Enomoto, H.
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THE CHARACTERS OF THE FINITE SYMPLECTIC GROUP Sp(4,q), q=2^f

HIKOE ENOMOTO

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The purpose of this note is to calculate all the (complex) irreducible characters of the finite symplectic group $Sp(4, q)$ where $q=2^f$, while B. Srinivasan [3] determined the character table of $Sp(4, q)$ for odd q. All the irreducible characters of $Sp(4, 2^f)$ are expressed as linear combinations of induced characters with integral coefficients. The conjugacy classes of various subgroups are easily determined by the same method described in [1], and only the results are given (cf. [4]). For purposes of convenience the character tables of various subgroups (and of $Sp(4, 2^f)$ itself) are given in the Appendix.

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By similar but a little more complicated calculations, the author has obtained the character table of the finite Chevalley group $G_2(2^f)$, which will appear elsewhere.

1. Notation and preliminary results

Let K be the finite field with q elements, where $q=p^f$ and p is a prime number. Let $\bar{K}$ be the algebraic closure of K, and put

$$K_i = \{x \in \bar{K} \mid x^{q^i} = x\} .$$

Then K_i is the subfield of $\bar{K}$ with q^i elements, and $K_1=K$. Let κ be a fixed generator of the multiplicative group K_4^*, and put $\tau=\kappa^{q^2-1}$, $\theta=\kappa^{q^2+1}$, $\eta=\theta^{q-1}$ and $\gamma=\theta^{q+1}$. Then we have $\langle\theta\rangle=K_2^*$ and $\langle\gamma\rangle=K^*$. Choose a fixed isomorphism from the multiplicative group K_4^* into the multiplicative group of complex numbers, and let $\tilde{\tau}$, $\tilde{\theta}$, $\tilde{\eta}$ and $\tilde{\gamma}$ be the images of τ, θ, η and γ respectively under this isomorphism.

Let G be the 4-dimensional symplectic group over K, that is,

$$G = \{A \in GL(4, K) \mid {}^tAJA = J\} ,$$

where $J=\begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}$ and ${}^tA$ is the transposed matrix of A. For $t\in K$, define

$$x_a(t)=\begin{bmatrix} 1 & t & & \\ & 1 & & \\ & & 1 & -t \\ & & & 1 \end{bmatrix},\quad x_b(t)=\begin{bmatrix} 1 & & & \\ & 1 & t & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

$$x_{a+b}(t)=\begin{bmatrix} 1 & & t & \\ & 1 & & t \\ & & 1 & \\ & & & 1 \end{bmatrix}\quad\text{and}\quad x_{2a+b}(t)=\begin{bmatrix} 1 & & & t \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

and put $\Delta^+=\{a,\ b,\ a+b,\ 2a+b\}$. Then for $r\in\Delta^+$, $\mathfrak{X}_r=\{x_r(t)\,|\,t\in K\}$ is a subgroup of G isomorphic to the additive group of K, and we have the following commutator relations, where the commutator $x^{-1}y^{-1}xy$ is denoted by $[x, y]$:

$$\begin{aligned} &[x_a(t),\ x_b(u)] = x_{a+b}(tu)x_{2a+b}(-t^2u)\,, \\ (1.1)\qquad &[x_a(t),\ x_{a+b}(u)] = x_{2a+b}(2tu)\,, \\ &[x_r(t),\ x_s(u)] = 1 \text{ for all other pairs of } r,\ s\in\Delta^+\,. \end{aligned}$$

Next, define $h(z_1, z_2)=\begin{bmatrix} z_1 & & & \\ & z_2 & & \\ & & z_2^{-1} & \\ & & & z_1^{-1} \end{bmatrix}$ for $z_i\in K^*_4$, and put $\mathfrak{U}=\mathfrak{X}_a\mathfrak{X}_b\mathfrak{X}_{a+b}\times\mathfrak{X}_{2a+b}$, $\mathfrak{H}=\{h(z_1, z_2)\,|\,z_i\in K^*\}$ and $B=\mathfrak{H}\mathfrak{U}$. Then $\mathfrak{U}$ is a Sylow p-subgroup of G, and B is the normalizer of $\mathfrak{U}$ in G (called the Borel subgroup of G). Put $\omega_r=x_r(1){}^tx_r(-1)x_r(1)$ for $r\in\Delta^+$. Especially,

$$\omega_a=\begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{bmatrix}\quad\text{and}\quad \omega_b=\begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix}.$$

Then G is generated by $B\cup\{\omega_a,\ \omega_b\}$. The maximal parabolic subgroups of G generated by $B\cup\{\omega_a\}$ and $B\cup\{\omega_b\}$ are denoted by P and Q respectively.

From now on, we shall assume that the characteristic p of K is 2. Then the commutator relations (1.1) are reduced to:

$$\begin{aligned} (1.1)'\qquad &[x_a(t),\ x_b(u)] = x_{a+b}(tu)x_{2a+b}(t^2u)\,, \\ &[x_r(t),\ x_s(u)] = 1 \text{ for all other pairs } r,\ s\in\Delta^+\,. \end{aligned}$$

Now we shall consider linear characters of the additive group K. Define the map $\mathcal{E}$ of K into $\{1, -1\}$ by $\mathcal{E}(t)=1$ if and only if X^2+X+t is reducible in $K[X]$ and $\mathcal{E}(t)=-1$ if and only if X^2+X+t is irreducible in $K[X]$, where $K[X]$

is the polynomial ring over K in the indeterminate X. In other words, $\mathcal{E}(t^2+t)=1$ and $\mathcal{E}(t^2+t+\xi)=-1$ for $t\in K$, where ξ is a fixed element of K such that $X^2+X+\xi$ is an irreducible polynomial of $K[X]$.

The following lemmas are used to calculate the characters of B.

Lemma 1.2. $\sum_{t\in K}\mathcal{E}(t)=0$, *hence* $\sum_{t\in K^*}\mathcal{E}(t)=-1$.

Lemma 1.3. *For* $x\in K$, *we have*

$$\sum_{t\in K,u\in K^*}\mathcal{E}(t^2u+tu+xu)=q\mathcal{E}(x)\,.$$

Proof.

$$\begin{aligned}\sum_{t\in K,u\in K^*}\mathcal{E}(t^2u+tu+xu)&=\sum_{t\in K,u\in K^*}\mathcal{E}(t^2(u+u^2)+xu)\\&=\sum_{u\in K^*}\Big(\sum_{t\in K}\mathcal{E}(t^2(u+u^2))\Big)\mathcal{E}(xu)\\&=q\sum_{u\in K^*,u+u^2=0}\mathcal{E}(xu)\\&=q\mathcal{E}(x)\,.\end{aligned}$$

For a finite group H and two characters χ_1, χ_2 of H, the scalar product $(\chi_1,\chi_2)_H$ of χ_1 and χ_2 is defined by

$$(\chi_1,\chi_2)_H=\frac{1}{|H|}\sum_{x\in H}\chi_1(x)\overline{\chi_2(x)}\,,$$

where $|H|$ is the order of H and $\overline{\chi_2(x)}$ is the complex conjugate of $\chi_2(x)$.

We use the following set of parameters both for the conjugacy classes and for the characters ($\boldsymbol{Z}$ mod m means that i and j ($i,j\in\boldsymbol{Z}$) give the same class or the same character if $i\equiv j \mod m$):

$$\begin{aligned}T_0&=\boldsymbol{Z}\bmod q-1\,,\\T_1&=\{i\in T_0\,|\,i\not\equiv 0 \bmod q-1\}\,,\\T_2&=\{i\in\boldsymbol{Z}\bmod q+1\,|\,i\not\equiv 0\bmod q+1\}\,,\\S_0&=\{(i,j)\in T_0\times T_0\,|\,i\not\equiv j\bmod q-1\}\,,\\S_1&=\{(i,j)\in T_1\times T_1\,|\,i\not\equiv\pm j\bmod q-1\}\,,\\S_2&=\{(i,j)\in T_2\times T_2\,|\,i\not\equiv\pm j\bmod q+1\}\,,\\R_1&=\{i\in\boldsymbol{Z}\bmod q^2-1\,|\,i\not\equiv qi\bmod q^2-1\}\,,\\R_2&=\{i\in\boldsymbol{Z}\bmod q^2-1\,|\,i\not\equiv\pm qi\bmod q^2-1\}\,,\text{ and}\\R_3&=\{i\in\boldsymbol{Z}\bmod q^2+1\,|\,i\not\equiv 0\bmod q^2+1\}\,.\end{aligned}$$

We also use the following abbreviations:

$$\begin{aligned}\alpha_k&=\tilde{\gamma}^k+\tilde{\gamma}^{-k}\,,\quad k\in\boldsymbol{Z}\bmod q-1\,,\text{ and}\\\beta_k&=\tilde{\eta}^k+\tilde{\eta}^{-k}\,,\quad k\in\boldsymbol{Z}\bmod q+1\,.\end{aligned}$$

2. Characters of the Borel subgroup B

In this section we show that every irreducible character of B is the induced character of some linear character of a subgroup, that is, B is an M-group. The character table of B is given in Table I of the Appendix.

