by u_F the righthand side of the above equality. Obviously $u_F \geq e_F$ ν—a.e. Since $e_F = \tilde{e}_F$ ν—a.e., it is enough to show that $u_F \leq \tilde{e}_F$ ν—a.e. According to the definition (2.3), $\tilde{e}_F$ is quasi-lower semicontinuous, that is, for any $\varepsilon > 0$, there exists an open set N_ε such that $\mathrm{Cap}(N_\varepsilon) < \varepsilon$ and $\tilde{e}_F$ is lower semicontinuous on $\mathcal{X} - N_\varepsilon$. Thus, for any $\delta > 0$, we can find an open set O_δ^ε such that $\{y;\, \tilde{e}_F(y) > 1 - \delta\} = O_\delta^\varepsilon \cap (\mathcal{X} - N_\varepsilon)$. Set $u_{m,n} = \{1/(1-(1/m))\}\, \tilde{e}_F + \tilde{e}_{N(1/n)}$. Then $u_{m,n} \in \mathcal{P}$ and $u_{m,n} \geq 1$ a.e. on O_δ^ε for $\varepsilon = 1/n$ and $\delta = 1/m$. Since $\tilde{e}_{N(1/n)}$ decreases q.e. and converges in $\mathcal{V}$ to zero by [8; (3.8)], $u_{m,m}$ converges q.e. to $\tilde{e}_F$. This implies the result.

Lemma 2.3. *For any Borel set B of $\mathcal{X}$,*

$$\tilde{e}_B(y) = E_y(e^{-\sigma_B}) \ \text{q.e.}, \tag{2.5}$$

where σ_B is the hitting time of B.

Proof. We shall first note that

$$\mathcal{E}_1(w, u) \leq \mathcal{A}_1(u, w) + \mathcal{A}_1(w, u) \tag{2.6}$$

holds for all $u \in \mathcal{P}$ and $w \in \mathcal{F}^+$. In fact, since $u \in \mathcal{P}$ and $w \geq 0$.

$$\mathcal{E}_1(u, w) = \left(\frac{\partial w}{\partial \tau}, u\right) + \mathcal{A}_1(u, w) \geq 0.$$

(2.6) follows from this since

$$\mathcal{E}_1(w, u) = -\left(\frac{\partial w}{\partial \tau}, u\right) + \mathcal{A}_1(w, u) \leq \mathcal{A}_1(u, w) + \mathcal{A}_1(w, u).$$

Now we shall show the lemma in the case that B is an open set such that $\mathrm{Cap}(B) < \infty$. Denote by h_B the righthand side of (2.5). Then it is clear that $pG_{p+1}(e_B \wedge h_B) \leq e_B \wedge h_B$ ν—a.e. Furthermore, since

$$\begin{aligned}
\mathcal{A}_1(pG_{p+1}(e_B \wedge h_B), pG_{p+1}(e_B \wedge h_B)) &= \mathcal{E}_1(pG_{p+1}(e_B \wedge h_B), pG_{p+1}(e_B \wedge h_B)) \\
&\leq p(e_B \wedge h_B - pG_{p+1}(e_B \wedge h_B), e_B \wedge h_B) \\
&\leq p(e_B \wedge h_B - pG_{p+1}(e_B \wedge h_B), e_B)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{E}_1(pG_{p+1}(e_B \wedge h_B), e_B) \\
&\leq \mathcal{A}_1(pG_{p+1}(e_B \wedge h_B), e_B) + \mathcal{A}_1(e_B, pG_{p+1}(e_B \wedge h_B)) \\
&\leq 2M \| pG_{p+1}(e_B \wedge h_B) \|_{\mathcal{CV}} \, \| e_B \|_{\mathcal{CV}} ,
\end{aligned}
$$

it follows that $\| pG_{p+1}(e_B \wedge h_B) \|_{\mathcal{CV}}$ is bounded and hence, by Lemma 2.1, $e_B \wedge h_B$ belongs to $\mathcal{CV}$. In particular, $e_B \wedge h_B \in \mathcal{P}$. Combining this with that $e_B \wedge h_B = 1$ ν—a.e. on B, we have $e_B \wedge h_B \geq e_B$, which is nothing but $e_B \leq h_B$. Operating nV_{n+1} both sides of the inequality and letting $n \to \infty$, we have $\tilde{e}_B \leq h_B$ q.e.

The converse inequality can be proved similarly to [8; Lemma 4.1]. Since the set $C \equiv \{y \in B; \tilde{e}_B(y) \neq 1\}$ is a ν-null set, for any fixed time $s > 0$,

$$
\begin{aligned}
&E_y\Big(\int_0^\infty e^{-(p+1)t} E_{Y(t)}(I_C(Y(s)))\, dt\Big) \\
&= E_y\Big(\int_0^\infty e^{-(p+1)t} I_C(Y(s+t))\, dt\Big) \\
&\leq e^{(p+1)s} E_y\Big(\int_0^\infty e^{-(p+1)t} I_C(Y(t))\, dt\Big) \\
&= e^{(p+1)s} V_{p+1} I_C(y) = 0 \text{ q.e.y.}
\end{aligned}
$$

In particular, if $D = \{a_1, a_2, \cdots, a_n\}$ is a finite subset of (a, b) and

$$
\sigma(D) = \inf \{t \in D;\ Y(t) \in B\} ,
$$

then with P_y probability one,

$$
\begin{aligned}
&P_{Y(t)}(\tilde{e}_B(Y(\sigma(D))) \neq 1) \\
&\leq \sum_{k=1}^{n} P_{Y(t)}(Y(a_k) \in C, \sigma(D) = a_k) = 0 \text{ a.e. } t .
\end{aligned}
$$

Hence, by noting that $(e^{-t}\, \tilde{e}_B(Y(t)), P_y)$ is a right continuous supermartingale, we have

$$
\begin{aligned}
&nE_y\Big(\int_0^\infty e^{-(n+1)t} E_{Y(t)}(e^{-\sigma(D)})\, dt\Big) \\
&= nE_y\Big(\int_0^\infty e^{-(n+1)t} E_{Y(t)}(e^{-\sigma(D)}\, \tilde{e}_B(Y(\sigma(D)))\, dt\Big) \\
&\leq nE_y\Big(\int_0^\infty e^{-(n+1)t}\, \tilde{e}_B(Y(t))\, dt\Big) \\
&= nV_{n+1}\, \tilde{e}_B(y) \text{ q.e.}
\end{aligned}
$$

By letting D increase to a dense subset of $(0, \infty)$, we see that $nV_{n+1} h_B \leq nV_{n+1} \tilde{e}_B$ q.e. and which implies that $h_B \leq \tilde{e}_B$ q.e.

Suppose next that B is a compact subset of $\mathcal{X}$. According to Lemma 2.2 and [10; Corollary I.2], there exist a sequence $\{u_n\}$ of functions of $\mathcal{P}$ and a sequence $\{O_n\}$ of open sets of $\mathcal{X}$ containing B such that $u_n \geq 1$ ν-a.e. on O_n and $\lim_{n \to \infty} u_n = e_B$ ν-a.e. We may suppose that $\{O_n\}$ is decreasing. Since $u_n \geq e_{O_n}$, we see that e_{O_n} decreases to e_B ν-a.e. Therefore,

$$\begin{aligned}\tilde{e}_B(y) &= \lim_{m\to\infty} mV_{m+1}\, e_B(y) = \lim_{m\to\infty}\lim_{n\to\infty} mV_{m+1}\, e_{O_n}(y) \\ &= \lim_{m\in\infty}\lim_{n\to\infty} mV_{m+1}\, h_{O_n}(y) \\ &= \lim_{m\to\infty} mV_{m+1}\, h_B(y) = h_B(y) \text{ q.e.}\end{aligned}$$

The proof of (2.5) for Borel sets can be done by the usual capacitability theorem.

3. Extension of $(\mathcal{E}, \mathcal{F})$

So far, we considered the form $\mathcal{E}$ defined on $\mathcal{F}\times\mathcal{V}$ and $\mathcal{V}\times\mathcal{F}$. But for later discussion, we need to extend the domain of $\mathcal{E}$ to a class including $\mathcal{P}\times\mathcal{P}$, $\mathcal{P}\times\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}\times\hat{\mathcal{P}}$. For any $u\in\mathcal{V}$, since $pV_p u$ belongs to $\mathcal{F}$ and converges to u strongly in $\mathcal{H}$, we can naturally consider that $\lim_{p\to\infty}\mathcal{E}(pV_p u, u)$ defines an extension of $\mathcal{E}$. In fact, if either u or v belongs to $\mathcal{F}$ then the limit exists and coincides with $\mathcal{E}(u, v)$ (see Lemma 2.1). Now, we shall set

$$\mathcal{E}(u, v) = \lim_{p\to\infty}\mathcal{E}(pV_p u, u) \quad\text{and}\quad \tilde{\mathcal{E}}(u, v) = \lim_{p\to\infty}\mathcal{E}(p\hat{V}_p u, v)\,, \tag{3.1}$$

if the limits exist. Note that $\mathcal{E}(u, v)=\tilde{\mathcal{E}}(u, v)$ if $u\in\mathcal{F}$ or $v\in\mathcal{F}$, and that $\mathcal{E}(u, v)$ satisfies

$$\mathcal{E}(u, v) = \lim_{p\to\infty}\mathcal{E}(u, p\hat{V}_p v) = \lim_{p\to\infty} p(u-pV_p u, v) \tag{3.2}$$

if it exists.

