Wada, J.
Osaka J. Math.
1 (1964), 153-164

# ON THE INTERPOLATION OF SOME FUNCTION ALGEBRAS

Dedicated to Professor H. Terasaka on his sixtieth birthday

JUNZO WADA

(Received October 7, 1964)

Let $C(X)$ be the algebra of all complex-valued continuous functions on a compact Hausdorff space $X$ and let $A$ be a function algebra on $X$, that is, a closed (by supremum norm) subalgebra in $C(X)$ containing constants and separating points of $X$. A closed set $F_0$ in $X$ is said to be an interpolation set of $A$ (or a closed restriction set of $A$) if $A|F_0=C(F_0)$ (or $A|F_0$ is closed in $C(F_0)$), where $A|F_0=\{f|F_0, f\in A\}$ and $f|F_0$ is the restriction of $f$ on $F_0$. In [4] I. Glicksberg characterized interpolation sets and closed restriction sets on general function algebras $A$, and also showed that in a Dirichlet algebra, any closed restriction set of $A$ is an intersection of peak sets, but we see that the above fact is false in the case of a non-Dirichlet algebra. The main purpose of this paper is to consider problems of interpolation and closed restriction on a function algebra $A$ which is not a Dirichlet algebra and which has the property that the restriction $A|\partial A$ of $A$ by its Šilov boundary is an essential maximal algebra. Our main theorems are the following: Let $A$ be a function algebra on a compact metric space having the above property and some additional properties (Properties (B) and (D), cf. §2, 3.). Then, (1) if $F_0$ is a closed restriction set of $A$ for a closed set $F_0$, $F_0$ contains $\partial A$ or $F_0\sim\partial A$ is a countable set whose cluster points are in $\partial A$ (Theorem 2.2). (2) if $F_0$ is an interpolation set of $A$, then $F_0\cap\partial A$ is an interpolation set of $A|\partial A$ and $F_0\sim\partial A$ is an $H^\infty$-interpolating sequence, and the converse is also true (Theorem 3.2). ((2) was pointed by ([7], p. 208) in the case of the function algebra of continuous functions on the unit closed disc which is analytic on its interior, and it is a generalization).

## 1. Preliminaries

Let $A$ be a linear subspace of $C(X)$. Then $A$ is said to be *a function algebra on a compact Hausdorff space* $X$ if it satisfies the following conditions; (i) $f\cdot g\in A$ for any $f, g\in A$, (ii) $A$ is closed with the supremum norm of $C(X)$, (iii) $A$ contains the constant function 1 and (iv) for any distinct two points $x, y$ in $X$, there is an $f \in A$ with $f(x) \neq f(y)$. First, we define the Šilov boundary $\partial A$ and the essential set $E$ of $A$ as follows. *The Šilov boundary of A* is the smallest closed subset $F$ of $X$ such that $|f|$ takes its maximum value on $F$ for any $f \in A$. *The essential set of A* is the minimal closed subset $E$ of $X$ such that if $f(E)=0$ for a continuous functions $f$ on $X$, then $f \in A$. $A$ is *an essential algebra* if the essential set of $A$ is $X$ (cf. [1]). $A$ is said to be *an antisymmetric algebra* (or *an analytic algebra*) if any real-valued function in $A$ is always constant (or any function in $A$ vanishing on a non-empty open set in $X$ is always identically zero) (cf. [6]). A function algebra $A$ is said to be *a sequentially analytic algebra* if any function $f$ in $A$ vanishing on an infinite closed set in $X \sim \partial A$ is always identically zero. If $X$ is a metric space, we can take a sequence of points converging to a point in $X \sim \partial A$ in place of an infinite closed set in the above definition. For a sequentially analytic algebra, we can easily prove the following.

