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NOTE ON SEMISIMPLE EXTENSIONS AND SEPARABLE EXTENSIONS

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1. H-separable extensions

K. Hirata introduced the notion of a type of the separable extension recently in [7], which we shall call H-separable extension in this paper.

Let $\Lambda\supseteq\Gamma$ be rings with the common identity element. Then we say that Λ is an H-separable extension of Γ if $\Lambda\otimes_{\Gamma}\Lambda$ is isomorphic to a direct summand of a finite direct sum of the copies of Λ as two sided Λ-module. Such an extnesion is necessarily a separable extension i.e., ${}_{\Lambda}\Lambda_{\Lambda}<\oplus{}_{\Lambda}\Lambda\otimes_{\Gamma}\Lambda_{\Lambda}$ by Th. 2.2 [7]. Let $\Lambda\supseteq\Gamma$ be an H-separable extension, $V_{\Lambda}(\Gamma)=\{\lambda\in\Lambda\,|\,\gamma\lambda=\lambda\gamma$ for all $\gamma\in\Gamma\}$, and C be the center of Λ. Then, $\Lambda\otimes_{\Gamma}\Lambda\cong$ Hom $_C(V_{\Lambda}(\Gamma), \Lambda)$ and $V_{\Lambda}(\Gamma)$ is a finitely generated projective generator as C-module (see §2 [7]). Now we give some characterizations of H-separable extension and H-separable algebra. We assume all rings have units and all subrings have the same 1.

Theorem 1.1. *Let $\Lambda\supseteq\Gamma$ be rings with the common* 1. *Then $\Lambda\supseteq\Gamma$ is an H-separable extension if and only if the map $\eta: \Lambda\otimes_{\Gamma}\Lambda\to Hom_C(\Delta, \Lambda)$ such that $\eta(x\otimes y)(d)=xdy$ is an isomorphism and Δ is a finitely generated projective C-module, where C is the center of Λ and $\Delta=V_{\Lambda}(\Gamma)$.*

Proof. The 'only if' part have been proved in [7]. So we need only to prove the converse. Since Δ is a finitely generated projective C-module, the map $\varphi:\Delta\otimes_C \mathrm{Hom}_{\Lambda^e}(\Lambda, \Lambda\otimes_{\Gamma}\Lambda)\to \mathrm{Hom}_{\Lambda^e}(\mathrm{Hom}_C(\Delta, \Lambda), \Lambda\otimes_{\Gamma}\Lambda)$ such that $\varphi(d\otimes f)(h)=f(h(d))$ is an isomorphism. On the other hand, we see $\mathrm{Hom}_{\Lambda^e}(\Lambda\otimes\Lambda, \Lambda)\cong\Delta$ by the map $f\to f(1)$. Since the map $\eta:\Lambda\otimes_{\Gamma}\Lambda\to \mathrm{Hom}_C(\Delta, \Lambda)$ is an isomorphism, the map

$$\psi:\mathrm{Hom}_{\Lambda^e}(\Lambda\otimes_{\Gamma}\Lambda, \Lambda)\otimes_C \mathrm{Hom}_{\Lambda^e}(\Lambda, \Lambda\otimes_{\Gamma}\Lambda)\to \mathrm{Hom}_{\Lambda^e}(\Lambda\otimes_{\Gamma}\Lambda, \Lambda\otimes_{\Gamma}\Lambda)$$

such that $\psi(f\otimes g)=g\circ f$ is an isomorphism. This means ${}_{\Lambda}\Lambda\otimes_{\Gamma}\Lambda_{\Lambda}<\oplus{}_{\Lambda}(\sum^{n}\oplus\Lambda)_{\Lambda}$. Hence Λ is an H-separable extension of Γ.

Proposition 1.1 *Let Λ be an algebra over a commutative ring R and C its center. Then, Λ is an H-separable R-algebra if and only if Λ is separable over C* *and* $C\otimes_R C\cong C$ *by the map* φ *such that* $\varphi(x\otimes y)=xy$.