Lemma 3.1. *If one of* $\lim_{p\to\infty}\mathcal{E}(pV_p u, v)$, $\lim_{p\to\infty}\mathcal{E}(v, pV_p u)$, $\lim_{p\to\infty}\mathcal{E}(u, p\hat{V}_p v)$ *and* $\lim_{p\to\infty}\mathcal{E}(p\hat{V}_o v, u)$ *exists for* $u, v\in\mathcal{V}$, *then the others exist. In this case,*

$$\mathcal{E}(u, v) = \lim_{p\to\infty}\mathcal{E}(pV_p u, v) = \lim_{p\to\infty}\mathcal{E}(u, p\hat{V}_p v)\,, \tag{3.3}$$

$$\tilde{\mathcal{E}}(v, u) = \lim_{p\to\infty}\mathcal{E}(p\hat{V}_p v, u) = \lim_{p\to\infty}\mathcal{E}(v, pV_p u)\,, \tag{3.4}$$

$$\mathcal{E}(u, v)+\tilde{\mathcal{E}}(v, u) = \mathcal{A}(u, v)+\mathcal{A}(v, u)\,. \tag{3.5}$$

Proof. Since

$$\mathcal{E}(w, u)+\mathcal{E}(u, w) = \mathcal{A}(w, u)+\mathcal{A}(u, w)$$

for all $u\in\mathcal{V}$ and $w\in\mathcal{F}$, we have

$$\mathcal{E}(pV_p u, v) = \mathcal{E}(u, p\hat{V}_p v) = \mathcal{A}(u, p\hat{V}_p v)+\mathcal{A}(p\hat{V}_p v, u)-\mathcal{E}(p\hat{V}_p v, u)\,, \tag{3.6}$$

$\forall v\in\mathcal{V}$. According to the fact that $\lim_{p\to\infty}\mathcal{A}(u, p\hat{V}_p v)=\mathcal{A}(u, v)$ and $\lim_{p\to\infty}\mathcal{A}$

$(p\hat{V}_p v, u) = \mathcal{A}(v, u)$ the equivalence of the existences of $\lim_{p\to\infty} \mathcal{E}(pV_p u, v)$, $\lim_{p\to\infty} \mathcal{E}(u, p\hat{V}_p v)$ and $\lim_{p\to\infty} \mathcal{E}(p\hat{V}_p v, u)$ as well as the equalities (3.3) and (3.5) follow from (3.2) and (3.6). The other part of the lemma is a consequence of (3.6) and

$$\mathcal{E}(pV_p u, v) = \mathcal{A}(pV_p u, v) + \mathcal{A}(v, pV_p u) - \mathcal{E}(v, pV_p u)\,.$$

Theorem 3.2. *The spaces $\mathcal{F} \oplus (\mathcal{P} \ominus \mathcal{P})$ and $\mathcal{F} \oplus (\hat{\mathcal{P}} \ominus \hat{\mathcal{P}})$ are identical. Moreover, if we denote the space by $\mathcal{I}$ then $\mathcal{I} \times \mathcal{I}$ is cantained in the domains of $\mathcal{E}$ and $\tilde{\mathcal{E}}$ extended by* (3.1).

Proof. If $u \in \mathcal{P}$, then there exists a positive Radon measure μ_u on $\mathcal{X}$ such that $\mathcal{E}_1(u, w) = \langle \mu_u, w \rangle$ for all $w \in \mathcal{F} \cap C_0(\mathcal{X})$. According to the definition of $\mathcal{E}$, it then holds that

$$\mathcal{E}_1(w, u) = -\langle \mu_u, w \rangle + \mathcal{A}_1(w, u) + \mathcal{A}_1(u, w)$$

for all $w \in \mathcal{F} \cap C_0(\mathcal{X})$. Since the linear functional f on $\mathcal{V}$ defined by $(f, w) = \mathcal{A}_1(w, u) + \mathcal{A}_1(u, w)$ belongs to $\mathcal{V}'$, there exists a function $\hat{G}_1 f \in \mathcal{F}$ such that $\mathcal{E}_1(w, \hat{G}_1 f) = (f, w)$ for all $w \in \mathcal{F}$. Hence the function $\hat{G}_1 f - u$ satisfies

$$\mathcal{E}_1(w, \hat{G}_1 f - u) = \langle \mu_u, w \rangle$$

for all $w \in \mathcal{F} \cap C_0(\mathcal{X})$. This implies that $\hat{G}_1 f - u = \hat{V}_1 \mu_u \in \hat{\mathcal{P}}$ and that $\mathcal{P} \subset \mathcal{F} \ominus \hat{\mathcal{P}}$. Similarly, we can see that $\hat{\mathcal{P}} \subset \mathcal{F} \ominus \mathcal{P}$. Thus the first assertion has been proved.

For the proof of the second assertion, we shall only prove that the limit $\mathcal{E}(u, v)$ in (3.1) exists for all $u, v \in \mathcal{P}$, since the other cases can be proved similarly by using Lemma 3.1. If $u, v \in \mathcal{P}$, then $\{pV_{p+1} u\}$ increases and, consequently, $\{\mathcal{E}_1(v, pV_{p+1} u)\}$ increases as $p \uparrow \infty$. Also, by noting that $pV_{p+1} u \in \mathcal{P}$ and $v \geq 0$, we have the uniform boundedness of $\{\mathcal{E}_1(v, pV_{p+1} u)\}$ with respect to $p > 0$, because

$$0 \leq \mathcal{E}_1(v, pV_{p+1} u) \leq \mathcal{A}_1(v, pV_{p+1} u) + \mathcal{A}_1(pV_{p+1} u, v)\,,$$

from (2.6). Then the existence of $\lim_{p\to\infty} \mathcal{E}(v, pV_{p+1} u)$ follows and which implies the existence of $\mathcal{E}(u, v)$ by Lemma 3.1.

REMARK 3.3. We noted after (3.1) that $\mathcal{E}(u, v) = \tilde{\mathcal{E}}(u, v)$ if either $u \in \mathcal{F}$ or $v \in \mathcal{F}$. But, for general u, v, the values $\mathcal{E}(u, v)$ and $\tilde{\mathcal{E}}(u, v)$ will not coincide. For example, if $u \in \mathcal{P}$ and $v \in \hat{\mathcal{P}}$, then they can be written as $u = V_1 \mu_u$ and $v = \hat{V}_1 \hat{\mu}_v$ for some positive Radon measures μ_u and $\hat{\mu}_v$ on $\mathcal{X}$, respectively, In this case,

$$\mathcal{E}_1(u, v) = \lim_{p\to\infty} \mathcal{E}_1(pV_p u, v) = \lim_{p\to\infty} \int_{\mathcal{X}} pV_p u(y)\, \hat{\mu}_v(dy)\,,$$

$$\tilde{\mathcal{E}}_1(u, v) = \lim_{p\to\infty} \mathcal{E}_1(p\hat{V}_p u, v) = \lim_{p\to\infty} \int_{\mathcal{X}} p\hat{V}_p u(y)\, \hat{\mu}_v(dy)\,.$$

In the last terms of these two equalities, $\tilde{u}=\lim_{p\to\infty} pV_p u$ is the 1-exsessive regularization of u and $\lim_{p\to\infty} p\hat{V}_p u(y)$ (if exists) is expected to be equal to the cofine limit of $\tilde{u}$ at y, which will be different from $\tilde{u}$ on a semipolar set.

REMARK 3.4. As we have seen in the above Remark 3.3, the relation

$$\mathcal{E}_1(u, v) = \int_{\mathcal{X}} \tilde{v}(y)\, \mu_u(dy)$$

holds for all $u\in\mathcal{P}$ and $v\in\mathcal{F}\oplus(\mathcal{P}\ominus\mathcal{P})$, where $\tilde{v}$ is the q.c. modification if $v\in\mathcal{F}$, and the 1-excessive regularization if $v\in\mathcal{P}$. Also the inequality (2.6) holds for u, $w\in\mathcal{P}$.

4. Decomposition of the Dirichlet forms

In the case of symmetric Dirichlet form (E, V) on $L^2(X; m)$, if we denote by V_p^X the resolvent of the associated Markov process, by $H_B^p(x, dz)$ the p-th order hitting distribution and by V_p^{X-B} the resolvent of the part process on $X-B$, then it is well known that the decomposition

$$V_p^X \varphi = V_p^{X-B} \varphi + H_B^p V_p^K \varphi$$

is the orthogonal decomposition with respect to E_p of $V_p^X \varphi$ into the elements of the subspace $V^{X-B}=\{\varphi\in V; \tilde{\varphi}=0 \text{ q.e. on } B\}$ and its orthogonal complement, where $\tilde{\varphi}$ is the q.c. modification of φ with respect to the E_1-capaaity (see [4]). In this section we shall show that the analogous result holds in our situation.