It is clear that $\chi_1(k, l)$ $(k, l\in T_0)$ is a linear character of B. Now, consider the following linear character of the subgroup $\{h(\gamma^i, \gamma^i)\,|\,i\in T_0\}\mathfrak{U}$:

$$h(\gamma^i, \gamma^i)x_a(t)x_b(u)x_{a+b}(v)x_{2a+b}(w) \rightarrow \tilde{\gamma}^{ik}\varepsilon(t)\,,$$

where $i, k\in T_0$ and $t, u, v, w\in K$. Using Lemma 1.2, the values of the induced character of this linear character to B are easily calculated, and we obtain $\chi_2(k)$ $(k\in T_0)$. Similarly, $\chi_3(k)$ $(k\in T_0)$ is the induced character of the following linear character of the subgroup $\{h(\gamma^i, 1)\,|\,i\in T_0\}\mathfrak{U}$:

$$h(\gamma^i, 1)x_a(t)x_b(u)x_{a+b}(v)x_{2a+b}(w) \rightarrow \tilde{\gamma}^{ik}\varepsilon(u), \quad i\in T_0, \quad t, u, v, w\in K\,.$$

Also, $\chi_4(k)$ $(k\in T_0)$ is the induced character of the following linear character of the subgroup $\{h(\gamma^i, \gamma^j)\,|\,i\in T_0\}\mathfrak{X}_b\mathfrak{X}_{a+b}\mathfrak{X}_{2a+b}$:

$$h(\gamma^i, \gamma^{-i})x_b(t)x_{a+b}(u)x_{2a+b}(v) \rightarrow \tilde{\gamma}^{ik}\varepsilon(u), \quad i\in T_0, \quad t, u, v\in K\,,$$

and $\chi_5(k)$ $(k\in T_0)$ is the induced character of the following linear character of the subgroup $\{h(1, \gamma^i)\,|\,i\in T_0\}\mathfrak{X}_b\mathfrak{X}_{a+b}\mathfrak{X}_{2a+b}$:

$$h(1, \gamma^i)x_b(t)x_{a+b}(u)x_{2a+b}(v) \rightarrow \tilde{\gamma}^{ik}\varepsilon(v), \quad i\in T_0, \quad t, u, v\in K\,.$$

$\theta_1=\chi_2(0)\cdot\chi_3(0)$ is the induced character of the following linear character of $\mathfrak{U}$:

$$x_a(t)x_b(u)x_{a+b}(v)x_{2a+b}(w) \rightarrow \varepsilon(t+u), \quad t, u, v, w\in K\,.$$

Next, consider the subgroup $\mathfrak{U}_1=\langle x_a(1)\rangle\mathfrak{X}_b\mathfrak{X}_{a+b}\mathfrak{X}_{2a+b}$ of order $2q^3$, and the following linear character $\varepsilon(k, x)$ $(k=0, 1, \ x\in K)$:

$$\begin{aligned}&x_a(1) \rightarrow (-1)^k\,,\\ &x_b(t)x_{a+b}(u)x_{2a+b}(v) \rightarrow \varepsilon(xt+u+v), \quad t, u, v\in K\,.\end{aligned}$$

Using Lemma 1.3, the values of the induced character $\tilde{\varepsilon}(k, x)$ are easily calculated. For example, we shall consider the class A_{42}. The elements of $\mathfrak{U}_1$ which are conjugate to $x_b(1)x_{2a+b}$ (1) in B are $\{x_b(t)x_{a+b}(ut)x_{2a+b}(u^2t+v)\,|\, t, v\in K^*, \ u\in K\}$. Therefore the value of $\tilde{\varepsilon}(k, x)$ on the class A_{42} is

$$\begin{aligned}&\frac{|C_B(x_b(1)x_{2a+b}(1))|}{|\mathfrak{U}_1|}\sum_{t,v\in K^*,u\in K}\varepsilon(xt+ut+u^2t+v)\\ &\quad=\frac{1}{2}\Big(\sum_{t\in K^*,u\in K}\varepsilon(xt+ut+u^2t)\Big)\Big(\sum_{v\in K^*}\varepsilon(v)\Big)\\ &\quad=-q\varepsilon(x)/2\,.\end{aligned}$$

Then we know that $\tilde{\varepsilon}(k, x)$ is irreducible and depends only on k and $\varepsilon(x)$. Hence four more irreducible characters $\theta_2(k)=\tilde{\varepsilon}(k, 0)$ and $\theta_3(k)=\tilde{\varepsilon}(k, \xi)$ $(k=0, 1)$ are obtained. Now we have obtained all the irreducible characters of B.

3. Characters of the maximal parabolic subgroups P and Q

First, we determine the character table of $P=\langle B, \omega_a\rangle=B\cup B\omega_a B$, which is given in Table II.

It is clear that $\chi_1(k)$ $(k\in T_0)$ is a linear character of P. Inducing the irreducible characters of B to P, we obtain the following irreducible characters of P:

$$\begin{aligned}
&\chi_2(k) = \tilde{\chi}_1(k, k)-\chi_1(k), \quad k\in T_0\,, \\
&\chi_3(k, l) = \tilde{\chi}_1(k, l), \quad (k, l)\in S_0\,, \\
&\chi_4(k) = \tilde{\chi}_3(k), \quad k\in T_0\,, \\
&\chi_5(k) = \tilde{\chi}_4(k), \quad k\in T_1\,, \\
&\theta_1 = \tilde{\theta}_1, \quad \text{and} \\
&\theta_2(k) = \tilde{\theta}_2(k) \;-\; \sum_{l=1}^{(q-2)/2} \chi_5(l), \quad k = 0, 1\,.
\end{aligned}$$

To obtain the remaining characters, we use the following characters of $\langle \mathfrak{H}\mathfrak{X}_a\mathfrak{X}_{a+b}, \omega_a\rangle$.

Characters of $\langle \mathfrak{H}\mathfrak{X}_a\mathfrak{X}_{a+b}, \omega_a\rangle$

Class representative	Number of classes	Order of centralizer	σ_1	σ_2	$\sigma_3(k)$ $k\in T_2$	$\sigma_4(k)$ $k\in R_1$	Class in P
$h(1, 1)$	1	$q^2(q-1)(q^2-1)$	$q(q-1)$	$q-1$	$(q-1)^2$	$q-1$	A_1
$x_{a+b}(1)$	1	$q^2(q^2-1)$	$-q$	-1	$-(q-1)$	$q-1$	A_{31}
$x_a(1)$	1	$q^2(q-1)$		$q-1$	$-(q-1)$	-1	A_{41}
$x_a(1)x_{a+b}(1)$	1	q^2		-1	1	-1	A_{42}
$h(\gamma^i, \gamma^j)$	$(q-2)^2/2$	$(q-1)^2$					B_1, C_1
$h(\theta^i, \theta^{qi})$	$q(q-2)/2$	q^2-1				$-(\tilde{\theta}^{ik}+\tilde{\theta}^{qik})$	B_2
$h(\gamma^i, \gamma^{-i})$	$(q-2)/2$	$q(q-1)^2$	$q-1$	$q-1$			C_2
$h(\gamma^i, \gamma^i)$	$q-2$	$q(q-1)(q^2-1)$				$(q-1)\tilde{\gamma}^{ik}$	C_3
$h(\eta^i, \eta^{-i})$	$q/2$	$q(q^2-1)$	$-(q-1)$	$q-1$	$-(q-1)\beta_{ik}$	$-\beta_{ik}$	C_4
$h(\gamma^i, \gamma^{-i})x_{a+b}(1)$	$(q-2)/2$	$q(q-1)$	-1	-1			D_2
$h(\gamma^i, \gamma^i)x_a(1)$	$q-2$	$q(q-1)$				$-\tilde{\gamma}^{ik}$	D_3
$h(\eta^i, \eta^{-i})x_{a+b}(1)$	$q/2$	$q(q+1)$	1	-1	β_{ik}	$-\beta_{ik}$	D_4

Then the remaining irreducible characters of P are obtained as follows:

$$\begin{aligned}
&\chi_7(k) = \tilde{\sigma}_4(k)-\theta_1, \quad k\in R_1\,, \\
&\theta_3(0) = \tilde{\sigma}_2-\theta_2(0)\,, \\
&\theta_3(1) = \theta_3(0)+\tilde{\theta}_3(1)-\tilde{\theta}_3(0), \quad \text{and} \\
&\chi_6(k) = \tilde{\sigma}_1-\tilde{\sigma}_3(k)-\theta_2(0)+\theta_3(1), \quad k\in T_2\,.
\end{aligned}$$

We have obtained all the irreducible characters of P.