Proof. Let Λ be an H-separable R-algebra. Then, by Th. 2.1 and Th. 2.3 [3] Λ is separable over C, and the map $\eta_C:\Lambda\otimes_C\Lambda\to\mathrm{Hom}_C(\Lambda, \Lambda)$ such that $\eta_C(x\otimes y)(\lambda)=x\lambda y$ is an isomorphism. On the other hand, we have the isomorphism $\eta_R:\Lambda\otimes_R\Lambda\to\mathrm{Hom}_C(\Lambda, \Lambda)$ with $\eta_R(x\otimes y)(\lambda)=x\lambda y$ by Prop. 1.1. Therefore, $\Lambda\otimes_R\Lambda$ is isomorphic to $\Lambda\otimes_C\Lambda$ by the map $\eta_C^{-1}\circ\eta_R(x\otimes y)=(x\otimes y)$. Then, since C is a C-direct summand of Λ, it follows $C\otimes_R C\cong C$. Conversely, assume Λ is separable over C and $C\otimes_R C\cong C$. Then $\Lambda\otimes_R\Lambda\cong(\Lambda\otimes_C C)\otimes_R(C\otimes_C\Lambda)\cong\Lambda\otimes_C(C\otimes_R C)\otimes_C\Lambda\cong\Lambda\otimes_C C\otimes_C\Lambda\cong\Lambda\otimes_C\Lambda$. On the other hand, since Λ is separable over C, $\Lambda=V_\Lambda(R)$ is a finitely generated projective C-module and $\mathrm{Hom}_C(V_\Lambda(R), \Lambda)=\mathrm{Hom}_C(\Lambda, \Lambda)\cong\Lambda\otimes_C\Lambda\cong\Lambda\otimes_R\Lambda$. Hence Λ is H-separable over R by Prop. 1.1.

EXAMPLE. Let R be a commutative ring and S a multiplicatively closed subset of R which does not contain 0. Then R_S, the ring of quatients of R with respect to S, enjoys the condition $R_S\otimes_R R_S\cong R_S$, since $r/s\otimes 1=r/s\otimes s/s=s/s\otimes r/s=1\otimes r/s$ for every $s\in S$ and $r\in R$. Therefore, every central separable R_S-algebra is an H-separable algebra over R but not a central separable R-algebra whenever S contains non unit elements.

Proposition 1.2. *If* Λ *is an H-separable extension of* Γ *such that* Γ *is a left* (*or right*) Γ*-direct summand of* Λ, *then* $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$.

Proof. Since Λ is H-separable over Γ,the map $\eta:\Lambda\otimes_\Gamma\Lambda\to\mathrm{Hom}_C(\Delta, \Lambda)$ such that $\eta(x\otimes y)(d)=xdy$ is an isomorphism. Let $x\in V_\Lambda(V_\Lambda(\Gamma))$. Then $\eta(x\otimes 1)(d)=xd=dx=\eta(1\otimes x)$ for all $d\in\Delta$. Hence $x\otimes 1=1\otimes x$. Then it is easy to show that $x\in\Gamma$, since Γ is a left (or right) Γ-direct summand of Λ.

Corollary 1.1. *An R-algebra* Λ *is central separable over R if and only if* Λ *is H-separable over R and R is an R-direct summand of* Λ.

Proposition 1.3. *Let* Λ *be an H-separable extension of* Γ *and B a subring of* Λ *which contains* Γ *and is a B-*Γ*-direct summand of* Λ *as left B and right* Γ *module. Then the map* $\eta_B:B\otimes_\Gamma\Lambda\to\mathrm{Hom}_D(\Delta, \Lambda)$, *where* $D=V_\Lambda(B)$ *and* $\Delta=V_\Lambda(\Gamma)$, *such that* $\eta_B(x\otimes y)(d)=xdy$ *is an isomorphism and* Δ *is a finitely generated projective left D-module, and* $V_\Lambda(V_\Lambda(B))=B$.

Proof. ${}_B B_\Gamma<\oplus_B\Lambda_\Gamma$ implies ${}_B B\otimes_\Gamma\Lambda_\Lambda<\oplus_B\Lambda\otimes_\Gamma\Lambda_\Lambda<\oplus_B(\sum^n\oplus\Lambda)_\Lambda$. On the other hand, $\mathrm{Hom}_{B\otimes_R\Lambda^0}(B\otimes_\Gamma\Lambda, \Lambda)=V_\Lambda(\Gamma)=\Delta$, where R is the center of Γ. Then, by Th. 1.2 (ii) [7] we see $\eta_B:B\otimes_\Gamma\Lambda\to\mathrm{Hom}_D(\Delta, \Lambda)$ is an isomorphism, while Th. 1.2 (iii) [7] shows $\mathrm{Hom}_{B\otimes_R\Lambda^0}(B\otimes_\Gamma\Lambda, \Lambda)=\Delta$ is a finitely generated projective left D-module. Now we have a commutative diagram of canonical maps