Now, as before, we shall denote by V_p the resolvent of the Markov process $\boldsymbol{M}=(Y(t), P_y)$ associated with $\mathcal{E}$, by H_B^p the p-th order hitting distribution of a Borel set B of $\mathcal{X}$, and by $V_p^{\mathcal{X}-B}$ the resolvent of the part process on $\mathcal{X}-B$, respectively, that is

$$H_B^p u(y) = E_y(e^{-p\sigma_B} u(Y(\sigma_B)))\,,$$

$$V_p^{\mathcal{X}-B} f(y) = E_y\Big(\int_0^{\sigma_B} e^{-pt} f(Y(t))\, dt\Big)\,.$$

For $h\in\mathcal{P}$ and a Borel set B of $\mathcal{X}$, set

$$h_B^1 = \inf\{u\in\mathcal{P};\ u\geq h\ \nu\text{—a.e. on } B\}\,. \tag{4.1}$$

Then as noted before, $h_B^1\in\mathcal{P}$ and the unique solution $h_\varepsilon^B\in\mathcal{P}\cap\mathcal{F}$ of

$$\mathcal{E}_1(h_\varepsilon^B, v) = \frac{1}{\varepsilon}\,((h_\varepsilon^B - I_B\, h)^-, v),\ \forall v\in\mathcal{V}$$

converges to h_B^1 ν-a.e. increasingly, strongly in $\mathcal{H}$ and weakly in $\mathcal{V}$ (see (2.1), (2.2)).

In the remainder of this paper, we shall consider that the elements of $\mathcal{F}$, $\mathcal{P}$ and $\hat{\mathcal{P}}$ are taken to be their q.c. modifications, 1-excessive and 1-coexcessive regularizations, respectively. The following lemma can be proved similarly to Lemma 2.3.

Lemma 4.1. *If $h\in\mathcal{P}$, then $H_B^1 h(y)=h_B^1(y)$, q.e.*

Theorem 4.2. *If $h\in\mathcal{P}$ [resp. $h\in\hat{\mathcal{P}}$], and B is a Borel subset of $\mathcal{X}$, then $H_B^1 h\in\mathcal{P}$ [resp. $\hat{H}_B^1 h\in\hat{\mathcal{P}}$] and*

$$\mathcal{E}_1(H_B^1 h, w)=0 \quad [resp.\ \mathcal{E}_1(w, \hat{H}_B^1 h)=0]\,, \tag{4.2}$$

for all $w\in\mathcal{G}$ such that $w=0$ ν-a.e. on B.

Proof. Since $H_B^1 h=h_B^1$ q.e., it belongs to $\mathcal{P}$ and ν-a.e. equals to the weak limit in $\mathcal{V}$ of functions $h_\varepsilon^B\in\mathcal{F}$. Hence, by noting that $(h_\varepsilon^B-I_B h)^-=0$ ν-a.e. outside of B, we have

$$\mathcal{E}_1(H_B^1 h, w)=\lim_{\varepsilon\to 0}\mathcal{E}_1(h_\varepsilon^B, w)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}((h_\varepsilon^B-I_B h)^-, w)=0\,,$$

for any function $w\in\mathcal{F}$ such that $w=0$ ν-a.e. on B. If $w\in\hat{\mathcal{P}}$ satisfies the condition of the theorem, then $\mathcal{E}_1(h_\varepsilon^B, p\hat{V}_{p+1}w)$ increases as $p\uparrow\infty$ and also as $\varepsilon\downarrow 0$. Hence

$$\begin{aligned}\mathcal{E}_1(H_B^1 h, w)&=\lim_{p\to\infty}\mathcal{E}_1(pV_{p+1}H_B^1 h, w)=\lim_{p\to\infty}\mathcal{E}_1(H_B^1 h, p\hat{V}_{p+1}w)\\&=\lim_{p\to\infty}\lim_{\varepsilon\to 0}\mathcal{E}_1(h_\varepsilon^B, p\hat{V}_{p+1}w)=\lim_{\varepsilon\to 0}\lim_{p\to\infty}\mathcal{E}_1(h_\varepsilon^B, p\hat{V}_{p+1}w)\\&=\lim_{\varepsilon\to 0}\mathcal{E}_1(h_\varepsilon^B, w)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}((h_\varepsilon^B-I_B h)^-, w)=0\,.\end{aligned}$$

Finally suppose that $w\in\mathcal{P}$ satisfies the condition in the theorem, then

$$\begin{aligned}\mathcal{E}_1(H_B^1 h, w)&=\lim_{p\to\infty}\mathcal{E}_1(pV_{p+1}H_B^1 h, w)\\&=\lim_{p\to\infty}\{\mathcal{A}_1(pV_{p+1}H_B^1 h, w)+\mathcal{A}_1(w, pV_{p+1}H_B^1 h)-\mathcal{E}_1(w, pV_{p+1}H_B^1 h)\}\\&=\mathcal{A}_1(H_B^1 h, w)+\mathcal{A}_1(w, H_B^1 h)-\lim_{p\to\infty}\mathcal{E}_1(w, pV_{p+1}H_B^1 h)\,.\end{aligned}$$

Since $\mathcal{E}_1(w, pV_{p+1}h_\varepsilon^B)$ increases as $p\uparrow\infty$ and $\varepsilon\downarrow 0$ and also as $\lim_{\varepsilon\to 0}\mathcal{E}_1(h_\varepsilon^B, w)=0$, we have

$$\begin{aligned}\lim_{p\to\infty}\mathcal{E}_1(w, pV_{p+1}H_B^1 h)&=\lim_{p\to\infty}\mathcal{E}_1(p\hat{V}_{p+1}w, H_B^1 h)\\&=\lim_{p\to\infty}\lim_{\varepsilon\to 0}\mathcal{E}_1(p\hat{V}_{p+1}w, h_\varepsilon^B)=\lim_{p\to\infty}\lim_{\varepsilon\to 0}\mathcal{E}_1(w, pV_{p+1}h_\varepsilon^B)\end{aligned}$$

$$= \lim_{\varepsilon\to 0}\lim_{p\to\infty} \mathcal{E}_1(w, pV_{p+1} h_\varepsilon^B) = \lim_{\varepsilon\to 0}\lim_{p\to\infty} \mathcal{E}_1(p\hat{V}_{p+1} w, h_\varepsilon^B)$$
$$= \lim_{\varepsilon\to 0} \mathcal{E}_1(w, h_\varepsilon^B) = \lim_{\varepsilon\to 0} \{\mathcal{A}_1(h_\varepsilon^B, w)+\mathcal{A}_1(w, h_\varepsilon^B)-\mathcal{E}_1(h_\varepsilon^B, w)\}$$
$$= \mathcal{A}_1(H_B^1 h, w)+\mathcal{A}_1(w, H_B^1 h)\,.$$

Hence $\mathcal{E}_1(H_B^1 h, w)=0$ for all w satisfying the condition in the theorem. The proof of the dual case is similar.

Corollary 4.3. *For any $f, g\in\mathcal{H}$ and $p>0$.*

$$\mathcal{E}_p(V_p f, \hat{V}_p g) = \mathcal{E}_p(H_B^p V_p f, \hat{H}_B^p \hat{V}_p g)+\mathcal{E}_p(V_p^{\mathcal{X}-H} f, \hat{V}_p^{\mathcal{X}-B} g) \tag{4.3}$$

$$\mathcal{E}_p(V_p f, V_p g) = \mathcal{E}_p(H_B^p V_p f, H_B^p V_p g)+\mathcal{E}_p(V_p^{\mathcal{X}-B} f, V_p^{\mathcal{X}-B} g) \tag{4.4}$$

$$\mathcal{E}_p(\hat{V}_p f, \hat{V}_p g) = \mathcal{E}_p(\hat{H}_B^p \hat{V}_p f, \hat{H}_B^p \hat{V}_p g)+\mathcal{E}_p(\hat{V}_p^{\mathcal{X}-B} f, \hat{V}_p^{\mathcal{X}-B} g)\,. \tag{4.5}$$

Proof. Since $\mathcal{E}_p(V_p^{\mathcal{X}-B} f, \hat{H}_B^p \hat{V}_p g)=0$ and $\varepsilon_p(H_B^p V_p f, \hat{V}_p^{\mathcal{X}-B} g)=0$ from Theorem 4.2, (4.3) follows obviously. The proofs of (4.4) and (4.5) are similar.

Corollary 4.4. *If $u=V_1 \mu_u\in\mathcal{P}$ and $v=\hat{V}_1 \hat{\mu}_v\in\hat{\mathcal{P}}$, then*

$$\langle\mu_u, \hat{H}_B^1 \hat{V}_1 \hat{\mu}_v\rangle = \langle\hat{\mu}_v, H_B^1 V_1 \mu_u\rangle\,. \tag{4.6}$$

Proof. Since

$$\mathcal{E}_1(H_B^1 u, v-\hat{H}_B^1 v) = 0 \quad\text{and}\quad \mathcal{E}_1(u-H_B^1 u, \hat{H}_B^1 v) = 0$$

are known from Theorem 4.2, it follows that

$$\mathcal{E}_1(H_B^1 u, v) = \mathcal{E}_1(H_B^1 u, \hat{H}_B^1 v) = \mathcal{E}_1(u, \hat{H}_B^1 v)\,.$$

The result is an easy consequence of this equality.