Induced characters of P[1)]

χ	$\tilde{\theta}_2(k)$ $k=0,1$	$\tilde{\theta}_2(k)$ $k=0,1$	$\tilde{\sigma}_1$	$\tilde{\sigma}_2$	$\sigma_3(k)$ $k\in T_2$	$\tilde{\sigma}_4(k)$ $k\in R_1$
A_1	$q(q-1)(q^2-1)/2$	$q(q-1)(q^2-1)/2$	$q^3(q-1)$	$q^2(q-1)$	$q^2(q-1)^2$	$q^2(q-1)$
A_2	$q(q-1)^2/2$	$-q(q^2-1)/2$				
A_{31}	$-q(q^2-1)/2$	$-q(q^2-1)/2$	$-q^3$	$-q^2$	$-q^2(q-1)$	$q^2(q-1)$
A_{32}	$-q(q-1)/2$	$q(q+1)/2$				
A_{41}	$(-1)^k q(q-1)/2$	$(-1)^k q(q-1)/2$		$q(q-1)$	$-q(q-1)$	$-q$
A_{42}	$-(-1)^k q/2$	$-(-1)^k q/2$		$-q$	q	$-q$
A_{51}	$(-1)^k q/2$	$-(-1)^k q/2$				
A_{52}	$-(-1)^k q/2$	$(-1)^k q/2$				
$B_2(i)$						$-(\tilde{\theta}^{ik}+\tilde{\theta}^{qik})$
$C_2(i)$			$q-1$	$q-1$		
$C_3(i)$						$(q-1)\tilde{\gamma}^{ik}$
$C_4(i)$			$-(q-1)$	$q-1$	$-(q-1)\beta_{ik}$	$-\beta_{ik}$
$D_2(i)$			-1	-1		
$D_3(i)$						$-\tilde{\gamma}^{ik}$
$D_4(i)$			1	-1	β_{ik}	$-\beta_{ik}$
$(\chi, \chi)_P$	$q/2$	$(q+2)/2$	$q+1$	2	q	2

Similarly, the character table of Q given in Table III is constructed.

4. Characters of $G=Sp(4, q)$

Inducing the irreducible characters of P and Q to G, the following irreducible characters are obtained:

$$\chi_1(k, l) = \tilde{\chi}_3(k, l), \quad (k, l)\in S_1,$$

$$\chi_2(k) = \tilde{\chi}_7(k), \quad k\in R_1,$$

$$\chi_3(k, l) = \tilde{\chi}_7'(k, l), \quad k\in T_1, \quad l\in T_2,$$

$$\chi_6(k) = \tilde{\chi}_1(k), \quad k\in T_1,$$

$$\chi_7(k) = \tilde{\chi}_1'(k), \quad k\in T_1,$$

$$\chi_{10}(k) = \tilde{\chi}_2(k), \quad k\in T_1, \quad \text{and}$$

$$\chi_{11}(k) = \tilde{\chi}_2'(k), \quad k\in T_1.$$

1) The rows corresponding to $B_1(i, j)$, $C_1(i)$ and $D_1(i)$ are omitted because the values of characters at these classes are 0. This convension is used throughout the paper.

Induced characters of G from P and Q

χ	$\tilde{\chi}_1(0)$	$\tilde{\chi}_1'(0)$	$\tilde{\chi}_2(0)$	$\tilde{\chi}_6(k)$ $k\in T_2$	$\tilde{\chi}_6'(k)$ $k\in T_2$	$\tilde{\chi}_7((q-1)k)$ $k\in T_2$
A_1	$(q+1)(q^2+1)$	$(q+1)(q^2+1)$	$q(q+1)(q^2+1)$	$q(q-1)(q^4-1)$	$q(q-1)(q^4-1)$	q^4-1
A_2	$q+1$	q^2+q+1	$q(q+1)$	$-q(q^2-1)$	$-q(q-1)$	q^2-1
A_{31}	q^2+q+1	$q+1$	q	$-q(q-1)$	$-q(q^2-1)$	$-(q^2+1)$
A_{32}	$q+1$	$q+1$	q	q	q	-1
A_{41}	1	1				-1
A_{42}	1	1				-1
$B_1(i,j)$	4	4	4			
$B_2(i)$	2		-2			$-2\beta_{ik}$
$B_3(i,j)$		2				
$C_1(i)$	$2(q+1)$	$q+3$	$2(q+1)$			
$C_2(i)$	$q+3$	$2(q+1)$	$3q+1$			$2(q-1)$
$C_3(i)$		$q+1$			$(q^2-1)\beta_{ik}$	
$C_4(i)$	$q+1$		$-(q+1)$	$(q^2-1)\beta_{ik}$		$-(q+1)\beta_{2ik}$
$D_1(i)$	2	3	2			
$D_2(i)$	3	2	1			-2
$D_3(i)$		1			$-\beta_{ik}$	
$D_4(i)$	1		-1	$-\beta_{ik}$		$-\beta_{2ik}$
$(\chi,\chi)_G$	3	3	3	q^2-q+1	q^2-q+1	2

χ	$\tilde{\theta}_1$	$\tilde{\theta}_2(1)$	$\tilde{\theta}_3(0)$	$\tilde{\theta}_3(1)$
A_1	$(q^2-1)(q^4-1)$	$q(q+1)(q^4-1)/2$	$q(q-1)(q^4-1)/2$	$q(q-1)(q^4-1)/2$
A_2	$-(q^2-1)$	$q(q^2-1)/2$	$-q(q^2-1)/2$	$-q(q^2-1)/2$
A_{31}	$-(q^2-1)$	$-q(q+1)(q^2-q+1)/2$	$q(q-1)(q^2+q-1)/2$	$-q(q-1)(q^2+q+1)/2$
A_{32}	$-(q^2-1)$	$q(q-1)/2$	$-q(q-1)/2$	$q(q+1)/2$
A_{41}	1	$-q/2$	$-q/2$	$q/2$
A_{42}	1	$q/2$	$q/2$	$-q/2$
$C_2(i)$		q^2-1		
$C_4(i)$			q^2-1	q^2-1
$D_2(i)$		-1		
$D_4(i)$			-1	-1
$(\chi,\chi)_G$	q^2	$(q^2+2q+2)/2$	$q^2/2$	$(q^2+2)/2$

To obtain the remaining characters of G, we use the following characters of $\langle \mathfrak{H}\mathfrak{X}_b\mathfrak{X}_{2a+b}, \omega_b, \omega_{2a+b}\rangle$, which is isomorphic to $SL(2, q)\times SL(2,q)$.

