Kyuno, S.
Osaka J. Math.
12 (1975), 639-645

ON THE RADICALS OF Γ-RINGS

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(Received October 17, 1974)

1. Introduction

N. Nobusawa [1] introduced the notion of a Γ-ring, more general than a ring, and proved analogues of the Wedderburn-Artin theorems for simple Γ-rings and for semi-simple Γ-rings; Barnes [2] obtained analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Γ-rings; Luh [3, 4] gave a generalization of the Jacobson structure theorem for primitive Γ-rings having minimum one-sided ideals, and obtained several other structure theorems for simple Γ-rings; Coppage-Luh [5] introduced the notions of Jacobson radical, Levitzki nil radical, nil radical and strongly nilpotent radical for Γ-rings and Barnes' [2] prime radical was studied further. Also, inclusion relations for these radicals were obtained, and it was shown that the radicals all coincide in the case of a Γ-ring which satisfies the descending chain condition on one-sided ideals.

In this paper the notions of semi-prime ideals are extended to Γ-rings, and it is shown that all of the following conditions are equivalent: (1) Q is a semi-prime ideal. (2) Q^c is an n-system. (3) The Γ-residue class ring M/Q contains no non-zero strongly nilpotent ideals. (4) The prime radical $P(Q)$ of the ideal Q coincides with Q. Also, the following characterization of $P(M)$ is obtained. $P(M)$ is a semi-prime ideal which is contained in every semi-prime ideal in M. Let R be the right operator ring of a Γ-ring M. For $P\subseteq R$ and for $Q\subseteq M$ we define $P^*=\{x\in M: [\Gamma, x]\subseteq P\}$ and $Q^{*\prime}=\{\sum_i [\alpha_i, x_i]\in R: M(\sum_i [\alpha_i, x_i])\subseteq Q\}$. In [5] the following theorem was proved. If $P(M)$ is the prime radical of the right operator ring R of the Γ-ring M, then $P(M)=P(R)^*$.

We show the following result dual to the above theorem, $P(R)=P(M)^{*\prime}$. As a result, it is obtained that $P(M)^{*\prime *}=P(M)$ and $P(R)^{**\prime}=P(R)$. The similar properties hold for the Levitzki nil radical and Jacobson radical. Also, some radical properties are cosidered.

2. Preliminaries

Let M and Γ be additive abelian groups. If for all $a, b, c\in M$, and $\alpha, \beta\in\Gamma$, the following conditions are satisfied, (1) $a\alpha b\in M$ (2) $a(\alpha+\beta)b=a\alpha b+a\beta b$, $a\alpha(b+c)=a\alpha b+a\alpha c$, $(a+b)\alpha c=a\alpha c+b\alpha c$ (3) $(a\alpha b)\beta c=a\alpha(b\beta c)$, then following Barnes [2], M is called a Γ-ring. If these conditions are strengthened to, (1′) $a\alpha b\in M$, $\alpha a\beta\in\Gamma$ (2′) same as (2) (3′) $(a\alpha b)\beta c=a(\alpha b\beta)c=a\alpha(b\beta c)$ (4) $x\gamma y=0$ for all $x, y\in M$ implies $\gamma=0$, then M is called a Γ-ring in the sense of Nobusawa [1]. If A and B are subsets of a Γ-ring M and $\Theta\subseteq\Gamma$, we denote $A\Theta B$, the subset of M consisting of all finite sums of the form $\sum_i a_i\alpha_i b_i$, where $a_i\in A$, $b_i\in B$, and $\alpha_i\in\Theta$. For singleton subsets we abbreviate this notation, for example, $\{a\}\Theta B=a\Theta B$. A right (left) ideal of a Γ-ring M is an additive subgroup I of M such that $I\Gamma M\subseteq I$ ($M\Gamma I\subseteq I$). If I is both a right ideal and a left ideal, then we say that I is an ideal, or a two-sided ideal of M. For each a of a Γ-ring M, the smallest right ideal containing a is called the principal right ideal generated by a and is denoted by $|a>$. Similarly we define $<a|$ and $\langle a\rangle$, the principal left and two-sided (respecitively) ideals generated by a. Let I be an ideal of a Γ-ring M. If for each $a+I$, $b+I$ in the factor group M/I, and each $\gamma\in\Gamma$, we define $(a+I)\gamma(b+I)=a\gamma b+I$, then M/I is a Γ-ring which we shall call the Γ-residue class ring of M with respect to I. Let M be a Γ-ring and F the free abelian group generated by $\Gamma\times M$. Then