(*a*) *Let A be a sequentially analytic algebra on X. Let X have no isolated point and let $\partial A$ be non dense in X. Then A is analytic.*

(*b*) *If A is a sequentially analytic algebra and if $X \sim \partial A$ has a non-isolated point, then A is an integral domain, that is, $fg \equiv 0$ implies $f \equiv 0$ or $g \equiv 0$ for $f, g \in A$.*

Let $A$ be a function algebra on $X$ and let $F_0$ be a closed set in $X$. We say "*$F_0$ determines A*" if $f \equiv 0$, whenever $f(F_0)=0$ for an $f \in A$. For arbitrary function algebra, $\partial A$ determines $A$. $F_0$ is said to be *an interpolation set of A* (or *a closed restriction set of A*) if $A|F_0 = C(F_0)$ (or $A|F_0$ is closed in $C(F_0)$), where $A|F_0$ denotes the set $\{f|F_0;\ f \in A\}$ and $f|F_0$ denotes the restriction of $f$ on $F_0$. If $F_0$ is an interpolation set of $A$, then any continuous function on $F_0$ can be extended to a function in $A$.

For a function algebra which is an integral domain, we have

**Theorem 1.1.** *Let A be a function algebra which is an integral domain and let $P \subset F_1 \cup \cdots \cup F_n$, where P determines A, $F_i$ is closed set in X and $F_i$ is a closed restriction set of A for any i. Then $F_k \supset \partial A$ for some k.*

Proof. If we assume that $f \equiv 0$ whenever $f(F_k)=0$ for an $f \in A$, the complex homomorphism $h \to h(x)$ of $A|F_k$ is well-defined for any $x \in X$. Since $A|F_k$ is a Banach algebra, $|h(x)| \leqq \|h\|_{F_k}$, so $\|h\|_X \leqq \|h\|_{F_k}$ for any $h \in A$, where $\|h\|_{F_k} = \sup_{x \in F_k} |h(x)|$ and $\|h\|_X = \sup_{x \in X} |h(x)|$. Therefore $F_k \supset \partial A$. If for any $i$, $F_i \not\supset \partial A$, there is an $f_i \in A$ such that $f_i(F_i)=0$ and $f_i \not\equiv 0$.

But, since $f_1 f_2 f_3 \cdots f_n \equiv 0$ on $P$, $f_1 f_2 \cdots f_n \equiv 0$ on $X$. This is a contradiction since $A$ is an integral domain.

REMARK. We see easily that the theorem is false in the case of antisymmetric algebras.

**Corollary 1. 2.** *Let $A$ be a sequentially analytic algebra on a compact metric space $X$, and let $F_0$ be a closed restriction set of $A$ for a closed set $F_0$ in $X$. Then either $F_0 \supset \partial A$ or $F_0 \sim \partial A$ is a countable set whose cluster points are in $\partial A$ (if it is an infinite set).*

Proof. Put $G_n = \{x\,;\ x \in X,\ d(x, \partial A) < \frac{1}{n}\}$, where $d(x, y)$ denotes the metric function on $X$. Then if $F_0 \sim G_n$ is an infinite set for some $n$, $F_0 \sim G_n$ is a set which determines $A$ since $A$ is a sequentially analytic algebra. Since $F_0 \sim G_n \subset F_0$ and $A|F_0$ is closed in $C(F_0)$, $F_0 \supset \partial A$ by Theorem 1. 1. (In the corollary, it is unnecessary that $A$ is an integral domain).

Let $A$ be a function algebra on a compact Hausdorff space $X$. By *the maximal ideal space* $\mathfrak{M}$ of $A$ we mean the set of all maximal ideal of $A$. $\mathfrak{M}$ can be regarded as the set of all non-zero complex homomorphisms of $A$. $\mathfrak{M}$ is a compact Hausdorff space for its weak topology and $\mathfrak{M} \supset X$. $A$ maximal ideal $M$ is said to *be a point $x$ in $X$* if $M = M_x = \{f : f(x) = 0,\ f \in A\}$. $A$ ideal $N$ in $A$ is said to be *a principal ideal* if $N = f_0 \cdot A = \{f_0 f\,;\ f \in A\}$ for an $f_0 \in A$.

Glicksberg [5] has proved the following theorem.

**Therem 1. 3.** *Let $A$ be a function algebra on a compact Hausdorff space $X$. If $F$ is a closed restriction set of $A$ for any closed $F$ in $X$, then $A = C(X)$.*

The following corollary is clear from Theorem 1. 3.