Characters of $\langle \mathfrak{H}\mathfrak{X}_b\mathfrak{X}_{2a+b},\ \omega_b,\ \omega_{2a+b}\rangle$

Class representative	Number of classes	Order of centralizer	σ_1	$\sigma_2(k)$ $k\in T_2$	$\sigma_3(k)$ $k\in T_2$	$\sigma_4(k, l)$ $k, l\in T_2$	Class in G
$h(1, 1)$	1	$q^2(q^2-1)^2$	q	$q-1$	$q(q-1)$	$(q-1)^2$	A_1
$x_{2a+b}(1)$	1	$q^2(q^2-1)$		-1	$-q$	$-(q-1)$	A_2
$h(\gamma^i, 1)$	$(q-2)/2$	$q(q-1)(q^2-1)$	1				C_1
$h(\eta^i, 1)$	$q/2$	$q(q+1)(q^2-1)$	-1	$-\beta_{ik}$	$-q\beta_{ik}$	$-(q-1)\beta_{ik}$	C_3
$x_b(1)$	1	$q^2(q^2-1)$	q	$q-1$		$-(q-1)$	A_2
$x_b(1)x_{2a+b}(1)$	1	q^2		-1		1	A_{32}
$h(\gamma^i, 1)x_b(1)$	$(q-2)/2$	$q(q-1)$	1				D_1
$h(\eta^i, 1)x_b(1)$	$q/2$	$q(q+1)$	-1	$-\beta_{ik}$		β_{ik}	D_3
$h(1, \gamma^j)$	$(q-2)/2$	$q(q-1)(q^2-1)$	q	$q-1$	$q-1$		C_1
$h(1, \gamma^j)x_{2a+b}(1)$	$(q-2)/2$	$q(q-1)$		-1	-1		D_1
$h(\gamma^i, \gamma^j)$	$(q-2)^2/4$	$(q-1)^2$	1				B_1, C_2
$h(\eta^i, \gamma^j)$	$q(q-2)/4$	q^2-1	-1	$-\beta_{ik}$	$-\beta_{ik}$		B_3
$h(1, \eta^j)$	$q/2$	$q(q+1)(q^2-1)$	q	$q-1$	$-(q-1)$	$-(q-1)\beta_{jl}$	C_3
$h(1, \eta^j)x_{2a+b}(1)$	$q/2$	$q(q+1)$		-1	1	β_{jl}	D_3
$h(\gamma^i, \eta^j)$	$q(q-2)/4$	q^2-1	1				B_3
$h(\eta^i, \eta^j)$	$q^2/4$	$(q+1)^2$	-1	$-\beta_{ik}$	β_{ik}	$\beta_{ik}\beta_{jl}$	B_4, C_4

Induced characters of G from $\langle \mathfrak{H}\mathfrak{X}_b\mathfrak{X}_{2a+b},\ \omega_b,\ \omega_{2a+b}\rangle$

χ	$\bar\sigma_1$	$\bar\sigma_2(k)$ $k\in T_2$	$\bar\sigma_3(k)$ $k\in T_2$	$\bar\sigma_4(k, l)$ $(k, l)\in S_2$
A_1	$q^3(q^2+1)$	$q^2(q-1)(q^2+1)$	$q^3(q-1)(q^2+1)$	$q^2(q-1)^2(q^2+1)$
A_2	q^3	$q^2(q-2)$	$-q^3$	$-2q^2(q-1)$
A_{32}		$-q^2$		
$B_1(i, j)$	2			
$B_3(i, j)$		$-\beta_{jk}$	$-\beta_{jk}$	
$B_4(i, j)$	-2	$-(\beta_{ik}+\beta_{jk})$	$\beta_{ik}+\beta_{jk}$	$\beta_{ik}\beta_{jl}+\beta_{il}\beta_{jk}$
$C_1(i)$	$q+1$	$q-1$	$q-1$	
$C_2(i)$	$q(q+1)$			
$C_3(i)$	$q-1$	$q-1-\beta_{ik}$	$-(q-1)-q\beta_{ik}$	$-(q-1)(\beta_{ik}+\beta_{il})$
$C_4(i)$	$-q(q-1)$	$-q(q-1)\beta_{ik}$	$q(q-1)\beta_{ik}$	$q(q-1)\beta_{ik}\beta_{il}$
$D_1(i)$		-1	-1	
$D_3(i)$	-1	$-1-\beta_{ik}$	1	$\beta_{ik}+\beta_{il}$
$(\chi, \chi)_G$	$q+2$	q	q^2-q+2	q^2-2q+3

Using these induced characters, almost all the irreducible characters of G are obtained:

$$\chi_4(k,\,l)=\tilde{\sigma}_4(k,\,l)-\tilde{\chi}_6(k+l)-\tilde{\chi}_6(k-l)+\tilde{\theta}_1,\quad (k,\,l)\in S_2\,,$$
$$\chi_{13}(k)=\tilde{\sigma}_3(k)-\tilde{\chi}_6(k),\quad k\in T_2\,,$$
$$\chi_9(k)=\tilde{\sigma}_2(k)+\tilde{\chi}_6(k)-\tilde{\theta}_1,\quad k\in T_2\,,$$
$$\chi_8(k)=\sum_{l=1}^{(q-2)/2}(\chi_3(l,\,k)-\chi_2((q-1)k+(q+1)l))+\sum_{\substack{1\le l\le q/2\\ l\not\equiv\pm k \bmod q+1}}(\chi_4(k,\,l)-\chi_4(k+l,\,k-l))$$
$$+\tilde{\chi}_6'(k)-\tilde{\chi}_6(2k)+\chi_{13}(k)-\chi_{13}(2k)+\chi_9(2k),\quad k\in T_2\,,$$
$$\chi_{12}(k)=\tilde{\chi}_7((q-1)k)-\chi_8(k),\quad k\in T_2\,,$$
$$\theta_2=\tilde{\theta}_3(0)-\sum_{k=2}^{(q-2)/2}\sum_{l=1}^{k-1}\chi_1(k,\,l)-\sum_{k=1}^{(q-2)/2}\sum_{l=1}^{q/2}\chi_3(k,\,l)-\sum_{k=2}^{q/2}\sum_{l=1}^{k-1}\chi_4(k,\,l)$$
$$-\sum_{k=1}^{(q-2)/2}\chi_6(k)-\sum_{k=1}^{(q-2)/2}\chi_{11}(k)-\sum_{k=1}^{q/2}\chi_{13}(k)\,,$$
$$\theta_1=\tilde{\chi}_1(0)-\theta_0-\theta_2\,,$$
$$\theta_3=\tilde{\chi}_1'(0)-\theta_0-\theta_1\,,$$
$$\theta_4=\tilde{\chi}_2(0)-\tilde{\chi}_1'(0)+\theta_0,\quad \text{and}$$
$$\theta_5=\tilde{\theta}_3(1)-\tilde{\theta}_2(1)+\tilde{\sigma}_1-\theta_1\,.$$

	$\chi_6'(k)-\chi_6(2k)$	$\chi_{13}(k)-\chi_{13}(2k)$	$\sum_{l=1}^{(q-2)/2}\{\chi_3(l,\,k)-\chi_2((q-1)k+(q+1)l)\}$	$\sum_{\substack{1\le l\le q/2\\ l\not\equiv\pm k \bmod q+1}}\{\chi_4(k,\,l)-\chi_4(k+l,\,k-l)\}$
A_2	$q^2(q-1)$		$-q^2(q-2)$	
A_{31}	$-q^2(b-1)$		$q^2(q-2)$	
$B_2(i)$			$-\beta_{ik}$	
$B_3(i,j)$		$-\beta_{jk}+\beta_{2jk}$	β_{jk}	
$B_4(i,j)$		$\beta_{ik}+\beta_{jk}-\beta_{2ik}-\beta_{2jk}$		$-\beta_{ik}-\beta_{jk}-\beta_{ik}\beta_{jk}+2\beta_{2ik}+2\beta_{2jk}$
$C_1(i)$			$-(q-1)$	
$C_2(i)$			$q-1$	
$C_3(i)$	$(q^2-1)\beta_{ik}$	$-q(\beta_{ik}-\beta_{2ik})$	$-(q+1)(q-2)/2\cdot\beta_{ik}$	$-(q-1)((q-2)/2\cdot\beta_{ik}+\beta_{2ik}+1)$
$C_4(i)$	$-(q^2-1)\beta_{2ik}$	$-(q-1)(\beta_{ik}-\beta_{2ik})$	$(q+1)(q-2)/2\cdot\beta_{2ik}$	$(q-1)((q-2)/2\cdot\beta_{2ik}+\beta_{ik}+1)$
$D_1(i)$			1	
$D_2(i)$			-1	
$D_3(i)$	$-\beta_{ik}$		$-(q-2)/2\cdot\beta_{ik}$	$(q-2)/2\cdot\beta_{ik}+\beta_{2ik}+1$
$D_4(i)$	β_{2ik}	$\beta_{ik}-\beta_{2ik}$	$(q-2)/2\cdot\beta_{2ik}$	$-(q-2)/2\cdot\beta_{2ik}-\beta_{ik}-1$