$$A=\{\sum_i n_i(\gamma_i, x_i)\in F\colon a\in M\Rightarrow\sum_i n_i a\gamma_i x_i=0\}$$

is a subgroup of F. Let $R=F/A$, the factor group, and denote the coset $(\gamma, x)+A$ by $[\gamma, x]$. It can be verified easily that $[\alpha, x]+[\alpha, y]=[\alpha, x+y]$ and $[\alpha, x]+[\beta, x]=[\alpha+\beta, x]$ for all $\alpha, \beta\in\Gamma$ and $x, y\in M$. We define a multiplication in R by

$$\sum_i[\alpha_i, x_i]\sum_j[\beta_j, y_j]=\sum_{i,j}[\alpha_i, x_i\beta_j y_j]\,.$$

If we define a composition on $M\times R$ into M by $a\sum_i[\alpha_i, x_i]=\sum_i a\alpha_i x_i$ for $a\in M$, $\sum_i[\alpha_i, x_i]\in R$, then M is a right R-module, and we call R the right operator ring of the Γ-ring M. Similarly we may construct a left operator ring L of M so that M is a left L-module. If A is a right (left) ideal of R (L), then MA (AM) is an ideal of M. For subsets $N\subseteq M$, $\Phi\subseteq\Gamma$, we denote by $[\Phi, N]$ the set of all finite sums $\sum_i[\gamma_i, x_i]$ in R, where $\gamma_i\in\Phi$, $x_i\in N$, and we denote by $[(\Phi, N)]$ the set of all elements $[\varphi, x]$ in R, where $\varphi\in\Phi$, $x\in N$. Thus, in particular, $R=[\Gamma, M]$. For $P\subseteq R$ we define $P^*=\{a\in M\colon [\Gamma, a]=[\Gamma, \{a\}]\subseteq P\}$. It then follows that if P is a right (left) ideal of R, then P^* is a right (left) ideal of M. Also, for any collection $\mathcal{C}$ of sets in R, $\bigcap_{P\in\mathcal{C}}P^*=(\bigcap_{P\in\mathcal{C}}P)^*$. For $Q\subseteq M$ we define $Q^{*\prime}=\{\sum_i[\alpha_i, x_i]\in R\colon M(\sum_i[\alpha_i, x_i])\subseteq Q\}$. Then it follows that if Q is a right (left) ideal of M, then $Q^{*\prime}$ is a right (left) ideal of R. Also for any collection $\mathscr{D}$ of sets in M, $\bigcap_{Q\in\mathscr{D}} Q^{*\prime}=(\bigcap_{Q\in\mathscr{D}} Q)^{*\prime}$. For other notions relevant to Γ-rings we refer to [5].

3. Semi-primeness

Following Barnes [2] an ideal P of a Γ-ring M is prime if for any ideals $A, B\subseteq M$, $A\Gamma B\subseteq P$ implies $A\subseteq P$ or $B\subseteq P$. A subset S of M is an m-system in M if $S=\phi$ or if $a, b\in S$ implies $\langle a\rangle\Gamma\langle b\rangle\cap S\neq\phi$. Barnes [2] has shown that an ideal P is prime if and only if its complement P^c is an m-system. The prime radical $P(A)$ of the ideal A in a Γ-ring M is the set consisting of those elements r of M with the property that every m-system in M which contains r meets A (that is, has nonempty intersection with A). The prime radical of the zero ideal in a Γ-ring M may be called the prime radical of the Γ-ring M which we denote by $P(M)$. Barnes [2] has characterized $P(M)$ as the intersection of all prime ideals of M. We now make the following definition. An ideal Q in a Γ-ring M is said to be a semi-prime ideal if and only if it has the following property: If A is an ideal in M such that $A\Gamma A\subseteq Q$, then $A\subseteq Q$. It is clear that a prime ideal is semi-prime. Moreover, the intersection of any set of semi-prime ideals is a semi-prime ideal. It follows easily by induction that if Q is a semi-prime ideal and A is an ideal and $(A\Gamma)^n A=(A\Gamma A\Gamma\cdots A\Gamma)\mathrm{A}\subseteq Q$ for an arbitary positive integer n, $A\subseteq Q$. Following Coppage-Luh [5] a subset S of M is strongly nilpotent if there exists a positive integer n such that $(S\Gamma)^n S=0$. We state the following theorem whose proof we omit since it can be established by very easy modifications of the proof of Theorem 4.11 in [6].