**Corollary 1. 4.** *Let $A$ be a function algebra on a compact Hausdorff space $X$, and let $F_0$ be a closed set in $X$. $F_0$ is an interpolation set of $A$ if and only if for any closed set $F \subset F_0$, $F$ is a closed restriction set of $A$.*

For any closed set $F$ containing $\partial A$, we see that $A|F$ is closed in $C(F)$. Conversely, we have (cf. [9])

**Theorem 1. 5.** *Let $A$ be an essential algebra on $X$ and let $F_0$ be a closed subset in $X$. If $F$ is a closed restriction set of $A$ for any closed subset $F$ containing $F_0$, then $F_0$ contains the Šilov boundary $\partial A$ of $A$.*

**Corollary 1. 6.** *Let $A$ be an arbitrary function algebra on $X$ and let $F_0$ be a closed subset in $X$ which is contained in the essential set $E$ of $A$. If $F$ is a closed resrtiction set of $A$ for any closed subset $F$ containig $F_0$,* *then $F_0$ contains the Šilov boundary $\partial_{A|E}$ of the function algebra $A|E$* (see the next Remark).

From Corollary 1.6 we can prove Theorem 1.3.

REMARK. (1) In Corollary 1.6 we can prove the converse, that is, for any closed set $F$ containing $\partial_{A|E}$ $F$ is a closed restriction set. For, let $F\supset\partial_{A|E}$, then $||f||_E=||f||_{\partial_{A|E}}\leqq||f||_F$ for any $f\in A$. Since $E\cup F\supset E$, we see that $E\cup F$ is a closed restriction set, that is, for any $f$ in $A$, there is a $g\in A$ such that $||g||_X\leqq\gamma||f||_{E\cup F}$ and $g=f$ on $E\cup F$. (Theorem 2.1) Therefore $||g||_X\leqq\gamma||f||_{E\cup F}\leqq\gamma||f||_F$ and $g=f$ on $F$, so $F$ is a closed restriction set by Theorem 2.1.

(2) $\partial A\cap E$ always contains $\partial_{A|E}$ (cf. [8]) and we can have an example with $\partial A\cap E\neq\partial_{A|E}$, so the conclusion $(F_0\supset\partial A\cap E)$ of Theorem 2 of [9] is false (see the above (1)).

(3) If $X$ is a compact metric space, we have that $F_0\supset P\cap E\neq\phi$ as the conclusion, under the hypothesis of Corollary 1.6. $P$ here denotes the minimal boundary of $A$ (it is the set of peak points of $A$ and is also equal to the Choquet boundary of $A$) (cf. [3]). For, it is clear since $\partial_{A|E}\supset P\cap E$.

## 2. Closed restriction sets

Let $A$ be a function algebra on a compact Hausdorff space $X$ and let $F_0$ be a closed subset in $X$. $A$ closed restriction set of $A$ is characterized as follows.

**Theorem 2.1.** *Let $A$ be a function algebra on a compact Hausdorff space $X$ and let $F_0$ be a closed subset in $X$. Then $F_0$ is a closed restriction set if and only if for any $f\in A$ there is a $g\in A$ such that $||g||_X\leqq\gamma||f||_{F_0}$ and $f=g$ on $F_0$, where $\gamma$ is a positive number which is independent of $f$.*

Proof. If $A|F_0$ is closed in $C(F_0)$, by Glicksberg ([4], P. 420), $A|F_0$ is isomorphic to $A/kF_0$ $(kF_0=\{f\in A: f(F_0)=0\})$, so the necessity of the theorem is clear. The sufficiency can also be proved easily.

After now, we consider function algebras satisfying the following properties:

$(A)$. *The function algebra $A|\partial A$ is an essential maximal algebra.*

$(B)$. *Any maximal ideal in $A$ which is not a point of $\partial A$ is always principal* (*cf.* § 1).

The main theorem of this paragraph is the following

**Theorem 2.2.** *Let $A$ be a function algebra on a compact metric space $X$ satisfying the properties $(A)$ and $(B)$, and let $F_0$ be a closed set in $X$.*

*If $F_0$ is a closed restriction set of $A$, then either $F_0 \supset \partial A$ or $F_0 \sim \partial A$ is a countable set whose cluster points are in $\partial A$ (if it is an infinite set).*

Proof. The proof is clear by Corollary 1.2 (§1) and the next lemma.