$$\begin{array}{ccc} B\otimes_{\Gamma}\Lambda & \xrightarrow{\eta_B} & \mathrm{Hom}\,_D(\Delta,\,\Lambda) \\ \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \tau'} \\ \Lambda\otimes_{\Gamma}\Lambda & \xrightarrow{\eta_{\Lambda}} & \mathrm{Hom}\,_C(\Delta,\,\Lambda) \end{array}$$

where τ, τ' are monomorphisms and η_{Λ}, η_B are isomorphisms. Let $x\in V_{\Lambda}(V_{\Lambda}(B))=V_{\Lambda}(D)$. Then $\eta_{\Lambda}(x\otimes 1)$ is a left D-homomorphism. Hence there exists $\sum b_i\otimes\lambda_i\in B\otimes_{\Gamma}\Lambda<\oplus\Lambda\otimes_{\Gamma}\Lambda$ such that $\eta_{\Lambda}(\sum b_i\otimes\lambda_i)=\eta_{\Lambda}(x\otimes 1)$, which implies $\sum b_i\otimes\lambda_i=x\otimes 1$. Since ${}_BB_{\Gamma}<\oplus_B\Lambda_{\Gamma}$ we see $x\in B$ by the map $\Lambda\otimes_{\Gamma}\Lambda\to\Lambda$: $x\otimes y\to xy$.

Proposition 1.4. *Let Λ, Γ and B be as in Prop. 1.3. Assume furthermore that B is a separable extension of Γ. Then D is a direct summand of Δ as two sided D-module, and Λ is an H-separable extension of B.*

Proof. Since B is separable over Γ, there exists $\sum x_i\otimes y_i\in B\otimes_{\Gamma}B$ such that $\sum x_iy_i=1$ and $\sum bx_i\otimes y_i=\sum x_i\otimes y_ib$ for every $b\in B$. Then, the map $f:\Delta\to D$ such that $f(d)=\sum x_idy_i$ $(d\in\Delta)$ is a D-D-homomorphism such that $f\circ i=1_D$, where i is the inclusion map. Hence, D is a D-D-direct summand of Δ. Let π be the projection of Δ onto D. Then we have a B-Γ-homomorphism φ' of ${}_B\Lambda_{\Gamma}$ into ${}_B\mathrm{Hom}\,_D(\Delta,\,\Lambda)_{\Gamma}$ such that $\varphi'(\lambda)=\lambda^r\circ\pi$, where λ^r means right multiplication of λ. Thus we have a commutative diagram

$$\begin{array}{ccc} B\otimes_{\Gamma}\Lambda & \xrightarrow{\eta_B} & \mathrm{Hom}\,_D(\Delta,\,\Lambda) \\ {\scriptstyle \eta_B}\big\downarrow & \overset{\varphi}{\swarrow} & \big\uparrow{\scriptstyle \varphi'} \\ \Lambda & \xleftarrow{1_{\Lambda}} & \Lambda \end{array}$$

where $\pi_B(b\otimes\lambda)=b\lambda$, $\varphi(h)=h(1)$ and η_B is an isomorphism, and all of them are right Λ and left B-maps. Since $\varphi'\circ\eta_B\circ\pi_B=1$, π_B splits as B-Λ-map. Consequently, we have $\Lambda\otimes_B\Lambda<\oplus\Lambda\otimes_B(B\otimes_{\Gamma}\Lambda)\cong\Lambda\otimes_{\Gamma}\Lambda$. Then, since $\Lambda\otimes_{\Lambda}\Lambda<\oplus\sum^{n}\oplus\Lambda$, ${}_{\Lambda}\Lambda\otimes_B\Lambda_{\Lambda}<\oplus_{\Lambda}\sum^{n}\oplus\Lambda_{\Lambda}$. This completes the proof.

Finally we shall give some formal properties of H-separable extensions.