Corollary 4.5. *Let $u=V_1 \mu\in\mathcal{P}$ and $H_B^1 u=V_1 \mu^B$ for some positive Radon measures μ and μ^B on $\mathcal{X}$, respectively. Then*

$$\mu^B(dy) = \mu\hat{H}_B^1(dy) \equiv \int_{\mathcal{X}} \hat{H}_B^1(z, dy)\, \mu(dz)\,. \tag{4.7}$$

In particular, μ^B is supported by the set of coregular points of B.

Proof. Let $v\in\hat{\mathcal{P}}-\hat{\mathcal{P}}$. Since $p\hat{V}_{p+1} v$ is dominated by an excessive function and converges q.e. to v, we have

$$\mathcal{E}_1(H_B^1 u, v) = \lim_{p\to\infty} \mathcal{E}_1(H_B^1 u, p\hat{V}_{p+1} v) = \lim_{p\in\infty} \int_{\mathcal{X}} p\hat{V}_{p+1} v(y)\, \mu^B(dy)$$
$$= \int_{\mathcal{X}} v(y)\, \mu^B(dy)\,.$$

On the other hand,

$$\mathcal{E}_1(H_B^1 u, v) = \mathcal{E}_1(u, \hat{H}_B^1 v) = \int_{\mathcal{X}} \hat{H}_B^1 v(z)\, \mu(dz)\,.$$

Hence

$$\int_{\mathcal{X}} v(y)\, \mu^B(dy) = \int_{\mathcal{X}} \hat{H}_B^1 v(z)\, \mu(dz)$$

for all $v \in \hat{\mathcal{P}} \ominus \hat{\mathcal{P}}$. In this equality, set $v = n\hat{V}_{n+1} f$ for $f \in C_0(\mathcal{X})$. If we take a 1-coexcessive function h such that $|f| \leq h$ a.e. (for example, $h = \|f\|_\infty\, \hat{e}_{(\mathrm{Supp}[f])}$), then $|n\hat{V}_{n+1} f| \leq h$ for all $n \geq 1$. This admits to use Lebesgue's theorem to get

$$\int_{\mathcal{X}} f(y)\, \mu^B(dy) = \int_{\mathcal{X}} \hat{H}_B^1 f(z)\, \mu(dz)\,.$$

Therefore we have the result.

REMARK 4.6. If $u \in \mathcal{F}$ is dominated by a 1-excessive function, then there exists a sequence $\{u_n\}$ of functions of $\mathcal{F} \cap (\mathcal{P} \ominus \mathcal{P})$ such that $\lim_{n\to\infty} u_n = u$ in $\mathcal{F}$ and $\lim_{n\to\infty} H_B^1 u_n = H_B^1 u$ in $\mathcal{V}$. In particular $H_B^1 u$ belongs to $\mathcal{V}$.

In fact, since $pV_p u$ converges to u weakly in $\mathcal{F}$ as $p \to \infty$, there exists a sequence $\{u_n\}$ of functions of $\mathcal{F} \cap (\mathcal{P} \ominus \mathcal{P})$ which can be written as convex combinations of functions of the form $pV_p u$ such that $u_n \to u$ in $\mathcal{F}$. Each function u_n has the form $V_1 f_n \in \mathcal{P} \ominus \mathcal{P}$, because $pV_p u = V_1(p(u-(p-1)\, V_p u))$. From Lemma 2.1, (3.2) and Theorem 4.2,

$$\begin{aligned}
\alpha\, \|H_B^1(u_n-u_m)\|_{\mathcal{V}}^2 &\leq \mathcal{A}_1(H_B^1(u_n-u_m), H_B^1(u_n-u_m)) \\
&\leq \mathcal{E}_1(H_B^1(u_n-u_m), H_B^1(u_n-u_m)) = \mathcal{E}_1(H_B^1(u_n-u_m), u_n-u_m) \\
&\leq K\, \|H_B^1(u_n-u_m)\|_{\mathcal{V}}\, \|u_n-u_m\|_{\mathcal{F}}\,,
\end{aligned}$$

for some constant K. Hence

$$\|H_B^1(u_n-u_m)\|_{\mathcal{V}} \leq \frac{K}{\alpha}\, \|u_n-u_m\|_{\mathcal{F}}\,.$$

The strong convergence of $\{H_B^1 u_n\}$ in $\mathcal{V}$ is an easy consequence of this inequality. The limit is equal to $H_B^1 u$ q.e. because $H_B^1 u_n \to H_B^1 u$ q.e. by the bounded convergence theorem.

DEFINITION. We say that $\mathcal{E}$ possesses the *local property* if $(E^{(\tau)}, V)$ possesses the local property for a.e. $\tau \in R^1$, that is, for a.e. $\tau \in R^1$, $E^{(\tau)}(\varphi, \psi) = 0$ for every $\varphi, \psi \in V$ such that their supports are disjoint compact subsets of X.

As in the case of symmetric Dirichlet forms, we can show the following

Theorem 4.7. *The following conditions are equivalent to each other.*

(i) *$\mathcal{E}$ possesses the local property.*

(ii) *The Markov process $\boldsymbol{M}$ on $\mathcal{X}$ associated with $\mathcal{E}$ is a diffusion process.*

(iii) *The dual Markov process* $\hat{\boldsymbol{M}}$ *of* $\boldsymbol{M}$ *is a diffusion process.*

The proof of the theorem is similar to [4; Theorem 4.5.1] according to the next lemma.

Lemma 4.8. *The following conditions are equivalent to each other.*
(i) $\mathcal{E}$ *possesses the local property.*
(ii) *For any relatively compact open set* O *of* $\mathcal{X}$, $H^1_{\mathcal{X}-O}(y, \cdot)$ *is concentrated on the boundary* ∂O *for q.e.* $y \in O$.

Proof. (i)$\Rightarrow$(ii): Let O be a relatively compact open set of $\mathcal{X}$, $u \in \mathcal{F} \cap C_0(\mathcal{X})$ be a function such that Supp $[u] \subset \mathcal{X}-\bar{O}$, and $\{u_n\}$ be the sequence in Remark 4.6. For any non-negative function f such that Supp $[f] \subset O$, set $v = \hat{V}_1 f - \hat{H}^1_{\mathcal{X}-O} \hat{V}_1 f$. Then v is supported by $\bar{O}$ and

$$\mathcal{E}_1(u_n - H^1_{\mathcal{X}-O} u_n, v) = \mathcal{E}_1(u_n, v) \to \mathcal{E}_1(u, v) = 0\,.$$

Therefore

$$\begin{aligned} 0 &= \lim_{n\to\infty} \mathcal{E}_1(u_n - H^1_{\mathcal{X}-O} u_n, v) = \lim_{n\to\infty} \mathcal{E}_1(u_n - H^1_{\mathcal{X}-O} u_n, \hat{V}_1 f) \\ &= \lim_{n\to\infty} (u_n - H^1_{\mathcal{X}-O} u_n, f) = (u - H^1_{\mathcal{X}-O} u, f)\,. \end{aligned}$$

This implies that $u = H^1_{\mathcal{X}-O} u$ q.e. on O. In particular $H^1_{\mathcal{X}-O} u = 0$ q.e. on O and (ii) follows.

(ii)$\Rightarrow$(i): Let φ and ψ be functions in V whose supports are compact subsets of $X-\bar{A}$ and A, respectively, for an open set A of X. For any interval (a, b) in R^1, set $O=(a, b)\times A$. Let $\xi(\tau)$ be a function belonging to $C^1_0(a, b)$. Then the support of $\xi\otimes\varphi$ is contained in $\mathcal{X}-\bar{O}$ and, by (ii), $H^1_{\mathcal{X}-O}(\xi\otimes\varphi)=0$ q.e. on O. Therefore we have $\xi\otimes\varphi - H^1_{\mathcal{X}-O}(\xi\otimes\varphi)=0$ a.e. on $\mathcal{X}$. Since $\xi\otimes\psi=0$ a.e. on $\mathcal{X}-\bar{O}$, it follows that

$$\begin{aligned} 0 &= \mathcal{E}_1(\xi\otimes\varphi - H^1_{\mathcal{X}-O}(\xi\otimes\varphi), \xi\otimes\psi) = \mathcal{E}_1(\xi\otimes\varphi, \xi\otimes\psi)\,. \\ &= \int E^{(\tau)}(\varphi, \psi)\, \xi(\tau)^2\, d\tau\,. \end{aligned}$$

This implies that $E^{(\tau)}(\varphi, \psi)=0$ for a.e. τ.

5. Measures of finite energy integrals and associated additive functionals

In this section, similarly to Fukushima [4], we shall define the notions of measures of finite energy integral as well as smooth measures and show a correspondence between such measures and positive natural additive functionals (PNAF's). Such a correspondence has also been studied by Revuz ([11], [12]) for Hunt processes satisfying the duality condition (see also [3]).

DEFINITION. A positive Radon measure μ on $\mathcal{X}$ charging no set of zero capacity is said to be a *measure of finite energy integral* if there exists a constant C such that

$$\int_{\mathcal{X}} |w(y)|\,d\mu(y) \leq C\|w\|_{\mathcal{F}}, \quad \text{for all} \quad w \in \mathcal{F} \cap C_0(\mathcal{X}). \tag{5.1}$$

The set of all measures of finite energy integral is denoted by $\mathcal{S}_0$.