**Lemma 2.3.** *Let $A$ be a function algebra on a compact Hausdorff space $X$ satisfying the properties $(A)$ and $(B)$. Then $A$ is a sequentially analytic algebra.*

Proof. Let $F_0$ be an infinite closed set in $X \sim \partial A$ and let $f_0(F_0)=0$ for an $f_0 \in A$. Then we have to prove that $f_0 \equiv 0$. Let $x_0$ be a point in $F_0$ which is not an isolated point. Since $M_0 = \{f : f(x_0)=0,\ f \in A\}$ is a maximal ideal in $A$, by the hypothesis, $M_0 = g_0 A$ for some $g_0 \in A$. Since $f_0(x_0)=0$, $f_0 = g_0 a_1$ for an $a_1 \in A$. We see here that $x_0$ is the sole point satisfying $g_0(x_0)=0$, so $a_1(F_0 \sim (x_0))=0$, $a_1(x_0)=0$ and $a_1 \in M_0$. Therefore, $a_1 = g_0 a_2$ for an $a_2 \in A$. By repeating the same argument, we have a sequence $\{a_n\}$ of functions in $A$ such that

$$f_0 = g_0 a_1 = g_0^2 a_2 = \cdots = g_0^k a_k = \cdots \qquad \cdots\cdots\cdots (1)$$

Now, since the function $g_0$ does not vanish on $\partial A$, $g_0^{-1} | \partial A \in C(\partial A)$. But $g_0^{-1} | \partial A$ can not be extended to any function in $A$ since $g_0(x_0)=0$, that is, $g_0^{-1} | \partial A \notin A | \partial A$. Since $A | \partial A$ is a maximal algebra, the closed subalgebra spanned by $g_0^{-1} | \partial A$ and $A | \partial A$ is identical to $C(\partial A)$, so for any $h \in C(\partial A)$ and for any $\varepsilon > 0$,

$$\| h - (\alpha_0 + \alpha_1 g_0^{-1} + \cdots + \alpha_k g_0^{-k}) \|_{\partial A} < \varepsilon \qquad \cdots\cdots\cdots (2)$$

, where $\alpha_i \in A$.

By (1), $$g_0^{-k} f_0 = a_k \text{ on } \partial A \qquad (k = 1, 2, 3, \cdots) \qquad \cdots\cdots\cdots (3)$$

By (2) and (3),

$$\| h f_0 - (\alpha_0 f_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k) \|_{\partial A} \leqq \varepsilon \| f_0 \|_{\partial A}$$

Since $\alpha_0 f_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k \in A$, $h f_0 \in A | \partial A$.

$$C(\partial A) \cdot f_0 \subset A | \partial A .$$

Put $\partial A = Y$ and $B = A | Y$. Then $B$ is an essential algebra on $Y$. $C(Y) \cdot f_0 \subset B$. From this we can prove that $f_0 = 0$ on $Y$, hence $f_0 \equiv 0$ on $X$. If $f_0 \not\equiv 0$ on $Y$, $Z = \{x : x \in Y,\ f_0(x)=0\}$ is a closed subset in $Y$ and $Z \neq Y$. We can take two open sets $U$, $V$ such that

$$Z \subset V \subset \bar{V} \subset U \subset \bar{U} \subsetneqq Y .$$

Since $A | Y$ is essential, there is a function $p \in C(Y)$ such that $p(\bar{U})=0$

and that $p$ cannot be extended to any function in $A$. We put

$$h_0(x) = p(x)/f_0(x) \quad \text{if} \quad x \in Y \sim V,$$
$$= 0 \quad \text{if} \quad x \in V,$$

then $h_0 f_0 = p$. Since $h_0$ is continuous on $Y$, this is a contradiction, so $f_0 \equiv 0$ on $X$.

**Corollary 2.4.** *Let $A$ be a function algebra on a compact metric space $X$ which has the property (A) and is generated by function $f_0$. Then if $F_0$ is a closed restriction set of $A$ for a closed set $F_0$ in $X$, either $F_0 \supset \partial A$ or $F_0 \sim \partial A$ is a countable set whose cluster points are in $\partial A$ (if it is an infinite set).*

Proof. This is clear by Theorem 2.1. and next lemma.