Theorem 1.2. *Let $\Lambda\supseteq\Gamma$ be a ring extension. Then the following statements are equivalent*:

(a) *Λ is an H-separable extension of Γ.*

(b) *The map $g:\Delta\otimes_C(\Lambda\otimes_{\Gamma}\Lambda)^{\Lambda}\to(\Lambda\otimes_{\Gamma}\Lambda)^{\Gamma}$ such that $g(d\otimes\alpha)=d\alpha$ is an epimorphism.*

(c) *For every two sided Λ-module M, the map $g:\Delta\otimes_CM^{\Lambda}\to M^{\Gamma}$ is an isomorphism, where $M^{\Omega}=\{m\in M\,|\,mx=xm$ for every $x\in\Omega\}$.*

Proof. (a)$\Rightarrow$(c). Since Λ is H-separable over Γ, Δ is C-finitely generated projective. Therefore we have $\Delta\otimes_CM^{\Lambda}\cong\Delta\otimes_C\mathrm{Hom}\,_{\Lambda^e}(\Lambda,\,M)\cong\mathrm{Hom}\,_{\Lambda^e}(\mathrm{Hom}_C(\Delta,\,\Lambda),\,M)\cong\mathrm{Hom}\,_{\Lambda^e}(\Lambda\otimes\Lambda,\,M)\cong M^{\Gamma}$.

As (c)⇒(b) is trivial, we will prove (b)⇒(a).

(b)⇒(a). Since $\Delta \cong \mathrm{Hom}_{\Lambda^e}(\Lambda \otimes_\Gamma \Lambda, \Lambda)$, we have $\Delta \otimes_C (\Lambda \otimes_\Gamma \Lambda)^\Lambda \cong \mathrm{Hom}_{\Lambda^e}(\Lambda \otimes_\Gamma \Lambda, \Lambda) \otimes_C \mathrm{Hom}_{\Lambda^e}(\Lambda, \Lambda \otimes_\Gamma \Lambda) \cong (\Lambda \otimes \Lambda)^\Gamma \cong \mathrm{Hom}_{\Lambda^e}(\Lambda \otimes_\Gamma \Lambda, \Lambda \otimes_\Gamma \Lambda)$. Hence Λ is an H-separable extension of Γ (see Prop. 1.1[7]).

Proposition 1.5. *Let f be a ring epimorphism from Λ_1 to Λ_2, $f(\Gamma_1)=\Gamma_2$ for a subring Γ_1 of Λ_1, C_i the center of Λ_i, and $\Delta_i = V_{\Lambda_i}(\Gamma_i)$ for $i=1, 2$. If Λ_1 is an H-separable extension of Γ_1, then Λ_2 is an H-separable extension of Γ_2 and $\Delta \otimes_{C_1} C_2 \cong \Delta_2$.*

Proof. Let M be an arbitrary two sided Λ_2-module. Then M becomes a two sided Λ_1-module by f, and $M^{\Lambda_1}=M^{\Lambda_2}$ and $M^{\Gamma_1}=M^{\Gamma_2}$. Therefore we have $\Delta_1 \otimes_{C_1} M^{\Lambda_2} = M^{\Gamma_2}$ by Theorem 1.2. Taking $M=\Lambda_2$, we have $\Delta_1 \otimes_{C_1} C_2 = \Delta_2$. Then $\Delta_2 \otimes_{C_2} M^{\Lambda_2} = \Delta_1 \otimes_{C_1} C_2 \otimes_{C_2} M^{\Lambda_2} \cong \Delta_1 \otimes_{C_1} M^{\Lambda_1} = M^{\Gamma_1} = M^{\Gamma_2}$ for any two sided Λ_2-module M, which means Λ_2 is an H-separable extension of Γ_2.

Proposition 1.6. *Let $\Omega \supseteq \Lambda \supseteq \Gamma$ be rings with the common 1. If both $\Omega \supseteq \Lambda$ and $\Lambda \supseteq \Gamma$ are H-separable extensions, $\Omega \supseteq \Gamma$ is also an H-separable extension. If furthermore $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$ and $V_\Omega(V_\Omega(\Lambda))=\Lambda$, then $V_\Omega(V_\Omega(\Gamma))=\Gamma$.*

Proof. Let $\Lambda \otimes_\Gamma \Lambda < \oplus \sum^m \oplus \Lambda$ and $\Omega \otimes_\Lambda \Omega < \oplus \sum^n \Omega$. Then $\Omega \otimes_\Gamma \Omega \cong \Omega \otimes_\Lambda (\Lambda \otimes_\Gamma \Lambda) \otimes_\Lambda \Omega < \oplus \sum^m \Omega \otimes_\Lambda \Lambda \otimes_\Lambda \Omega \cong \sum^m \Omega \otimes_\Lambda \Omega < \oplus \sum^{mn} \Omega$ as two sided-module. Hence Ω is H-separable over Γ. Assume $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$ and $V_\Omega(V_\Omega(\Lambda))=\Lambda$. Since $V_\Omega(\Gamma)=V_\Omega(\Lambda)\cdot V_\Lambda(\Gamma)$ by Theorem 1.2, $V_\Omega(V_\Omega(\Gamma)) = V_\Omega(V_\Omega(\Lambda)) \cap V_\Omega(V_\Lambda(\Gamma)) = \Lambda \cap V_\Omega(V_\Lambda(\Gamma)) = V_\Lambda(V_\Lambda(\Gamma)) = \Gamma$.