If $\mu \in \mathcal{S}_0$, then (5.1) holds for all $w \in \mathcal{F}$ by taking its q.c. modification. To give a characterization of the measures of finite energy integral, we need the following two lemmas.

Lemma 5.1. (cf. Pierre [10]) *If $w \in \mathcal{F}$, then there exists a constant K such that*

$$\|e_{|w|}\|_{\mathcal{V}} \leq K\|w\|_{\mathcal{F}}. \tag{5.2}$$

Proof. We shall first note that there exists a constant K_1 such that

$$\|e_{|v|}\|_{\mathcal{V}} \leq K_1(\|e_v\|_{\mathcal{V}} + \|e_{(-v)}\|_{\mathcal{V}}), \ \forall v \in \mathcal{V}. \tag{5.3}$$

To show it, let $|v|_\varepsilon \in \mathcal{F} \cap \mathcal{P}$ be the solution of (2.2) corresponding to $h = |v|$. Since

$$|v|_\varepsilon \leq e_{|v|} \leq e_{v^+} + e_{v^-} = e_v + e_{(-v)},$$

and $e_v + e_{(-v)} \in \mathcal{P}$, we have from (2.6)

$$\begin{aligned}\mathcal{A}_1(|v|_\varepsilon, |v|_\varepsilon) &= \mathcal{E}_1(|v|_\varepsilon, |v|_\varepsilon) \leq \mathcal{E}_1(|v|_\varepsilon, e_v + e_{(-v)}) \\ &\leq \mathcal{A}_1(|v|_\varepsilon, e_v + e_{(-v)}) + \mathcal{A}_1(e_v + e_{(-v)}, |v|_\varepsilon) \\ &\leq 2M\| |v|_\varepsilon\|_{\mathcal{V}} \|e_v + e_{(-v)}\|_{\mathcal{V}}.\end{aligned}$$

Then the condition (E3) shows that

$$\| |v|_\varepsilon\|_{\mathcal{V}} \leq K_1(\|e_v\|_{\mathcal{V}} + \|e_{(-v)}\|_{\mathcal{V}})$$

for some constant K_1. The inequality (5.3) follows from this by letting $\varepsilon \to 0$. Therefore to show (5.2), it is enough to prove that there exists a constant K_2 such that $\|e_w\|_{\mathcal{V}} \leq K_2\|w\|_{\mathcal{F}}$ for all $w \in \mathcal{F}$. To this end, let $w_\varepsilon \in \mathcal{F} \cap \mathcal{P}$ be the solution of (2.2) for $h = w$. Then,

$$\begin{aligned}\mathcal{A}_1(w_\varepsilon, w_\varepsilon) &= \mathcal{E}_1(w_\varepsilon, w_\varepsilon) = \frac{1}{\varepsilon}((w_\varepsilon - w)^-, w_\varepsilon) \\ &= \frac{1}{\varepsilon}((w_\varepsilon - w)^-, w_\varepsilon - w) + \frac{1}{\varepsilon}((w_\varepsilon - w)^-, w) \\ &\leq \frac{1}{\varepsilon}((w_\varepsilon - w)^-, w) = \mathcal{E}_1(w_\varepsilon, w) \leq K_3\|w_\varepsilon\|_{\mathcal{V}} \|w\|_{\mathcal{F}},\end{aligned}$$

for some constant K_3. Use again (E.3) to show

$$\|w_\varepsilon\|_{\mathcal{V}} \leq \frac{K_3}{\alpha} \|w\|_{\mathcal{F}} .$$

Now the result follows by letting $\varepsilon \to 0$.

Lemma 5.2. *For any $f \in \mathcal{V}'$,*

$$\|G_1 f\|_{\mathcal{F}} \leq \left(1+\frac{M+1}{\alpha}\right) \|f\|_{\mathcal{V}'} . \tag{5.4}$$

Proof. Set $w = G_1 f$ for $f \in \mathcal{V}'$. Then $\|w\|_{\mathcal{V}} \leq \frac{1}{\alpha} \|f\|_{\mathcal{V}'}$ (see [8; (2.12)]). Hence, for any $v \in \mathcal{V}$,

$$\begin{aligned} \left|\left(-\frac{\partial w}{\partial \tau}, v\right)\right| &= |(f, v) - \mathcal{A}_1(w, v)| \leq (\|f\|_{\mathcal{V}'} + M\|w\|_{\mathcal{V}}) \|v\|_{\mathcal{V}} \\ &\leq \left(1+\frac{M}{\alpha}\right) \|f\|_{\mathcal{V}'} \|v\|_{\mathcal{V}} . \end{aligned}$$

Therefore we have $\|\frac{\partial w}{\partial \tau}\|_{\mathcal{V}'} \leq \left(1+\frac{M}{\alpha}\right) \|f\|_{\mathcal{V}'}$ and

$$\|w\|_{\mathcal{F}} \leq \left\|\frac{\partial w}{\partial \tau}\right\|_{\mathcal{V}'} + \|w\|_{\mathcal{V}} \leq \left(1+\frac{M+1}{\alpha}\right) \|f\|_{\mathcal{V}'} .$$

Now we can show the following theorem which characterizes the measures of finite energy integral.

Theorem 5.3. *The following conditions are equivalent to each other.*

(i) $\mu \in \mathcal{S}_0$.

(ii) *There exists $u \in \mathcal{P}$ such that $\mu = \mu_u$.*

(iii) *There exists $v \in \mathcal{P}$ such that $\mu = \hat{\mu}_v$.*

Proof. (ii)$\Rightarrow$(i): If $\mu = \mu_u$ for $u \in \mathcal{P}$, then, by using the solution $|w|_\varepsilon$ of (2.2) corresponding to $h = |w|$, we have

$$\begin{aligned} \int_{\mathcal{X}} |w(y)| \mu(dy) &\leq \int_{\mathcal{X}} e_{|w|}(y) \mu(dy) = \lim_{\varepsilon \downarrow 0} \int_{\mathcal{X}} |w|_\varepsilon(y) \mu(dy) \\ &= \lim_{\varepsilon \downarrow 0} \mathcal{E}_1(u, |w|_\varepsilon) \leq \lim_{\varepsilon \downarrow 0} \{\mathcal{A}_1(u, |w|_\varepsilon) + \mathcal{A}(|w|_\varepsilon, u)\} \\ &= \mathcal{A}_1(u, e_{|w|}) + \mathcal{A}_1(e_{|w|}, u) \leq 2M \|u\|_{\mathcal{V}} \|e_{|w|}\|_{\mathcal{V}} , \end{aligned}$$

for all $w \in \mathcal{F}$. To show (i) it is enough to apply Lemma 5.1 to this inequality.

The proof of (iii)$\Rightarrow$(i) is similar.

To show (i)$\Rightarrow$(ii), suppoes that $\mu \in \mathcal{S}_0$ and set

$$L(f) = \int_{\mathcal{X}} \hat{V}_1 f(y) \mu(dy) \quad \text{for} \quad f \in \mathcal{H} .$$

Then $L(f) \geq 0$ if $f \geq 0$ ν-a.e. and

$$|L(f)| \leq C \|\hat{V}_1 f\|_{\mathcal{F}} \leq C(1+\frac{M+1}{\alpha}) \|f\|_{\mathcal{V}'} \leq CK_1(1+\frac{M+1}{\alpha}) \|f\|_{\mathcal{H}},$$

where K_1 is a constant such that $\|f\|_{\mathcal{V}'} \leq K_1 \|f\|_{\mathcal{H}}$. Hence there exists a non-negative function $u \in \mathcal{H}$ such that

$$L(f) = \int_{\mathcal{X}} f(y)\, u(y)\, \nu(dy)\,.$$

Moreover, since

$$\left|\int_{\mathcal{X}} f(y)\, u(y)\, \nu(dy)\right| = |L(f)| \leq C(1+\frac{M+1}{\alpha}) \|f\|_{\mathcal{V}'},$$

it is extended to a continuous lineaı functional on $\mathcal{V}'$. This shows that $u \in \mathcal{V}$ and

$$\int_{\mathcal{X}} \hat{V}_1 f(y)\, \mu(dy) = \int_{\mathcal{X}} f(y)\, u(y)\, \nu(dy) = \mathcal{E}_1(u, \hat{V}_1 f),\ \forall f \in \mathcal{H}\,. \tag{5.5}$$

For any $w \in \mathcal{F}$, since $n\hat{V}_n w$ converges to w weakly in $\mathcal{F}$ (see Prieıre [10]), there exists a sequence $\{w_n\}$ constituted of convex combinations of $\{n\hat{V}_n w\}$ and converging to w q.e. uniformly and strongly in $\mathcal{F}$. Then, $\{w_n\}$ converges to w in $L^1(\mu)$ by (5.1). Noting that each w_n can be wıitten in the form $\hat{V}_1 f_n$ for some $f_n \in \mathcal{H}$, we have from (5.5),

$$\int_{\mathcal{X}} w_n(y)\, \mu(dy) = \mathcal{E}_1(u, w_n)\,, \quad \forall n \geq 1\,. \tag{5.6}$$

Letting $n \to \infty$ we see that (5.6) also holds for w in place of w_n and which implies that $\mu = \mu_u$.

The proof of (i)$\Rightarrow$(iii) is similar.