**Lemma 2.5.** *If $A$ is generated by a function $f_0$, then any maximal ideal in $A$ which is not a point in $\partial A$ is principal, that is, $A$ satisfies the property (B).*

Proof. Let $M$ be a maximal ideal in $A$ which is not a point in $\partial A$. Then $M = \{f : \varphi_0(f) = 0\}$ for some non-zero complex homomorphism $\varphi_0$. Since $A$ is generated by a function $f_0$, for any $f \in M$ and for any $\varepsilon > 0$, there is a polynomial of $f_0$ such that

$$\|f - (a_0 + a_1 f_0 + \cdots + a_n f_0^n)\| < \varepsilon \qquad \cdots\cdots\cdots (1)$$

, where $a_i$ is a complex number.
If we put $\varphi_0(f_0) = \alpha$, $\varphi_0(f_0 - \alpha) = 0$. For the above polynomial, we set $g = a_0 + a_1 f_0 + \cdots + a_n f_0^n = (f_0 - \alpha)\psi(f_0) + \beta$, where $\psi(f_0)$ is a polynomial of $f_0$ and $\beta$ is a complex number. We easily see that $\varphi_0(g) = \beta$. By (1) we have $|\beta| < \varepsilon$. By (1) again,

$$\|f - (f_0 - \alpha)\psi(f_0)\| = \|f - g + \beta\| \leqq \|f - g\| + |\beta| < 2\varepsilon \qquad \cdots\cdots (2)$$

Now, the function $f_0$ cannot take the value $\alpha$ on $\partial A$. For, if $f_0(x_0) = \alpha$ for some $x_0 \in \partial A$, by (2) $f(x_0) = 0$ for any $f \in M$. Since $M$ is not a point in $\partial A$, this is a contradiction. Therefore, by (2) we have

$$\left\| \frac{f}{f_0 - \alpha} - \psi_n(f_0) \right\|_{\partial A} \leqq \frac{1}{n \cdot \delta}$$

, where $\delta = \min_{x \in \partial A} |f_0(x) - \alpha|$ and $\psi_n(f_0)$ is a polynomial of $f_0$ for any $n$. Since $\psi_n(f_0) \in A$, $f/f_0 - \alpha \in A|\partial A$, so $f = (f_0 - \alpha)h$ on $\partial A$ $(h \in A)$. $f = (f_0 - \alpha)h$ on $X$. This shows that $M$ is principal.

**Corollary 2. 6.** *Let $A$ be the algebra of all continuous functions on the closed unit disc (in the complex plane) which are analytic in its interior and let $F_0$ be a closed restriction set of $A$. Then $F_0$ contains $\partial A$ (=the unit circle) or $F_0$ is an interpolation set of $A$.*

Proof. Let $F_0$ be a closed restriction set of $A$. By Corollary 2. 4., $F_0$ contains the unit circle $K$ in the unit disc or $F_0 \sim K$ is a countable set whose cluster points are in $K$. Therefore, if $F_0$ does not contain $K$, $F_0 \cap K \subsetneqq K$ and $F_0 \sim K$ is a countable set whose cluster points are in $K$, so $F_0 = (F_0 \cap K) \cup (F_0 \sim K)$ does not divide the complex plane and is also a non dense set in the complex plane. It follows that $A|F_0$ is dense in $C(F_0)$ by the Lavrent'ev approximation theorem, so $A|F_0 = C(F_0)$, that is, $F_0$ is an interpolation set.

Corollary 2. 6. can be extended to the case which $A$ is a more general algebra.

## 3. Interpolation sets

Let $A_0$ be the function algebra of all continuous functions on the unit disc which are analytic in its interior. Then Hoffman ([7], P. 208) has pointed that the following two statement are equivalent for a sequence of distinct points $\{z_k\}$ in the open unit disc: (a). If $g$ is any continuous function on the closed unit disc, there exists $f \in A_0$ such that $f(z_k) = g(z_k)$, $k = 1, 2, 3, \cdots$. (b). $\{z_k\}$ is an interpolating sequence for $H^\infty$, and the set of accumulation points of $\{z_k\}$ on the unit circle has Lebesgue measure zero.