	$\sum_{k=2}^{(q-2)/2} \sum_{l=1}^{k-1} \chi_1(k,l)$	$\sum_{k=1}^{(q-2)/2} \sum_{l=1}^{q/2} \chi_3(k,l)$	$\sum_{k=2}^{q/2} \sum_{l=1}^{k-1} \chi_4(k,l)$
A_1	$(q+1)^2(q^2+1)(q-2)(q-4)/8$	$(q^4-1)q(q-2)/4$	$(q-1)^2(q^2+1)q(q-2)/8$
A_2	$(q+1)^2(q-2)(q-4)/8$	$-(q^2+1)q(q-2)/4$	$(q-1)^2q(q-2)/8$
A_{31}	$(q+1)^2(q-2)(q-4)/8$	$(q^2-1)q(q-2)/4$	$(q-1)^2q(q-2)/8$
A_{32}	$(2q+1)(q-2)(q-4)/8$	$-q(q-2)/4$	$-(2q-1)q(q-2)/8$
A_{41}	$(q-2)(q-4)/8$	$-q(q-2)/4$	$q(q-2)/8$
A_{42}	$(q-2)(q-4)/8$	$-q(q-2)/4$	$q(q-2)/8$
$B_1(i,j)$	3		
$B_3(i,j)$		-1	
$B_4(i,j)$			3
$C_1(i)$	$-(q+1)(q-4)/8$	$-q(q-1)/2$	
$C_2(i)$	$-(q+1)(q-4)/8$		
$C_3(i)$		$(q+1)(q-2)/2$	$(q-1)(q-2)/2$
$C_4(i)$			$(q-1)(q-2)/2$
$D_1(i)$	$-(q-4)/2$	$q/2$	
$D_2(i)$	$-(q-4)/2$		
$D_3(i)$		$(q-2)/2$	$-(q-2)/2$
$D_4(i)$			$-(q-2)/2$

	$\sum_{k=1}^{(q-2)/2} \chi_6(k)$	$\sum_{k=1}^{(q-2)/2} \chi_{11}(k)$	$\sum_{k=1}^{q/2} \chi_{13}(k)$
A_1	$(q+1)(q^2+1)(q-2)/2$	$q(q+1)(q^2+1)(q-2)/2$	$q^2(q-1)(q^2+1)/2$
A_2	$(q+1)(q-2)/2$	$q(q-2)/2$	$-q^2/2$
A_{31}	$(q^2+q+1)(q-2)/2$	$q(q+1)(q-2)/2$	$q^2(q-1)/2$
A_{32}	$(q+1)(q-2)/2$	$q(q-2)/2$	$-q^2/2$
A_{41}	$(q-2)/2$		
A_{42}	$(q-2)/2$		
$B_1(i,j)$	-2	-2	
$B_2(i)$	-1		
$B_3(i,j)$		1	1
$B_4(i,j)$			-2
$C_1(i)$	$-(q+1)$	$-q+(q+1)(q-2)/2$	$q(q-1)/2$
$C_2(i)$	$-1+(q+1)(q-2)/2$	$-(q+1)$	
$C_3(i)$		$-(q+1)(q-2)/2$	$q-q(q-1)/2$
$C_4(i)$	$(q+1)(q-2)/2$		$q-1$
$D_1(i)$	-1	$(q-2)/2$	$-q/2$
$D_2(i)$	$-1+(q-2)/2$	-1	
$D_3(i)$		$-(q-2)/2$	$q/2$
$D_4(i)$	$(q-2)/2$		-1

Finally, we consider the following linear character $\rho\ (k)$ of $M=\{h(\tau^i, \tau^{qi})\,|\, 0\leq i\leq q^2\}$[1]:

$$h(\tau^i, \tau^{qi}) \to \tilde{\tilde{\tau}}^{ik}\,.$$

Then we obtain the following family of irreducible characters:

$$\chi_5(k) = \tilde{\rho}(k)-\tilde{\rho}(0)+\theta_0-\theta_1+\theta_4+\theta_5\,,\quad k\in R_2\,.$$

Now we have obtained all the irreducible characters of G.

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References

[1] H. Enomoto: *The conjugacy classes of Chevalley groups of type* (G_2) *over finite fields of characteristic* 2 *or* 3, J. Fac. Sci. Univ. Tokyo Sect. I. **16** (1970), 497–512.

[2] H. Enomoto: *The characters of Chevalley groups of type* (G_2) *over finite fields of characteristic* 2, to appear.

[3] B. Srinivasan: *The characters of the finite symplectic group* $Sp(4, q)$, Trans. Amer. Math. Soc. **131** (1968), 488–525.

[4] G.E. Wall: *On the conjugacy classes in the unitary, symplectic and orthogonal groups,* J. Austral. Math. Soc. **3** (1963), 1–62.

Appendix

The character tables of various subgroups of $G=Sp(4,\ q)$ (and G itself) are given in this Appendix. We remark that the class representatives are not necessarily elements of G, but canonical forms in $Sp(4, \bar{K})$ are given. We use the convension that the value is zero if there is no entry in the character table.

1) To be exact, we should say that we consider the subgroup of G which is *conjugate* to M in Sp(4, $\bar{K}$), since $h(\tau^i, \tau^{qi})$ is not an element of G unless $h(\tau^i, \tau^{qi})=1$.

Table I-1 Conjugacy classes of B

Notation	Class representative	Number of classes	Order of centralizer	Class in P	Class in Q
A_1	$h(1,1)$	1	$q^4(q-1)^2$	A_1	A_1
A_2	$x_{2a+b}(1)$	1	$q^4(q-1)$	A_2	A_2
A_{31}	$x_{a+b}(1)$	1	$q^4(q-1)$	A_{31}	A_{31}
A_{32}	$x_{a+b}(1)x_{2a+b}(1)$	1	q^4	A_{32}	A_{32}
A_{41}	$x_b(1)$	1	$q^3(q-1)$	A_2	A_{41}
A_{42}	$x_b(1)x_{2a+b}(1)$	1	q^3	A_{32}	A_{42}
A_{51}	$x_a(1)$	1	$q^3(q-1)$	A_{41}	A_{31}
A_{52}	$x_a(1)x_{2a+b}(1)$	1	q^3	A_{42}	A_{32}
A_{61}	$x_a(1)x_b(1)$	1	$2q^2$	A_{51}	A_{51}
A_{62}	$x_a(1)x_b(1)x_{2a+b}(\xi)$	1	$2q^2$	A_{52}	A_{52}
$B(i,j)$	$h(\gamma^i,\gamma^j)$	$(i,j)\in S_1$ $(q-2)(q-4)$	$(q-1)^2$	B_1	B_1
$C_1(i)$	$h(1,\gamma^i)$	$i\in T_1$ $q-2$	$q(q-1)^2$	C_1	C_1
$C_2(i)$	$h(\gamma^i,\gamma^{-i})$	$i\in T_1$ $q-2$	$q(q-1)^2$	C_2	C_2
$C_3(i)$	$h(\gamma^i,1)$	$i\in T_1$ $q-2$	$q(q-1)^2$	C_1	C_3
$C_4(i)$	$h(\gamma^i,\gamma^i)$	$i\in T_1$ $q-2$	$q(q-1)^2$	C_3	C_2
$D_1(i)$	$h(1,\gamma^i)x_{2a+b}(1)$	$i\in T_1$ $q-2$	$q(q-1)$	D_1	D_1
$D_2(i)$	$h(\gamma^i,\gamma^{-i})x_{a+b}(1)$	$i\in T_1$ $q-2$	$q(q-1)$	D_2	D_2
$D_3(i)$	$h(\gamma^i,1)x_b(1)$	$i\in T_1$ $q-2$	$q(q-1)$	D_1	D_3
$D_4(i)$	$h(\gamma^i,\gamma^i)x_a(1)$	$i\in T_1$ $q-2$	$q(q-1)$	D_3	D_2