Now we shall concern the problem of the correspondence between the measures in $\mathcal{S}_0$ and NAF's. To show it, we need the following

Lemma 5.4. *If $u \in \mathcal{P}$, then, for any decreasing sequence $\{B_n\}$ of nearly Borel sets such that $P_y(\lim_{n\to\infty} \sigma_{B_n} \geq \zeta) = 1$ q.e.,*

$$\lim_{n\to\infty} H^1_{B_n} u = 0 \text{ q.e.} \tag{5.7}$$

Proof. Since $\{H^1_{B_n} u\}$ is a decreasing sequence of excessive functions, it is enough to show that $\lim_{n\to\infty} H^1_{B_n} u = 0$ a.e. (see [1; Proposition VI. 3.2]). For any $f \in C_0^+(\mathcal{X})$,

$$\begin{aligned}\lim_{n\to\infty} (H^1_{B_n} u, f) &= \lim_{n\to\infty} \mathcal{E}_1(H^1_{B_n} V_1 \mu_u, \hat{V}_1 f) \\ &= \lim_{n\to\infty} \mathcal{E}_1(V_1 \mu_u, \hat{H}^1_{B_n} \hat{V}_1 f) = \lim_{n\to\infty} \langle \mu_u, \hat{H}^1_{B_n} \hat{V}_1 f \rangle\,.\end{aligned}$$

The righthand side is equal to zero because

$$\hat{H}^1_{B_n}\hat{V}_1 f(y) = E_y\left(\int_{\sigma_{B_n}}^{\zeta} e^{-t} f(Y(t))\, dt\right) \to 0 \quad \text{q.e.}$$

from the hypothesis.

By using this lemma, we can prove the following:

Theorem 5.5. *If $u \in \mathcal{P}$, then there exists a unique PNAF A_t such that*

$$u(y) = E_y\left(\int_0^\infty e^{-t}\, dA_t\right) \quad q.e. \tag{5.8}$$

Proof. Set $B_n = \{y; u(y) > n\}$. Then, from the fine continuity of u,

$$u(y) \geq E_y(\exp(-\sigma_{B_n})\, u(Y(\sigma_{B_n}))) \geq n E_y(\exp(-\sigma_{B_n}); \sigma_{B_n} < \zeta)\,.$$

Hence $P_y(\lim_{n\to\infty} \sigma_{B_n} \geq \zeta) = 1$ q.e. Therefore, for any stopping time T,

$$\begin{aligned}
&E_y(e^{-T} u(Y(T)); u(Y(T)) > n) \\
&\quad \leq E_y(e^{-T} u(Y(T)); \sigma_{B_n} \leq T) \leq E_y(\exp(-\sigma_{B_n})\, u(Y(\sigma_{B_n}))) \\
&\quad = H^1_{B_n} u(y) \to 0 \text{ q.e. as } n \to \infty\,.
\end{aligned}$$

This implies that $\{e^{-t} u(Y(t))\}$ is a potential of class (D) with respect to the e^{-t}-subprocess of $\boldsymbol{M}$. Hence the theorem is a consequence of Meyer's theorem (see [2]).

From Theorems 5.3 and 5.5, for any $\mu \in \mathcal{S}_0$, there exists a PNAF $A(t)$ such that

$$V_1 \mu(y) = E_y\left(\int_0^\infty e^{-t}\, dA_t\right) \quad \text{q.e.} \tag{5.9}$$

Due to this relation, it holds that

$$\langle \mu, f \rangle = \lim_{p\to\infty} p E_\nu\left(\int_0^\infty e^{-pt} f(Y(t))\, dA(t)\right), \tag{5.10}$$

for all non-negative measurable function f on $\mathcal{X}$ (see [4; Lemma 5.1.3]). Furthermore, this correspondence can be extended to the class of all smooth measures defined as follows. A positive Borel measure μ on $\mathcal{X}$ is called a *smooth measure* if it satisfies the following conditions:
(i) μ charges no set of zero capacity,
(ii) there exists an increasing sequence $\{F_n\}$ of compact sets of $\mathcal{X}$ such that

$$\mu(F_n) < \infty\,, \quad \mu(\mathcal{X} - \cap_{n=1}^\infty F_n) = 0\,, \quad \lim_{n\to\infty} \mathrm{Cap}(K - F_n) = 0\,,$$

for any compact set K of $\mathcal{X}$.

We will not give the proof of the following theorem because it can be shown similarly to [4; §3.2, §5.1].

Theorem 5.6. *The following conditions are equivalent to each other.*
(i) *μ is a smooth measure.*
(ii) *There exists an increasing sequence $\{F_n\}$ of compact sets of $\mathcal{X}$ such that*

$$\mu(\mathcal{X}-\cup_{n=1}^{\infty} F_n) = 0\,, \quad \lim_{n\to\infty} \mathrm{Cap}(K-F_n) = 0\,,$$

for any compact set K of $\mathcal{X}$ and $I_{F_n}\cdot\mu\in\mathcal{S}_0$ for each $n\geq 1$.
(iii) *There exists a PNAF $A(t)$ of $\boldsymbol{M}$ satisfying* (5.10).

6. Decomposition of additive functionals

In the case of symmetric Markov processes $(X(t))$ associated with regular Dirichlet forms, it is well known that the additive functional (AF in abbreviation) of the form $u(X(t))-u(X(0))$ can be decomposed uniquely into the sum of martingale AF and continuous AF of zero energy (see [4]). In this section, we shall show that the analogous result remains valid in our situation. As in [4] (see also [7], [13]), for an AF $A(t)$ of $\boldsymbol{M}$, define the energy $e(A)$ of A by

$$e(A) = \frac{1}{2}\lim_{p\to\infty} p^2 E_\nu\Big(\int_0^\infty e^{-pt} A^2(t)\, dt\Big)\,. \tag{6.1}$$

An AF $M(t)$ of $\boldsymbol{M}$ is called a *martingale* AF (MAF in abbreviation) if, for all $t\geq 0$,

$$E_y(M^2(t))<\infty\,, \quad E_y(M(t)) = 0 \text{ q.e.}$$

The set of all MAF's of finite energy is denoted by $\mathring{\mathcal{M}}$. A NAF $N(t)$ is called a NAF *of zero energy* if

$$E_y(|N(t)|)<\infty \text{ q.e.} \quad \text{and} \quad e(N) = 0\,.$$

The set of all NAF's of zero energy is denoted by $\mathcal{N}$. For any function $u\in\mathcal{F}\oplus(\mathcal{P}\ominus\mathcal{P})$, we shall define the AF $A^{[u]}$ by

$$A^{[u]}(t) = u(Y(t))-u(Y(0))\,,$$

where u is taken to be the version stated in §4.

Lemma 6.1. *If $w\in\mathcal{F}$, then there exists a positive Radon measure k on $\mathcal{X}$ such that*

$$e(A^{[w]}) = \mathcal{A}(w, w)-\frac{1}{2}\int_{\mathcal{X}} w^2(y)\, k(dy)\,. \tag{6.2}$$

Proof. For any $p>0$ and $u\in\mathcal{H}$,

$$\begin{aligned} 0 &\leq p^2 E_\nu\Big(\int_0^\infty e^{-pt}(u(Y(t))-u(Y(0)))^2\, dt\Big) \\ &= p^2\langle \nu, V_p u^2-2uV_p u+\frac{1}{p} u^2\rangle = 2p(u-pV_p u, u)-(u^2, 1-p\hat{V}_p 1)\,. \end{aligned}$$

Hence

$$p(u^2, 1-p\hat{V}_p 1) \leq 2p(u-pV_p u, u), \ \forall u \in \mathcal{H} . \tag{6.3}$$

In particular, when $w \in \mathcal{F}$, since $2p(w-pV_p w, w)$ converges to $2\mathcal{E}(w, w)=2\mathcal{A}(w, w)$ as $p \to \infty$, there exists a sequence $p_n \uparrow \infty$ such that the measure $k_n(dy) \equiv p_n(1-p_n \hat{V}_{p_n} 1)(y)\, \nu(dy)$ converges vaguely to a positive Radon measure $k(dy)$ on $\mathcal{X}$. Then (6.3) tells us that

$$\int_{\mathcal{X}} w^2(y)\, k(dy) \leq 2\mathcal{A}(w, w), \ \forall w \in \mathcal{F} \cap C_0(\mathcal{X}) , \tag{6.4}$$

holds. For general $w \in \mathcal{F}$, by taking a sequence $w_n \in \mathcal{F} \cap C_0(\mathcal{X})$ such that $w_n \to w$ in $\mathcal{F}$, we can see from (6.4) that w_n converges to w in $L^2(\mathcal{X}; k)$ and (6.4) holds for all $w \in \mathcal{F}$. Moreover (6.3), (6.4) and the triangle inequality give us the following inequality.