In this paragraph we consider a generalization of the above fact (Theorem 3. 2.).

Let $A$ be a function algebra on $X$. $A$ is said to be *a Dirichlet algebra* if the set of all real parts of $A$, $Re\, A$ is dense in $C_R(X)$, where $C_R(X)$ is the set of all real-valued continuous functions on $X$. In §2 we see that if a function algebra $A$ satisfies the property ($A$) and has a function $f_0$ as its generator and if $F_0$ is a closed restriction set of $A$ (and hence, if $F_0$ is an interpolation set of $A$), then either $F_0 \supset \partial A$ or $F_0 \sim \partial A$ is a sequence of points whose cluster points are in $\partial A$. Let $A$ have a function $f_0$ as its generator. Then we can assume that $A$ satisfies the following property: Let $P$ be a compact set in the complex plane having a connected complement and let $\Gamma$ be the boundary of $P$.

(*). *$A$ is generated by a function $f_0$ such that $\Gamma \subset f_0(X) \subset P$, where $f_0(X) = \{f_0(x) : x \in X\}$.*[1)]

---

1) Put $P = C \sim U_\infty$, where $C$ is the complex plane and $U_\infty$ is the connected component of $C \sim f_0(X)$ containing $\infty$. Then $P$ satisfies (*). If $A \neq C(X)$, $P^i \neq \phi$.

Let $Y=\{y_1, y_2, \cdots, y_n, \cdots\}$ be a sequence of points in $X\sim\partial A$. $Y$ is said to be *an $H^\infty$-interpolating sequence* if for any bounded sequence of complex numbers $\{\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots\}$, there is an $f\in H^\infty$ such that $f(y_i)=\alpha_i$ for any $i$, where $H^\infty$ denotes the set of all bounded function $f$ on $X\sim\partial A$ such that there is a sequence of function $\{f_i\}$ in $A$ and $f_i$ converges uniformly to $f$ on any compact set in $X\sim\partial A$. We easily see that if $X$ is a compact metric space $H^\infty$ is a Banach algebra with the norm $||f||_\infty=\sup_{x\in X\sim\partial A}|f(x)|$. We call $H^\infty$ *the $\infty$-Hardy class relative to A.*

We consider the following property for $H^\infty$:

($D$). *For any $f\in H^\infty$, there is a sequence of functions $\{f_n\}$ in $A$ such that $||f_n||\leqq\delta||f||$ and $f_n$ converges to $f$ on any compact set in $X\sim\partial A$, where $\gamma$ is independent of $f$.*

Let $A_0$ be the set of all continuous functions on the unit disc which are analytic in its interior, and let $H^\infty$ be its $\infty$-Hardy class, that is, the set of all bounded analytic functions on the open disc. For any $f\in H^\infty$, we put $f_n(z)=f\left\{\left(1-\dfrac{1}{n}\right)z\right\}$ $(n=1, 2, 3, \cdots)$. Then $f_n$ can be defined as a function in $A_0$. We easily see that $||f_n||\leqq||f||_\infty$ and $f_n$ converges to $f$ on any compact set in the open unit disc.

First, we shall prove the following theorem.

**Theorem 3.1.** *Let A be a function algebra on a compact metric space X which has the property* ($A$) *and is generated by a function $f_0$. Then if $F_0$ is an interpolation set of A for a closed set $F_0$ in X, the following conditions are satisfied:*

( $i$ ) *$F_0\cap\partial A$ is an interpolation set of $A|\partial A$.*

($ii$) *For any finite set $\{y_1, y_2, \cdots, y_n\}$ in $F_0\sim\partial A$ and for any finite set $\{c_1, c_2, \cdots, c_n\}$ of comlex numbers, there is an $f\in A$ such that $f(y_i)=c_i$ $(i=1, 2, \cdots, n)$ and $||f||\leqslant\gamma\sup_{i\leqslant n}|c_i|$, where $\gamma$ is a positive number which is independent of $\{y_1, y_2, \cdots, y_n\}$ and of $\{c_1, c_2, \cdots, c_n\}$.*

*Conversely, the conditions* ($i$) *and* ($ii$) *imply that $F_0$ is an interpolation set of A.*