Proposition 1.7. *Let Λ_i, Γ_i be algebras over a commutative ring R for $i=1, 2$. If Λ_i is an H-separable extention of Γ_i for $i=1, 2$, $\Lambda_1 \otimes_R \Lambda_2$ is an H-separable extension of $Im\,(\Gamma_1 \otimes_R \Gamma_2)$.*

Proof. Since $(\Lambda_1 \otimes_R \Lambda_2) \otimes_{\Gamma_1 \otimes_R \Gamma_2} (\Lambda_1 \otimes_R \Lambda_2) \cong (\Lambda_1 \otimes_{\Gamma_1} \Lambda_1) \otimes_R (\Lambda_2 \otimes_\Gamma \Lambda_2)$, if $\Lambda_1 \otimes_{\Gamma_1} \Lambda_1 < \oplus \sum^m \oplus \Lambda_1$ and $\Lambda_2 \otimes_{\Gamma_2} \Lambda_2 < \oplus \sum^n \oplus \Lambda_2$, $(\Lambda_1 \otimes_R \Lambda_2) \otimes_R (\Lambda_2 \otimes_R \Lambda_2) < \oplus \sum^{mn} \oplus \Lambda_1 \otimes_R \Lambda_2$. This comptetes the proof.

2. Semisimple extensions

Again let $\Lambda \supseteq \Gamma$ be rings with common 1 in this section. We say that Λ is a left semisimple extension over Γ if every left Λ-module is (Λ, Γ)-projective, and that Λ is a weak left semisimple extension over Γ if every finitely generated Λ-module is (Λ, Γ)-projective. An algebra over a commutative ring R is said to be a left semisimple algebra over R if it is a weak left semisimple extension over $R\cdot 1$. In the previous paper [6] we showed.

Lemma 2.1. (Prop. 1.6 [6]). *Let* Λ *be a left semisimple extension over* Γ. *If* Λ *is left* Γ *-projective or right* Γ*-flat, then l. gl. dim* $\Lambda \leqq$ *l. gl. dim* Γ. *If a weak left semisimple extension* Λ *of* Γ *is right* Γ*-flat, we have also l. gl. dim* Λ $\leqq$ *l. gl. dim* Γ.

Lemma 2.2. *If a ring* Λ *is left projective over its subring* Γ, *and if* Γ *is* Γ-Γ*-isomorphic* to Γ' *a two sided* Γ*-direct summand of* Λ, *l. gl. dim* $\Lambda \geqq$ *l. gl. dim* Γ.

Proof. Let ${}_\Gamma\Lambda_\Gamma = {}_\Gamma\Gamma_\Gamma' \oplus {}_\Gamma\Lambda_\Gamma'$ as two sided Γ-module and $\mathfrak{I}$ be an arbitrary left ideal of Γ. Since $\Lambda\mathfrak{I} = \Gamma'\mathfrak{I} \oplus \Lambda'\mathfrak{I} \cong \mathfrak{I} \oplus \Lambda'\mathfrak{I}$ as left Γ-module, $\Lambda/\Lambda\mathfrak{I} \cong \Gamma/\mathfrak{I} \oplus \Lambda'/\Lambda'\mathfrak{I}$ as left Γ-module. Suppose 1. gl. dim $\Lambda \leqq n$. Then dim ${}_\Lambda\Lambda/\Lambda\mathfrak{I} \leqq$ n. As Λ is Γ-projective, dim ${}_\Gamma\Lambda/\Lambda\mathfrak{I} \leqq$ dim ${}_\Lambda\Lambda/\Lambda\mathfrak{I}$. Since $\Lambda/\Lambda\mathfrak{I} \cong \Gamma/\mathfrak{I} \oplus \Lambda'/\Lambda'\mathfrak{I}$, dim ${}_\Gamma\Lambda/\Lambda\mathfrak{I} =$ max (dim ${}_\Gamma\Gamma/\mathfrak{I}$, dim ${}_\Gamma\Lambda'/\Lambda'\mathfrak{I}) \geqq$ dim ${}_\Gamma\Gamma/\mathfrak{I}$. Thus we see 1. dim $\Gamma/\mathfrak{I}$ $\leqq n$ for every left ideal $\mathfrak{I}$ of Γ. Since 1. gl. dim $\Gamma =$ sup 1. dim ${}_\Gamma\Gamma/\mathfrak{I}$ where $\mathfrak{I}$ runs over all left ideals of Γ, 1. gl. dim $\Gamma \leqq n$. Hence 1. gl. dim $\Gamma \leqq$ 1. gl. dim Λ.