Table I-2 Characters of B

Character	$\chi_1(k, l)$	$\chi_2(k)$	$\chi_3(k)$	$\chi_4(k)$	$\chi_5(k)$	θ_1	$\theta_2(k)$	$\theta_3(k)$
Number of characters	$k, l \in T_0$ $(q-1)^2$	$k \in T_0$ $q-1$	$k \in T_0$ $q-1$	$k \in T_0$ $q-1$	$k \in T_0$ $q-1$	1	$k=0, 1$ 2	$k=0, 1$ 2
A_1	1	$q-1$	$q-1$	$q(q-1)$	$q(q-1)$	$(q-1)^2$	$q(q-1)^2/2$	$q(q-1)^2/2$
A_2	1	$q-1$	$q-1$	$q(q-1)$	$-q$	$(q-1)^2$	$-q(q-1)/2$	$-q(q-1)/2$
A_{31}	1	$q-1$	$q-1$	$-q$	$q(q-1)$	$(q-1)^2$	$-q(q-1)/2$	$-q(q-1)/2$
A_{32}	1	$q-1$	$q-1$	$-q$	$-q$	$(q-1)^2$	$q/2$	$q/2$
A_{41}	1	$q-1$	-1			$-(q-1)$	$q(q-1)/2$	$-q(q-1)/2$
A_{42}	1	$q-1$	-1			$-(q-1)$	$-q/2$	$q/2$
A_{51}	1	-1	$q-1$			$-(q-1)$	$(-1)^k q(q-1)/2$	$(-1)^k q(q-1)/2$
A_{52}	1	-1	$q-1$			$-(q-1)$	$-(-1)^k q/2$	$-(-1)^k q/2$
A_{61}	1	-1	-1			1	$(-1)^k q/2$	$-(-1)^k q/2$
A_{62}	1	-1	-1			1	$-(-1)^k q/2$	$(-1)^k q/2$
$B(i,j)$	$\tilde{\gamma}^{ik+jl}$							
$C_1(i)$	$\tilde{\gamma}^{il}$				$(q-1)\tilde{\gamma}^{ik}$			
$C_2(i)$	$\tilde{\gamma}^{i(k-l)}$			$(q-1)\tilde{\gamma}^{ik}$				
$C_3(i)$	$\tilde{\gamma}^{ik}$		$(q-1)\tilde{\gamma}^{ik}$					
$C_4(i)$	$\tilde{\gamma}^{(k+l)}$	$(q-1)\tilde{\gamma}^{ik}$						
$D_1(i)$	$\tilde{\gamma}^{il}$				$-\tilde{\gamma}^{ik}$			
$D_2(i)$	$\tilde{\gamma}^{i(k-l)}$			$-\tilde{\gamma}^{ik}$				
$D_3(i)$	$\tilde{\gamma}^{ik}$		$-\tilde{\gamma}^{ik}$					
$D_4(i)$	$\tilde{\gamma}^{i(k+l)}$	$-\tilde{\gamma}^{ik}$						

Table II-1 Conjugacy classes of P

Notation	Class representative	Number of classes	Order of centralizer	Class in G
A_1	$h(1,1)$	1	$q^4(q-1)(q^2-1)$	A_1
A_2	$x_{2a+b}(1)$	1	$q^4(q-1)$	A_2
A_{31}	$x_{a+b}(1)$	1	$q^4(q^2-1)$	A_{31}
A_{32}	$x_{a+b}(1)x_{2a+b}(1)$	1	q^4	A_{32}
A_{41}	$x_a(1)$	1	$q^3(q-1)$	A_{31}
A_{42}	$x_a(1)x_{2a+b}(1)$	1	q^3	A_{32}
A_{51}	$x_a(1)x_b(1)$	1	$2q^2$	A_{41}
A_{52}	$x_a(1)x_b(1)x_{2a+b}(\xi)$	1	$2q^2$	A_{42}
$B_1(i,j)$	$h(\gamma^i,\gamma^j)$	$(i,j)\in S_1$ $(q-2)(q-4)/2$	$(q-1)^2$	B_1
$B_2(i)$	$h(\theta^i,\theta^{qi})$	$i\in R_1$ $q(q-2)/2$	q^2-1	B_2
$C_1(i)$	$h(1,\gamma^i)$	$i\in T_1$ $q-2$	$q(q-1)^2$	C_1
$C_2(i)$	$h(\gamma^i,\gamma^{-i})$	$i\in T_1$ $(q-2)/2$	$q(q-1)^2$	C_2
$C_3(i)$	$h(\gamma^i,\gamma^i)$	$i\in T_1$ $q-2$	$q(q-1)(q^2-1)$	C_2
$C_4(i)$	$h(\eta^i,\eta^{-i})$	$i\in T_2$ $q/2$	$q(q^2-1)$	C_4
$D_1(i)$	$h(1,\gamma^i)x_{2a+b}(1)$	$i\in T_1$ $q-2$	$q(q-1)$	D_1
$D_2(i)$	$h(\gamma^i,\gamma^{-i})x_{a+b}(1)$	$i\in T_1$ $(q-2)/2$	$q(q-1)$	D_2
$D_3(i)$	$h(\gamma^i,\gamma^i)x_a(1)$	$i\in T_1$ $q-2$	$q(q-1)$	D_2
$D_4(i)$	$h(\eta^i,\eta^{-i})x_{a+b}(1)$	$i\in T_2$ $q/2$	$q(q+1)$	D_4

$$B_1(i,j)=B_1(j,i),$$
$$B_2(i)=B_2(qi),$$
$$C_2(i)=C_2(-i),\qquad C_4(i)=C_4(-i),$$
$$D_2(i)=D_2(-i),\qquad D_4(i)=D_4(-i).$$

Table II-2 Characters of P

Character	$\chi_1(k)$	$\chi_2(k)$	$\chi_3(k, l)$	$\chi_4(k)$	$\chi_5(k)$	$\chi_6(k)$	$\chi_7(k)$	θ_1	$\theta_2(k)$	$\theta_3(k)$
Number of characters	$k\in T_0$ $q-1$	$k\in T_0$ $q-1$	$(k, l)\in S_0$ $(q-1)(q-2)/2$	$k\in T_0$ $q-1$	$k\in T_1$ $(q-2)/2$	$k\in T_2$ $q/2$	$k\in R_1$ $q(q-1)/2$	1	$k=0, 1$ 2	$k=0, 1$ 2
A_1	1	q	$q+1$	q^2-1	$q(q^2-1)$	$q(q-1)^2$	$q-1$	$(q-1)(q^2-1)$	$q(q^2-1)/2$	$q(q-1)^2/2$
A_2	1	q	$q+1$	-1	$q(q-1)$	$-q(q-1)$	$q-1$	$-(q-1)$	$q(q-1)/2$	$-q(q-1)/2$
A_{31}	1	q	$q+1$	q^2-1	$-q(q+1)$	$-q(q-1)$	$q-1$	$(q-1)(q^2-1)$	$-q(q+1)/2$	$-q(q-1)/2$
A_{32}	1	q	$q+1$	-1	$-q$	q	$q-1$	$-(q-1)$	$-q/2$	$q/2$
A_{41}	1		1	$q-1$			-1	$-(q-1)$	$(-1)^k q(q-1)/2$	$(-1)^k q(q-1)/2$
A_{42}	1		1	$q-1$			-1	$-(q-1)$	$-(-1)^k q/2$	$-(-1)^k q/2$
A_{51}	1		1	-1			-1	1	$(-1)^k q/2$	$-(-1)^k q/2$
A_{52}	1		1	-1			-1	1	$-(-1^k)q/2$	$(-1)^k q/2$
$B_1(i, j)$	$\tilde{\gamma}^{(i+j)k}$	$\tilde{\gamma}^{(i+j)k}$	$\tilde{\gamma}^{ik+jl}+\tilde{\gamma}^{il+jk}$							
$B_2(i)$	$\tilde{\gamma}^{ik}$	$-\tilde{\gamma}^{ik}$					$-(\tilde{\theta}^{ik}+\tilde{\theta}^{qik})$			
$C_1(i)$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{ik}+\tilde{\gamma}^{il}$	$(q-1)\tilde{\gamma}^{ik}$						
$C_2(i)$	1	1	$\alpha_{i(k-l)}$		$(q-1)\alpha_{ik}$				$q-1$	
$C_3(i)$	$\tilde{\gamma}^{2ik}$	$q\tilde{\gamma}^{2ik}$	$(q+1)\tilde{\gamma}^{i(k+l)}$				$(q-1)\tilde{\gamma}^{ik}$			
$C_4(i)$	1	-1				$(q-1)\beta_{ik}$	$-\beta_{ik}$			$q-1$
$D_1(i)$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{ik}+\tilde{\gamma}^{il}$	$-\tilde{\gamma}^{ik}$						
$D_2(i)$	1	1	$\alpha_{i(k-l)}$		$-\alpha_{ik}$				-1	
$D_3(i)$	$\tilde{\gamma}^{2ik}$		$\tilde{\gamma}^{i(k+l)}$				$-\tilde{\gamma}^{ik}$			
$D_4(i)$	1	-1				$-\beta_{ik}$	$-\beta_{ik}$			-1

$$\chi_3(k, l) = \chi_3(l, k),$$
$$\chi_5(k) = \chi_5(-k),$$
$$\chi_6(k) = \chi_6(-k),$$
$$\chi_7(k) = \chi_7(-k).$$