Theorem 3.1. *An ideal Q in a Γ-ring M is a semi-prime ideal in M if and only if the Γ-residue class ring M/Q contains no nonzero strongly nilpotent ideals.*

The following result is easy to prove.

Theorem 3.2. *If Q is an ideal in a Γ-ring M, the following conditions are equivalent*: (1) *Q is a semi-prime ideal.* (2) *If $a\in M$ such that $\langle a\rangle\Gamma\langle a\rangle\subseteq Q$, then $a\in Q$.*

A set N of elements of a Γ-ring M is said to be an n-system if $N=\phi$ or if $a\in N$ implies $\langle a\rangle\Gamma\langle a\rangle\cap N\neq\phi$. The equivalence of conditions (1) and (2) of Theorem 3.2 assures us that an ideal Q in a Γ-ring M is semi-prime if and only if its complement Q^c is an n-system. By proofs analogous to Lemma 4.14 and Theorem 4.15 in McCoy [6] we have the following results.

Lemma 3.3. *If N is an n-system in a Γ-ring M and $a\in N$, there exists an m-system L such that $a\in L$ and $L\subseteq N$.*

Theorem 3.4. *An ideal Q in a Γ-ring M is a semi-prime ideal in M if and* *only if* $P(Q)=Q$.

In view of Barnes' characterization of $P(M)$ as the intersection of all prime ideals of M (Theorem 7 in [2]) we have the following immediate corollary to the preceding theorem.

Corollary 3.5. *If Q is an ideal in a Γ-ring M, then $P(Q)$ is the smallest semi-prime ideal in M which contains Q.*

We have the following characterization of $P(M)$ which follows immediately from Corollary 3.5 and Theorem 7 in [2].

Theorem 3.6. *$P(M)$ is a semi-prime ideal which is contained in every semi-prime ideal in M.*

4. The prime radical

Coppage-Luh [5] have proved the following Lemma 4.1 and Theorem 4.2.

Lemma 4.1. *If P is a prime ideal of R, then P^* is a prime ideal of M.*

Theorem 4.2. *If $P(R)$ is the prime radical of the right operator ring R of the Γ-ring M, then $P(M)=P(R)^*$.*

We prepare the following lemma.

Lemma 4.3. *If Q is a prime ideal of M, then $Q^{*\prime}$ is a prime ideal of R.*

This proof is found in the proof of Theorem 4.1 in [5]. We now prove the following theorem dual to Theorem 4.2.

Theorem 4.4. *If $P(R)$ is the prime radical of the right operator ring R, then $P(R)=P(M)^{*\prime}$.*

Proof. Let P be a prime ideal of R, by Lemma 4.1 P^* is a prime ideal of M. Let us set $P^*=Q$. Then by Lemma 4.3 $Q^{*\prime}$ is a prime ideal of R. Since $Q^{*\prime}=\{r\in R\colon Mr\subseteq Q\}=\{r\in R\colon [\Gamma, Mr]\subseteq P\}$, it follows that $RQ^{*\prime}=[\Gamma, M]Q^{*\prime}=[\Gamma, MQ^{*\prime}]\subseteq P$. Thus by the primeness of P, $Q^{*\prime}\subseteq P$. Also, $[\Gamma, MP]=[\Gamma, M]P=RP\subseteq P$. Hence, $P\subseteq Q^{*\prime}$. Therefore $P=Q^{*\prime}$. It follows that $P(R)$, which is the intersection of all prime ideals of R, contains $\bigcap_{Q\in\mathcal{D}} Q^{*\prime}=(\bigcap_{Q\in\mathcal{D}} Q)^{*\prime}$, where $\mathcal{D}$ is a certain collection of prime ideals of M. But $(\bigcap_{Q\in\mathcal{D}} Q)^{*\prime}\supseteq P(M)^{*\prime}$, so we conclude that $P(R)\supseteq P(M)^{*\prime}$. On the other hand, $P(M)^{*\prime}=(\bigcap Q)^{*\prime}=\bigcap Q^{*\prime}$, where the intersection is taken over all prime ideals of M. Since, by Lemma 4.3 each $Q^{*\prime}$ is a prime ideal of R, and $P(R)$ is the intersection of all prime ideals of R, it follows $P(R)\subseteq P(M)^{*\prime}$. Thus $P(R)=P(M)^{*\prime}$.