Combining Lemma 2.1 and Lemma 2.2, we have

Proposition 2.1. *If* $\Lambda \supseteq \Gamma$ *be a left semisimple extension such that* Γ *is* Γ-Γ*-isomorphic to a two sided* Γ*-direct summand of* Λ *and* Λ *is left* Γ*-projective, then l. gl. dim* $\Lambda =$ *l. gl. dim* Γ.

Theorem 2.1. *If an R-algebra* Λ *is a finitely generated R-projective and left semisimple R-algebra, l. gl. dim* $\Lambda =$ *l. gl. dim* $R/\mathfrak{a}$, *where* $\mathfrak{a}$ *is the annihilator of* Λ *in R. Consequently, when* Λ *is (two sided) semisimple over R, l. gl. dim* Λ $=$ *r. gl. dim* Λ.

Proof. If Λ is R-finitely generated projective, Λ is $R/\mathfrak{a}$-finitely generated projective, and Λ is an $R/\mathfrak{a}$-generator. Hence $R/\mathfrak{a} < \oplus\Lambda$ as $R/\mathfrak{a}$-module. Since Λ is $R/\mathfrak{a}$-projective, it is $R/\mathfrak{a}$-flat. Therefore, the proof is straightforward by Lemma 2.1 and Lemma 2.2.

REMARK. Th. 2.1 shows that if Λ is a central separable R-algebra, l. gl. dim $\Lambda =$ r. gl. dim $\Lambda =$ gl. dim R. Th. 2.1 induces the well known fact that l. gl. dim $\Lambda = 0$ if and only if r. gl. dim $\Lambda = 0$ in case R is a field or a semisimple ring.

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Added in proof. K. Hirata kindly advised me that Proposition 1.1 can be stated in noncommutative case as follows.

Theorem 1.3′. *Let $\Lambda\supseteq\Gamma$ be an H-separable extension. Then Λ is H-separable extension of $\Gamma'=V_\Lambda(V_\Lambda(\Gamma))$. If Γ' is left and right Γ'-direct summands of Λ, then Λ is H-separable over Γ if and only if Λ is H-separable over Γ' and $\Gamma'\otimes_\Gamma\Gamma'\cong\Gamma'$.*

Proof. If Λ is H-separable over Γ, we have a commutative diagram

$$\begin{array}{ccc} \Lambda\otimes_\Gamma\Lambda & \xrightarrow{\varphi} & \Lambda\otimes_{\Gamma'}\Lambda \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta'} \\ & \mathrm{Hom}_C(\Delta,\Lambda) & \end{array}$$

where η is an isomorphism and $\varphi(x\otimes y)=x\otimes y$ is an epimorphism. Hence φ is an isomorphism, and Λ is an H-separable extension of Γ'. The rest of the proof is same as Theorem 1.1.

The next is a corollary to Theorem 1.1.

Corollary 1.2. *Let Λ be a faithful R-algebra. Then Λ is a central separable R-algebra, if and only if Λ is H-separable over R and a finitely generated R-module.*

Proof. The 'only if' part is clear, so we need only to prove the converse. Let C be the center of Λ. Since Λ is H-separable over R, $C<\oplus\Lambda$. Hence C is a finitely generated R-module, as Λ is R-finitely generated. Since $C\otimes_R C\cong C$ by Theorem 1.1, $C/\mathfrak{m}C\otimes_{R/\mathfrak{m}}C/\mathfrak{m}C\cong C/\mathfrak{m}C$ for every maximal ideal $\mathfrak{m}$ of R. Therefore we have $C/\mathfrak{m}C=R/\mathfrak{m}$, and $C=R+\mathfrak{m}C$ for every maximal ideal $\mathfrak{m}$ of R. Hence $C=R$, and Λ is central separable over R.