$$\begin{aligned}
&|\{\int_{\mathcal{X}} w^2(y)\, k_n(dy)\}^{1/2} - \{\int_{\mathcal{X}} w^2(y)\, k(dy)\}^{1/2}| \\
&\leq |\{\int_{\mathcal{X}} w_m^2(y)\, k_n(dy)\}^{1/2} - \{\int_{\mathcal{X}} w_m^2(y)\, k(dy)\}^{1/2}| \\
&\quad + |\int_{\mathcal{X}} (w_m-w)^2(y)\, k_n(dy)|^{1/2} + |\int_{\mathcal{X}} (w_m-w)^2(y)\, k(dy)|^{1/2} \\
&\leq |\{\int_{\mathcal{X}} w_m^2(y)\, k_n(dy)\}^{1/2} - \{\int_{\mathcal{X}} w_m^2(y)\, k(dy)\}^{1/2}| \\
&\quad + \{2p_n(w_m-w-p_n V_{p_n}(w_m-w), w_m-w)\}^{1/2} + \{2\mathcal{A}(w_m-w, w_m-w)\}^{1/2} .
\end{aligned}$$

Let $n \to \infty$ and use Lemma 3.1 to get

$$\begin{aligned}
&\lim_{n\to\infty} |\{\int_{\mathcal{X}} w^2(y)\, k_n(dy)\}^{1/2} - \{\int_{\mathcal{X}} w^2(y)\, k(dy)\}^{1/2}| \\
&\leq 2\sqrt{2}\ \mathcal{A}(w_m-w, w_m-w)^{1/2} .
\end{aligned}$$

Since the righthand side tends to zero as $m \to \infty$, we have

$$\lim_{n\to\infty} \int_{\mathcal{X}} w^2(y) k_n(dy) = \int_{\mathcal{X}} w^2(y)\, k(dy), \ \forall w \in \mathcal{F} .$$

Now, to prove (6.2), it is enough to take $p=p_n$ in the equality of the first paragraph of the proof and let $n \to \infty$.

To show the decomposition theorem, we need the following lemma (see [4: Theorem 5.2.2]).

Lemma 6.2. *If $w \in \mathcal{F}$, then for any $T>0$ there exists a constant $K(T)$ depending on T such that*

$$P_\mu(\sup_{0\leq t\leq T} |w(Y(t))| > \varepsilon) \leq \frac{K(T)}{\varepsilon} \|V_1 \mu\|_{\mathcal{CV}} \|w\|_{\mathcal{F}} , \tag{6.5}$$

for all $\varepsilon>0$ *and* $\mu\in\mathcal{S}_0$.

Proof. Let $B=\{y;\ |w(y)|>\varepsilon\}$. Then, from Lemma 2.3, we have

$$\begin{aligned}
&P_\mu(\sup_{0\leq t\leq T}|w(Y(t))|>\varepsilon)=P_\mu(\sigma_B\leq T)\\
&\leq e^T E_\mu(e^{-\sigma_B})=e^T\int_{\mathcal{X}}e_B(y)\,\mu(dy)\leq e^T\int_{\mathcal{X}}e_{(|w|/\varepsilon)}(y)\,\mu(dy)\\
&=\frac{e^T}{\varepsilon}\int_{\mathcal{X}}e_{|w|}(y)\,\mu(dy)
\end{aligned}$$

for all $\mu\in\mathcal{S}_0$. Let $|w|_\varepsilon$ be the function defined in the proof of Theorem 5.3. Since $nV_{n+1}|w|_{(1/m)}$ increases q.e. with respect to both of m and n, we have

$$\begin{aligned}
e_{|w|}&=\lim_{n\to\infty}nV_{n+1}\,e_{|w|}=\lim_{n\to\infty}\lim_{m\to\infty}nV_{n+1}|w|_{(1/m)}\\
&=\lim_{m\to\infty}\lim_{n\to\infty}nV_{n+1}|w|_{(1/m)}=\lim_{m\to\infty}|w|_{(1/m)}\ \text{q.e.}
\end{aligned}$$

Combining this with (2.6) and the property that $|w|_{(1/m)}$ converges to $e_{|w|}$ weakly in $\mathcal{V}$, we have

$$\begin{aligned}
&\int_{\mathcal{X}}e_{|w|}(y)\,\mu(dy)=\lim_{m\to\infty}\int_{\mathcal{X}}|w|_{(1/m)}(y)\,\mu(dy)\\
&=\lim_{m\to\infty}\mathcal{E}_1(V_1\,\mu,\ |w|_{(1/m)})\\
&\leq\lim_{m\to\infty}\{\mathcal{A}_1(V_1\,\mu,\ |w|_{(1/m)})+\mathcal{A}_1(|w|_{(1/m)},\ V_1\,\mu)\}\\
&=\mathcal{A}_1(V_1\,\mu,\ e_{|w|})+\mathcal{A}_1(e_{|w|},\ V_1\,\mu)\\
&\leq 2M\,\|V_1\,\mu\|_{\mathcal{V}}\,\|e_{|w|}\|_{\mathcal{V}}\,.
\end{aligned}$$

Thus, to prove the lemma, it is enough to apply Lemma 5.1.

Similarly to [4: Lemma 5.1.2], this lemma implies the following

Corollary 6.3. *If* $\{w_n\}$ *is a Cauchy sequence in* $\mathcal{F}$, *then there exists a subsequence* $\{w_{n_k}\}$ *such that* $w_{n_k}(Y(t))$ *converges uniformly on each compact interval of* $[0,\infty)$ *a.s.* P_y *for q.e.* y.

Also similarly to [4: Theorem 5.2.1], it holds that if $\{M_n\}$ is an e-Cauchy sequence of elements of $\mathring{\mathcal{M}}$ then there exists a subsequence $\{M_{n_k}\}$ such that $M_{n_k}(t)$ converges uniformly on each compact interval of $[0,\infty)$ to an element $M\in\mathcal{M}$ a.s. P_y for q.e.y. These facts will be used to prove the following decomposition theorem.

Theorem 6.4. *If* $u\in\mathcal{F}\oplus(\mathcal{P}\ominus\mathcal{P})$, *then there exist uniquely* $M^{[u]}\in\mathring{\mathcal{M}}$ *and* $N^{[u]}\in\mathcal{N}$ *such that*

$$A^{[u]}=M^{[u]}+N^{[u]}\,. \tag{6.6}$$

In partucular, if $u \in \mathcal{F}$ then $N^{[u]}(t)$ becomes continuous.

Proof. (Uniqueness): If $M \in \mathring{\mathcal{M}}$, then it is easy to see that $E_\nu(M_t^2)$ is subadditive with respect to t and hence $e(M) = \sup_{t>0}(1/2t)\, E_\nu(M_t^2)$. This shows that $M=0$ if $e(M)=0$. The uniqueness of the decomposition follows easily from this.

(existence): Assume first that $u \in \mathcal{F}$. Since $pV_p u \to u$ weakly in $\mathcal{F}$, there exists a sequence $\{u_n\}$ of convex combinations of functions of the form $pV_p u$, which sequence converges to u strongly in $\mathcal{F}$. We may suppose that $u_n = V_1 f_n$ for some $f_n \in \mathcal{H}$. Define

$$N_n(t) \equiv \int_0^t (u_n - f_n)(Y(s))\, ds\,, \quad M_n(t) = u_n(Y(t)) - u_n(Y(0)) - N_n(t)\,.$$

As in [4; (5.2.11)], it is easy to see that $N_n \in \mathcal{N}$ and $M_n \in \mathring{\mathcal{M}}$. Also, from Lemma 6.1.

$$\begin{aligned} 0 \le e(M_n - M_m) &= e(A^{[u_n - u_m]}) \\ &= \mathcal{A}(u_n - u_m, u_n - u_m) - \int_{\mathcal{X}} (u_n - u_m)^2(y)\, k(dy) \\ &\le \mathcal{A}(u_n - u_m, u_n - u_m)\,. \end{aligned}$$

Hence $\{M_n(t)\}$ forms an e—Cauchy sequence and consequently a subsequence $\{M_{n_k}\}$ of $\{M_n\}$ converges to some $M^{[u]} \in \mathcal{M}$ uniformly on each compact interval a.s. P_y for q.e. y. According to Corollary 6.3, we may suppose that $\{u_{n_k}(Y(t))\}$ converges to $u(Y(t))$ uniformly on each compact interval of $[0, \infty)$ a.s. P_y for q.e. y. Hence $N^{[u]}(t) \equiv \lim_{l\to\infty} N_{n_k}(t)$ exists as a uniform convergence limit on each compact interval and satisfies

$$u(Y(t)) - u(Y(0)) = M^{[u]}(t) + N^{[u]}(t)\,.$$

Since $N_{n_k}(t)$ is continuous, $N^{[u]}(t)$ is also continuous. From the inequality

$$\begin{aligned} &E_\nu((N^{[u]}(t))^2) \\ &\le E_\nu(\{A^{[u-u_n]}(t) + N_n(t) - (M^{[u]}(t) - M_n(t))\}^2) \\ &\le 3\{E_\nu((A^{[u-u_n]})^2(t) + E_\nu((N_n(t))^2) + E_\nu((M^{[u]}(t) - M_n(t))^2)\}\,, \end{aligned}$$

we have

$$\begin{aligned} e(N^{[u]}) &\le 3\{e(A^{[u-u_n]}) + e(M^{[u]} - M_n)\} \\ &\le 6\, \mathcal{A}(u - u_n, u - u_n)\,. \end{aligned}$$

By taking $n = n_k$ in this inequality and letting $k \to \infty$ we see that $e(N^{[u]}) = 0$.