Table III-1 Conjugacy classes of Q

Notation	Class representative	Number of classes		Order of centralizer	Class in G
A_1	$h(1,1)$	1		$q^4(q-1)(q^2-1)$	A_1
A_2	$x_{2a+b}(1)$	1		$q^4(q^2-1)$	A_2
A_{31}	$x_{a+b}(1)$	1		$q^4(q-1)$	A_{31}
A_{32}	$x_{a+b}(1)x_{2a+b}(1)$	1		q^4	A_{32}
A_{41}	$x_b(1)$	1		$q^3(q-1)$	A_2
A_{42}	$x_b(1)x_{2a+b}(1)$	1		q^3	A_{32}
A_{51}	$x_a(1)x_b(1)$	1		$2q^2$	A_{41}
A_{52}	$x_a(1)x_b(1)x_{2a+b}(\xi)$	1		$2q^2$	A_{42}
$B_1(i,j)$	$h(\gamma^i,\gamma^j)$	$(i,j)\in S_1$	$(q-2)(q-4)/2$	$(q-1)^2$	B_1
$B_2(i,j)$	$h(\gamma^i,\eta^j)$	$i\in T_1,\ j\in T_2$	$q(q-2)/2$	q^2-1	B_3
$C_1(i)$	$h(1,\gamma^i)$	$i\in T_1$	$(q-2)/2$	$q(q-1)^2$	C_1
$C_2(i)$	$h(\gamma^i,\gamma^{-i})$	$i\in T_1$	$q-2$	$q(q-1)^2$	C_2
$C_3(i)$	$h(\gamma^i,1)$	$i\in T_1$	$q-2$	$q(q-1)(q^2-1)$	C_1
$C_4(i)$	$h(1,\eta^i)$	$i\in T_2$	$q/2$	$q(q^2-1)$	C_3
$D_1(i)$	$h(1,\gamma^i)x_{2a+b}(1)$	$i\in T_1$	$(q-2)/2$	$q(q-1)$	D_1
$D_2(i)$	$h(\gamma^i,\gamma^{-i})x_{a+b}(1)$	$i\in T_1$	$q-2$	$q(q-1)$	D_2
$D_3(i)$	$h(\gamma^i,1)x_b(1)$	$i\in T_1$	$q-2$	$q(q-1)$	D_1
$D_4(i)$	$h(1,\eta^i)x_{2a+b}(1)$	$i\in T_2$	$q/2$	$q(q+1)$	D_3

$$B_1(i,j)=B_1(i,-j)\,,$$
$$B_2(i,j)=B_2(i,-j)\,,$$
$$C_1(i)=C_1(-i)\,,\qquad C_4(i)=C_4(-i)\,,$$
$$D_1(i)=D_1(-i)\,,\qquad D_4(i)=D_4(-i)\,.$$

Table III-2 Characters of Q

Character	$\chi_1'(k)$	$\chi_2'(k)$	$\chi_3'(k, l)$	$\chi_4'(k)$	$\chi_5'(k)$	$\chi_6'(k)$	$\chi_7'(k, l)$	θ_1'	$\theta_2'(k)$	$\theta_3'(k)$
Number of characters	$k \in T_0$ $q-1$	$k \in T_0$ $q-1$	$k \in T_0, l \in T_1$ $(q-1)(q-2)/2$	$k \in T_0$ $q-1$	$k \in T_1$ $(q-2)/2$	$k \in T_2$ $q/2$	$k \in T_0,\ l \in T_2$ $q(q-1)/2$	1	$k=0, 1$ 2	$k=0, 1$ 2
A_1	1	q	$q+1$	q^2-1	$q(q^2-1)$	$q(q-1)^2$	$q-1$	$(q-1)(q^2-1)$	$q(q^2-1)/2$	$q(q-1)^2/2$
A_2	1	q	$q+1$	q^2-1	$-q(q+1)$	$-q(q-1)$	$q-1$	$(q-1)(q^2-1)$	$-q(q+1)/2$	$-q(q-1)/2$
A_{31}	1	q	$q+1$	-1	$q(q-1)$	$-q(q-1)$	$q-1$	$-(q-1)$	$q(q-1)/2$	$-q(q-1)/2$
A_{32}	1	q	$q+1$	-1	$-q$	q	$q-1$	$-(q-1)$	$-q/2$	$q/2$
A_{41}	1		1	$q-1$			-1	$-(q-1)$	$(-1)^k q(q-1)/2$	$(-1)^\kappa q(q-1)/2$
A_{42}	1		1	$q-1$			-1	$-(q-1)$	$-(-1)^k q/2$	$-(-1)^k q/2$
A_{51}	1		1	-1			-1	1	$(-1)^k q/2$	$-(-1)^k q/2$
A_{52}	1		1	-1			-1	1	$-(-1)^\kappa q/2$	$(-1)^k q/2$
$B_1(i, j)$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{ik}\alpha_{jl}$							
$B_2(i, j)$	$\tilde{\gamma}^{ik}$	$-\tilde{\gamma}^{ik}$					$-\tilde{\gamma}^{ik}\beta_{jl}$			
$C_1(i)$	1	1	α_{il}		$(q-1)\alpha_{ik}$				$q-1$	
$C_2(i)$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{ik}\alpha_{il}$	$(q-1)\tilde{\gamma}^{ik}$						
$C_3(i)$	$\tilde{\gamma}^{ik}$	$q\tilde{\gamma}^{ik}$	$(q+1)\tilde{\gamma}^{ik}$				$(q-1)\tilde{\gamma}^{ik}$			
$C_4(i)$	1	-1				$(q-1)\beta_{ik}$	$-\beta_{il}$			$q-1$
$D_1(i)$	1	1	α_{il}		$-\alpha_{ik}$				-1	
$D_2(i)$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{ik}$	$\tilde{\gamma}^{ik}\alpha_{il}$	$-\tilde{\gamma}^{ik}$						
$D_3(i)$	$\tilde{\gamma}^{ik}$		$\tilde{\gamma}^{ik}$				$-\tilde{\gamma}^{ik}$			
$D_4(i)$	1	-1				$-\beta_{ik}$	$-\beta_{il}$			-1

$$\chi_3'(k, l) = \chi_3'(k, -l)\,, \qquad \chi_5'(k) = \chi_5'(-k)\,,$$

$$\chi_6'(k) = \chi_6'-(k)\,, \qquad \chi_7'(k, l) = \chi_7'(k, -l)\,.$$

Table IV-1 Conjugacy classes of $G=Sp(4, 2^f)$

Notation	Class representative	Number of classes	Order of centralizer
A_1	$h(1, 1)$	1	$q^4(q^2-1)(q^4-1)$
A_2	$x_{2a+b}(1)$	1	$q^4(q^2-1)$
A_{31}	$x_{a+b}(1)$	1	$q^4(q^2-1)$
A_{32}	$x_{a+b}(1)x_{2a+b}(1)$	1	q^4
A_{41}	$x_a(1)x_b(1)$	1	$2q^2$
A_{42}	$x_a(1)x_b(1)x_{2a+b}(\xi)$	1	$2q^2$
$B_1(i,j)$	$h(\gamma^i, \gamma^j)$	$(i,j)\in S_1$ $(q-2)(q-4)/8$	$(q-1)^2$
$B_2(i)$	$h(\theta^i, \theta^{qi})$	$i\in R_2$ $q(q-2)/4$	q^2-1
$B_3(i,j)$	$h(\gamma^i, \eta^j)$	$i\in T_1, j\in T_2$ $q(q-2)/4$	q^2-1
$B_4(i,j)$	$h(\eta^i, \eta^j)$	$(i,j)\in S_2$ $q(q-2)/8$	$(q+1)^2$
$B_5(i)$	$h(\tau^i, \tau^{qi})$	$i\in R_3$ $q^2/4$	q^2+1
$C_1(i)$	$h(1, \gamma^i)$	$i\in T_1$ $(q-2)/2$	$q(q-1)(q^2-1)$
$C_2(i)$	$h(\gamma^i, \gamma^{-i})$	$i\in T_1$ $(q-2)/2$	$q(q-1)(q^2-1)$
$C_3(i)$	$h(1, \eta^i)$	$i\in T_2$ $q/2$	$q(q+1)(q^2-1)$
$C_4(i)$	$h(\eta^i, \eta^{-i})$	$i\in T_2$ $q/2$	$q(q+1)(q^2-1)$
$D_1(i)$	$h(1, \gamma^i)x_{2a+b}(1)$	$i\in T_1$ $(q-2)/2$	$q(q-1)$
$D_2(i)$	$h(\gamma^i, \gamma^{-i})x_{a+b}(1)$	$i\in T_1$ $(q-2)/2$	$q(q-1)$
$D_3(i)$	$h(1, \eta^i)x_{2a+b}(1)$	$i\in T_2$ $q/2$	$q(q+1)$
$D_4(i)$	$h(\eta^i, \eta^{-i})x_{a+b}(1)$	$i\in T_2$ $q/2$	$q(q+1)$

The 8 classes $B_\mu(\pm i, \pm j)$, $B_\mu(\pm j, \pm i)$ are the same one, for $\mu=1,4$.