The following result is a consequence of Theorem 4.2 and Theorem 4.4.

Theorem 4.5. *If $P(R)$ is the prime radical of the right operator ring R, then $P(M)=P(M)^{*'*}$ and $P(R)=P(R)^{**'}$.*

The next theorem follows immediately from previous Theorem 4.4 and Theorem 4.2 in [5].

Theorem 4.6. *If I is an ideal of a Γ-ring M, then $P(I)^{*'}=I^{*'}\cap P(R)$, where $P(I)$ denotes the prime radical of I considered as a Γ-ring.*

5. The Levitzki nil radical

Following Coppage-Luh [5] a subset S of a Γ-ring M is locally nilpotent if for any finite set $F\subseteq S$ and any finite set $\Phi\subseteq\Gamma$, there exists a positive integer n such that $(F\Phi)^nF=0$. Also the Levitzki nil radical of M is the sum of all locally nilpotent ideals of M and is denoted by $\mathcal{L}(M)$.

Coppage-Luh [5] have proved the next theorem.

Theorem 5.1. *If M is a Γ-ring then $\mathcal{L}(M)=\mathcal{L}(R)^*$, where $\mathcal{L}(R)$ is the Levitzki nil radical of the right operator ring R of M.*

We know the following result whose proof will be found in Jacobson [7], p. 163.

Theorem 5.2. *$\mathcal{L}(R)$ is a locally nilpotent ideal.*

We prove the following two lemmas.

Lemma 5.3. *If J is a locally nilpotent ideal of a Γ-ring M, then $J^{*'}$ is a locally nilpotent ideal of R, where R is the right operator ring of M.*

Proof. A finite subset of $J^{*'}$ is a subset of $[\phi_1, F_1]$, where ϕ_1 is a finite subset of Γ and F_1 is a finite subset of M. Since $M[\phi_1, F_1]\subseteq J$, $M\phi_1F_1\subseteq J$. Thus $F_1\phi_1F_1$ is a finite subset of J. Since J is locally nilpotent, $(F_1\phi_1F_1\phi_1)^nF_1\phi_1F_1=0$ for some n. Hence, $(F_1\phi_1)^{2n+1}F_1=0$. Thus $[\phi_1, F_1]^{2n+2}=[\phi_1, (F_1\phi_1)^{2n+1}F_1]=[\phi_1, 0]=0$. Then $J^{*'}$ is locally nilpotent.

Lemma 5.4. *If M is a Γ-ring then $\mathcal{L}(R)\subseteq\mathcal{L}(M)^{*'}$, where $\mathcal{L}(R)$ is the Levitzki nil radical of the right operator ring R of M.*

Proof. Since $\mathcal{L}(R)$ is an ideal by Theorem 5.2., we have $R\mathcal{L}(R)\subseteq\mathcal{L}(R)$, so $[\Gamma, M]\mathcal{L}(R)\subseteq\mathcal{L}(R)$, that is, $[\Gamma, M\mathcal{L}(R)]\subseteq\mathcal{L}(R)$. Thus, $M\mathcal{L}(R)\subseteq\mathcal{L}(R)^*$. By Theorem 5.1., we get $M\mathcal{L}(R)\subseteq\mathcal{L}(M)$. Therefore $\mathcal{L}(R)\subseteq\mathcal{L}(M)^{*'}$.

Theorem 5.5. *If M is a Γ-ring then $\mathcal{L}(M)^{*'}=\mathcal{L}(R)$, where $\mathcal{L}(R)$ is the*

Levitzki nil radical of the right operator ring R of M.

Proof. In view of Lemma 5.3 and the fact that $\mathcal{L}(M)$ is a locally nilpotent ideal (Theorem 7.1 in [5]), we know that $\mathcal{L}(M)^{*\prime}$ is a locally nilpotent ideal of R. By the definition of $\mathcal{L}(R)$, $\mathcal{L}(M)^{*\prime}\subseteq\mathcal{L}(R)$. On the other hand by Lemma 5.4, $\mathcal{L}(R)\subseteq\mathcal{L}(M)^{*\prime}$. Hence $\mathcal{L}(M)^{*\prime}=\mathcal{L}(R)$.

The next theorem follows from Theorem 5.1 and Theorem 5.5.