When $u \in \mathcal{P}$, we saw in Theorem 5.5 that there exists a PNAF $A(t)$ which has u as its 1-potential. Then

$$e^{-t} u(Y(t)) - u(Y(0)) - \int_0^t e^{-s}\, dA(s) = E_y(\int_0^\infty e^{-s}\, dA(s) \,|\, \mathcal{F}_t) \equiv M^{(1)}(t)$$

is a martingale. Defining the MAF $M^{[u]}$ by $M^{[u]}(t)=\int_0^t e^s\, dM^{(1)}(s)$, we have

$$u(Y(t))-u(Y(0)) = M^{[u]}(t)+A(t)$$

(see [2]). Hence what remains is only to show that $e(A)=0$. The proof of this can be done similarly to the proof of $e(N^{[u]})=0$ in the above case because it has only used the property that $\{u_n\}$ converges to u in $\mathcal{V}$ which holds in the present case.

Corollary 6.5. *If* $u\in\mathcal{F}\oplus(\mathcal{P}\ominus\mathcal{P})$, *then*

$$e(M^{[u]}) = \mathcal{A}(u, u)-\frac{1}{2}\int_{\mathcal{X}}u^2(y)\, k(dy)\,. \tag{6.7}$$

Proof. If $u\in\mathcal{F}\oplus(\mathcal{P}\ominus\mathcal{P})$, then there exists a sequence $\{u_n\}$ of functions of $\mathcal{F}$ which converges to u strongly in $\mathcal{V}$. Then (6.2) shows that $\{u_n\}$ converges to u in $L^2(\mathcal{X}; k)$ and, furthermore $M^{[u_n]}$ converges to $M^{[u]}$ with respect to the e-norm. Then (6.7) holds because it holds for each u_n.

We know that most of the stochastic calculi of Fukushima [4; Chapter 5] related to the MAF's are based upon the equality (6.7). Hence one can naturally expect that the similar calculi would be possible in our setting. Moreover, it seems to be advantageous for the calculi that the equality (6.7) does not (at least explicitly) depend upon the derivative $\partial u/\partial\tau$ or, in other words, it does not depend on $\mathcal{E}$ but does on $\mathcal{A}$. This property will be natural if one thinks of the Itô's formula applied to smooth functions (recall that the time derivative appears in the bounded variation term). From this property, most of the stochastic calculi in [4; §5] hold in our case by replacing $\mathcal{A}$ instead of $\mathcal{E}$, but we will not go into details because the calculations are similar. Instead we shall only mention the following formulas characteiizing the smooth measure $\mu_{\langle M^{[u]}\rangle}$ associated with the CAF $\langle M^{[u]}\rangle$ and the CAF $N^{[u]}$ of zero energy for $u\in\mathcal{F}$;

$$\int_{\mathcal{X}}f(y)\, d\mu_{\langle M^{[u]}\rangle}(dy) = 2\mathcal{A}(u, uf)-\mathcal{A}(u^2, f)-\int_{\mathcal{X}}u^2 f(y)\, k(dy)\,, \tag{6.8}$$

$$\lim_{p\to\infty} p^2\, E_{f\cdot\nu}\Big(\int_0^\infty e^{-pt}\, N^{[u]}(t)\, dt\Big) = -\mathcal{E}(u, f)\,, \tag{6.9}$$

for all $f\in\mathcal{F}_b$ (see [4; Theorems 5.2.3 and 5.3.1], [7], [13]).

Finally we shall give an example of a time dependent Dirichlet form. In the example we give the explicit forms of the quadıatic variation of $M^{[u]}$ and $N^{[u]}$ for $u\in\mathcal{F}$ and, furthermore show the equivalence of the $E^{(\tau)}$-polarity of a subset B of R^d and the $\mathcal{E}$-polarity of the subset $(a, b)\times B$ of $R^1\times R^d$ for any interval (a, b) of R^1.

Example 6.6. Let $(E^{(\tau)}, V)$ be the symmetric form on $C_0^\infty(R^d)$ defined by

$$(6.10)\qquad E^{(\tau)}(\varphi, \psi) = \sum_{i,j=1}^{d} \int_{R^d} a_{ij}(\tau, x) \frac{\partial \varphi(x)}{\partial x_i} \frac{\partial \psi(x)}{\partial x_j} dx, \quad \varphi, \psi \in C_0^\infty(R^d),$$

for the family $(a_{ij}(\tau, x))$ such that, for some positive constants λ and Λ,

$$(6.11)\qquad \lambda \sum_{i,j=1}^{d} \xi_i \xi_j a_{ij}(\tau, x) \le \sum_{i,j=1}^{d} \xi_i \xi_j a_{ij}(\sigma, x) \le \Lambda \sum_{i,j=1}^{d} \xi_i \xi_j a_{ij}(\tau, x),$$

for all $x \in R^d$ and $\tau, \sigma \in R^1$. We assume that $(E^{(\tau)}, C_0^\infty(R^d))$ is closable on $H \equiv L^2(R^d; dx)$ for some $\tau \in R^1$. Then $(E^{(\sigma)}, C_0^\infty(R^d))$ is also closable on H for any $\sigma \in R^1$. Let $(E^{(\tau)}, V)$ be the Dirichlet form determined by the closure. Obviously a subset B of R^d is $E^{(\tau)}$-polar if and only if it is $E^{(\sigma)}$-polar. Furthermore, we can show that a set B is an $E^{(\tau)}$-polar set if and only if $(a, b) \times B$ is an $\mathcal{E}$-polar set for some (or equivalently for all) interval (a, b) in R^1.

In fact, if B is a polar set of X, then there exists a sequence $\{\varphi_n\}$ of functions of V such that $\varphi_n = 1$ a.e. on B and $\lim_{n\to\infty} E_1^{(\tau)}(\varphi_n, \varphi_n) = 0$ uniformly with respect to τ. For any interval (a, b) take a $C_0^1(R^1)$-function $\xi(\tau)$ such that $\xi(\tau) = 1$ on (a, b) and set $w_n = \xi \otimes \varphi_n$. Then $w_n \in \mathcal{F}$, $w_n \ge 1$ a.e. on $(a, b) \times B$ and

$$(6.12)\qquad \|w_n\|_{\mathcal{F}}^2 \le K \|\varphi_n\|_V^2 \int_{-\infty}^{\infty} \{(\xi'(\tau))^2 + (\xi(\tau))^2\} d\tau,$$

for some constant K. Since $\|e_{(a,b)\times B}\|_{\mathcal{C}V} \le \|e_{|w_n|}\|_{\mathcal{C}V} \le K \|w_n\|_{\mathcal{F}}$ for all n from Lemma 5.1, we see that

$$\|e_{(a,b)\times B}\|_{\mathcal{C}V} \le K \lim_{n\to\infty} \|w_n\|_{\mathcal{F}} = 0.$$

This implies that the set $(a, b) \times B$ is an $\mathcal{E}$-polar set.

Conversely if $(a, b) \times B$ is an $\mathcal{E}$-polar set, then $e_{(a,b)\times B} = 0$ ν-a.e. Hence by Fubini's theorem, for a.e. $\tau \in (a, b)$, $e_{(a,b)\times B}(\tau, \cdot) = 0$ dx-a.e. Furthermore, for a.e. $\tau \in R^1$, $e_{(a,b)\times B}(\tau, \cdot)$ belongs to V and $e_{(a,b)\times B}(\tau, \cdot) \ge 1$ a.e. on B. This shows that B is a polar set.

In this case, the AF $\langle M^{[u]} \rangle$ for $u \in \mathcal{F}$ is given by

$$(6.13)\qquad \langle M^{[u]} \rangle (t) = 2 \sum_{i,j=1}^{d} \int_0^t \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} a_{ij}(Y(s)) ds,$$

because

$$\int_{R^1 \times R^d} f(y) \mu_{\langle M^{[u]} \rangle}(dy) = 2\mathcal{A}(u, uf) - \mathcal{A}(u^2, f)$$
$$= 2 \sum_{i,j=1}^{d} \int_{R^d} a_{ij}(\tau, x) \frac{\partial u(\tau, x)}{\partial x_i} \frac{\partial u(\tau, x)}{\partial x_j} f(\tau, x) d\tau dx,$$

from (6.8). Furthermore, if $a_{ij} \in C(R^1; C^1(R^d))$ then, for any $u \in C^{1,2}(R^1 \times R^d)$, it holds that

$$(6.14)\qquad N^{[u]}(t)=\int_0^t \frac{\partial u}{\partial \tau}(Y(s))\,ds+\sum_{i,j=1}^{d}\int_0^t \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j})(Y(s))\,ds\,,$$

because the righthand side of (6.14) satisfies (6.9). In this case, (6.14) can aslo be proved by Itô's foımula, but for geneıal $u\in\mathcal{F}$, $N^{[u]}$ is not locally of bounded variation. In that case, let $\rho^{(\varepsilon)}$ be the molifier in R^d supported by the ball of radius ε. Then the function $u^{(\varepsilon)}(\tau,\cdot)\equiv u(\tau,\cdot)*\rho^{(\varepsilon)}$ converges in $\mathcal{F}$ to u as $\varepsilon\downarrow 0$. Then, by Corollaries 6.3 and 6.5, $N^{[u^{(\varepsilon)}]}$ converges to $N^{[u]}$ uniformly on each compact interval of $[0,\infty)$ a.s. P_y for q.e. y.

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