The 4 classes $B_\mu(\pm i)$, $B_\mu(\pm qi)$ are the same one for $\mu=2,5$.

The 4 classes $B_\mu(\pm i, \pm j)$ are the same one.

$C_\mu(i) = C_\mu(-i)$, $D_\mu(i) = D_\mu(-i)$ for $1\leq\mu\leq 4$.

Table IV-2 Characters of G

Character	θ_0	θ_1	θ_2	θ_3	θ_4	θ_5	$\chi_1(k,l)$	$\chi_2(k)$	$\chi_3(k,l)$	$\chi_4(k,l)$
Number of characters							$(k,l)\in S_1$ $(q-2)(q-4)/8$	$k\in R_2$ $q(q-2)/4$	$k\in T_1, l\in T_2$ $q(q-2)/4$	$(k,l)\in S_2$ $q(q-2)/8$
A_1	1	$q(q^2+1)/2$	$q(q+1)^2/2$	$q(q^2+1)/2$	q^4	$q(q-1)^2/2$	$(q+1)^2(q^2+1)$	q^4-1	q^4-1	$(q-1)^2(q^2+1)$
A_2	1	$q(q+1)/2$	$-q(q-1)/2$	$q(q+1)/2$		$-q(q-1)/2$	$(q+1)^2$	q^2-1	$-(q^2+1)$	$(q-1)^2$
A_{31}	1	$q(q+1)/2$	$q(q+1)/2$	$-q(q-1)/2$		$-q(q-1)/2$	$(q+1)^2$	$-(q^2+1)$	q^2-1	$(q-1)^2$
A_{32}	1	$q/2$	$q/2$	$q/2$		$q/2$	$2q+1$	-1	-1	$-(2q-1)$
A_{41}	1	$q/2$	$-q/2$	$-q/2$		$q/2$	1	-1	-1	1
A_{42}	1	$-q/2$	$q/2$	$q/2$		$-q/2$	1	-1	-1	1
$B_1(i,j)$	1	2	1	1	1		$\alpha_{ik}\alpha_{jl}+\alpha_{il}\alpha_{jk}$	$-(\tilde{\theta}^{ik}+\tilde{\theta}^{-ik}+\tilde{\theta}^{qik}+\tilde{\theta}^{-qik})$		
$B_2(i)$	1		1	-1	-1				$-\alpha_{ik}\beta_{jl}$	
$B_3(i,j)$	1		-1	1	-1					
$B_4(i,j)$	1		-1	-1	1	-2				$\beta_{ik}\beta_{jl}+\beta_{il}\beta_{jk}$
$B_5(i)$	1	-1			1					
$C_1(i)$	1		q	1	q	1	$(q+1)(\alpha_{ik}+\alpha_{il})$		$(q-1)\alpha_{ik}$	
$C_2(i)$	1	$q+1$	1	q	q		$(q+1)\alpha_{ik}\alpha_{il}$	$(q-1)\alpha_{ik}$		
$C_3(i)$	1	$q+1$	-1	q	$-q$	$q-1$			$-(q+1)\beta_{il}$	$-(q-1)(\beta_{ik}+\beta_{il})$
$C_4(i)$	1		q	-1	$-q$	$q-1$		$-(q+1)\beta_{ik}$		$-(q-1)\beta_{ik}\beta_{il}$
$D_1(i)$	1	1	1	1			$\alpha_{ik}+\alpha_{il}$		$-\alpha_{ik}$	
$D_2(i)$	1	1	-1				$\alpha_{ik}\alpha_{il}$	$-\alpha_{ik}$		
$D_3(i)$	1					-1			$-\beta_{il}$	$\beta_{ik}+\beta_{il}$
$D_4(i)$	1			-1		-1		$-\beta_{ik}$		$\beta_{ik}\beta_{il}$

The 8 characters $\chi_\mu(\pm k, \pm l)$, $\chi_\mu(\pm l, \pm k)$ are the same one for $\mu=1,4$.

The 4 characters $\chi_\mu(\pm k)$, $\chi_\mu(\pm qk)$ are the same one for $\mu=2,5$.

The 4 characters $\chi_3(\pm k, \pm l)$ are the same one.

$\chi_\mu(k) = \chi_\mu(-k)$ for $6\le\mu\le 13$.

Character	$\chi_5(k)$	$\chi_6(k)$	$\chi_7(k)$	$\chi_8(k)$	$\chi_9(k)$	$\chi_{10}(k)$	$\chi_{11}(k)$	$\chi_{12}(k)$	$\chi_{13}(k)$
Number of characters	$k \in R_3$ $q^2/4$	$k \in T_1$ $(q-2)/2$	$k \in T_1$ $(q-2)/2$	$k \in T_2$ $q/2$	$k \in T_2$ $q/2$	$k \in T_1$ $(q-2)/2$	$k \in T_1$ $(q-2)/2$	$k \in T_2$ $q/2$	$k \in T_2$ $q/2$
A_1	$(q^2-1)^2$	$(q+1)(q^2+1)$	$(q+1)(q^2+1)$	$(q-1)(q^2+1)$	$(q-1)(q^2+1)$	$q(q+1)(q^2+1)$	$q(q+1)(q^2+1)$	$q(q-1)(q^2+1)$	$q(q-1)(q^2+1)$
A_2	$-(q^2-1)$	$q+1$	q^2+q+1	$q-1$	$-(q^2-q+1)$	$q(q+1)$	q	$q(q-1)$	$-q$
A_{31}	$-(q^2-1)$	q^2+q+1	$q+1$	$-(q^2-q+1)$	$q-1$	q	$q(q+1)$	$-q$	$q(q-1)$
A_{32}	1	$q+1$	$q+1$	$q-1$	$q-1$	q	q	$-q$	$-q$
A_{41}	1	1	1	-1	-1				
A_{42}	1	1	1	-1	-1				
$B_1(i,j)$		$\alpha_{ik}\alpha_{jk}$	$\alpha_{ik}+\alpha_{jk}$			$\alpha_{ik}\alpha_{jk}$	$\alpha_{ik}+\alpha_{jk}$		
$B_2(i)$		α_{ik}		$-\beta_{ik}$		$-\alpha_{ik}$		$-\beta_{ik}$	
$B_3(i,j)$			α_{ik}		$-\beta_{jk}$		$-\alpha_{ik}$		
$B_4(i,j)$				$-\beta_{ik}\beta_{jk}$	$-(\beta_{ik}+\beta_{jk})$			$\beta_{ik}\beta_{kj}$	$-\beta_{jk}$
$B_5(i)$	$\tau^{ik}+\tau^{-ik}+\tau^{qik}+\tau^{-qik}$								$\beta_{ik}+\beta_{jk}$
$C_1(i)$		$(q+1)\alpha_{ik}$	$q+1+\alpha_{ik}$		$q-1$	$(q+1)\alpha_{ik}$	$q+1+q\alpha_{ik}$		$q-1$
$C_2(i)$		$q+1+\alpha_{2ik}$	$(q+1)\alpha_{ik}$	$q-1$		$q+1+q\alpha_{2ik}$	$(q+1)\alpha_{ik}$	$q-1$	
$C_3(i)$			$q+1$	$(q-1)\beta_{ik}$	$q-1-\beta_{ik}$		$-(q+1)$	$-(q-1)\beta_{ik}$	$-(q-1)-q\beta_{ik}$
$C_4(i)$		$q+1$		$q-1-\beta_{2ik}$	$(q-1)\beta_{ik}$	$-(q+1)$		$-(q-1)-q\beta_{2ik}$	$-(q-1)\beta_{ik}$
$D_1(i)$		α_{ik}	$1+\alpha_{ik}$		-1	α_{ik}	1		-1
$D_2(i)$		$1+\alpha_{2ik}$	α_{ik}	-1		1	α_{ik}	-1	
$D_3(i)$			1	$-\beta_{ik}$	$-1-\beta_{ik}$		-1	β_{ik}	1
$D_4(i)$		1		$-1-\beta_{2ik}$	$-\beta_{ik}$	-1		1	β_{ik}