Theorem 5.6. *If $\mathcal{L}(R)$ is the Levitzki nil radical of the right operator ring R of a Γ-ring M, then $\mathcal{L}(M)^{*\prime *}=\mathcal{L}(M)$ and $\mathcal{L}(R)^{**\prime}=\mathcal{L}(R)$.*

The next theorem follows immediately from previous Theorem 5.5 and Theorem 7.3 in [5].

Theorem 5.7. *If I is an ideal of a Γ-ring M, then $\mathcal{L}(I)^{*\prime}=I^{*\prime}\cap\mathcal{L}(R)$.*

6. The Jacobson radical

Following Copppage-Luh [5] an element a of a Γ-ring M is right quasi-regular (abbreviated *rqr*) if, for any $\gamma\in\Gamma$, there exists $\eta_i\in\Gamma$, $x_i\in M$, $i=1,2,\cdots,n$ such that

$$x\gamma a+\sum_{i=1}^{n} x\eta_i x_i-\sum_{i=1}^{n} x\gamma a\eta_i x_i = 0 \qquad \text{for all } x\in M\,.$$

A subset S of M is *rqr* if every element in S is *rqr*. $\mathcal{J}(M)=\{a\in M\colon \langle a\rangle$ is *rqr*$\}$ is the right Jacobson radical of M. Copppage-Luh [5] have shown the following Lemma 6.1 and Theorem 6.2.

Lemma 6.1. *An element a of a Γ-ring M is rqr if and only if, for all $\gamma\in\Gamma$, $[\gamma, a]$ is rqr in the right operator ring R of M.*

Theorem 6.2. *If M is a Γ-ring then $\mathcal{J}(M)=\mathcal{J}(R)^*$, where $\mathcal{J}(R)$ denotes the Jacobson radical of the right operator ring R of M.*

We prove the following theorem dual to Theorem 6.2.

Theorem 6.3. *If M is a Γ-ring, then $\mathcal{J}(R)=\mathcal{J}(M)^{*\prime}$.*

Proof. If $\sum_i[\gamma_i, x_i]\in\mathcal{J}(M)^{*\prime}$, then for all $x\in M$

$$x(\sum_i[\gamma_i, x_i]) = \sum_i x\gamma_i x_i\in\mathcal{J}(M)\,.$$

By Theorem 6.2, for all $\gamma\in\Gamma$

$$[\gamma, \sum_i x\gamma_i x_i] = [\gamma, x]\sum_i[\gamma_i, x_i]\in\mathcal{J}(R).$$

Hence $R\sum_i[\gamma_i, x_i]$ *is rqr*. By the difinition of $\mathcal{J}(R)$, $\sum_i[\gamma_i, x_i]\in\mathcal{J}(R)$ and then

$\mathcal{J}(M)^{*\prime} \subseteq \mathcal{J}(R)$. If $\sum_i [\delta_i, x_i] \in \mathcal{J}(R)$, for all $\gamma \in \Gamma$ and all $x \in M$

$$[\gamma, x]\sum[\delta_i, x_i] = [\gamma, \sum x\delta_i x_i] \in \mathcal{J}(R)\,.$$

Thus

$$\langle[\gamma, \sum x\delta_i x_i]\rangle = \{[\gamma, a] \in R: a \in \langle\sum x\delta_i x_i\rangle\} \subseteq \mathcal{J}(R)\,.$$

Hence, $[\gamma, a]$ is *rqr* for all $\gamma \in \Gamma$ and for $a \in \langle\sum x\delta_i x_i\rangle$. By Lemma 6.1 $\langle\sum x\delta_i x_i\rangle$ is *rqr* and then $x(\sum[\delta_i, x_i]) = \sum x\delta_i x_i \in \mathcal{J}(M)$, that is, $M(\sum[\delta_i, x_i]) \subseteq \mathcal{J}(M)$. Therefore $\sum[\delta_i, x_i] \in \mathcal{J}(M)^{*\prime}$. Then $\mathcal{J}(R) \subseteq \mathcal{J}(M)^{*\prime}$. Thus the proof is completed.

By Theorem 6.2 and Theorem 6.3 we have the following theorem.

Theorem 6.4. $\mathcal{J}(M) = \mathcal{J}(M)^{*\prime *}$ *and* $\mathcal{J}(R) = \mathcal{J}(R)^{**\prime}$.

The next result follows from Theorem 6.3 and Theorem 8.4 in [5].

Theorem 6.5. *If I is an ideal in a Γ-ring M,* $\mathcal{J}(I)^{*\prime} = I^{*\prime} \cap \mathcal{J}(R)$.

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