The main theorem of this paragraph is the following

**Theorem 3.2.** *Let A be a function algebra on a compact metric space X satisfying the property* ($A$) *and having a generator $f_0$. Then if $F_0$ is an interpolation set of A, the following conditions are satisfied:*

( $i$ ) *$F_0\cap\partial A$ is an interpolation set of $A|\partial A$.*

($ii'$) *$F_0\sim\partial A$ is an $H^\infty$-interpolating sequence.*

*Conversely, if the $\infty$-Hardy class $H^\infty$ relative to A has the property*

*$(D)$, then the conditions $(i)$ and $(ii')$ imply that $F_0$ is an interpolation set of $A$.*

Proof of Theorem 3.1. (i) Let $f$ be any continuous function on $M=F_0\cap\partial A$, and let $f^*$ be a continuous extension of $f$ on $F_0$. Since $A|F_0=C(F_0)$, there is a function $g\in A$ such that $g=f^*$ on $F_0$, so $g=f$ on $M$.

(ii) Let $\{c_1, c_2, \cdots, c_n\}$ be a sequence of complex numbers and let $h$ be a continuous function on $F_0$ such that

$$\begin{aligned} h(y_i) &= c_i && (i=1, 2, \cdots, n) \\ h(x) &= 0 && (x\in F_0\sim\{y_1, y_2, \cdots, y_n\}). \end{aligned}$$

Since $A|F_0=C(F_0)$, there is an $f\in A$ such that $f(x)=h(x)$ on $F_0$. We here can assume that $\|f\|_X\leqq\gamma\|h\|_{F_0}$ ($\gamma$ is independent of $h$) by Theorem 2.1, so $f(y_i)=c_i$ $(i=1, 2, \cdots, n)$ and $\|f\|_X\leqq\gamma\sup_{i\leqq n}|c_i|$.

Conversely, let $A$ satisfy the conditions (i) and (ii). We will show that $A|F_0=C(F_0)$. Put $M=\partial A\cap F_0$. For any continuous function $f$ on $F_0$, $f|M\in C(M)$. By (i) there is an $f'\in A$ such that $f'=f$ on $M$. If we put $f_1=f'-f$, then $f_1(M)=0$. If we prove that $f_1=g$ on $F_0$ for a function $g\in A$, then $f=h$ on $F_0$ for some $h\in A$, so the theorem will be proved. Therefore, for any $f_1\in C(F_0)$, $f_1(M)=0$, we are only to prove that $f_1=g$ on $F_0$ for some $g\in A$. We can assume that $\|f_1\|_{F_0}=1$. By Theorem 2.2 we put $F_0\sim\partial A=\{y_1, y_2, y_3, \cdots\}$. Since $f_1(M)=0$, there is a positive integer $n_1$ such that

$$\{y_i \,|\, |f_1(y_i)|\geqq 1/4\}\subset\{y_1, y_2, y_3, \cdots, y_{n_1}\}.$$

Since $A|\partial A$ is a maximal essential algebra, $M\neq\partial A$. And since $M$ is an interpolation set of $A|\partial A$, there is a function $\psi\in A|\partial A$ such that $\psi(M)=1$, $\psi\not\equiv 1$ and $\|\psi\|_{\partial A}=1$.[2] (cf. [4]). Since $y_i\notin\partial A$, by ([2] or [6], §5), $|\psi(y_i)|<1$ $(i=1, 2, 3, \cdots)$. By taking a sufficiently large integer $m$, the value of $1-\psi^m$ on $y_i$ $(i=1, 2, \cdots, n_1)$ can be arbitrarily near 1. Also, since $\psi(M)=1$, there is a positive integer $n_2$ $(n_2\geqq n_1)$ such that $\{y_i : |(1-\psi^m)(y_i)|<1/4\gamma\}\subset\{y_{n_2+1}, y_{n_2+2}, \cdots\}$, where $\gamma$ is that in the condition (ii). By (ii) there is a function $p\in A$, such that $p(y_i)=f_1(y_i)$ $(i=1, 2, \cdots, n_1)$, $p(y_i)=0$ $(i=n_1+1, \cdots, n_2)$ and $\|p\|\leqq\gamma\|f_1\|_{F_0}=\gamma$. For a sufficiently large integer $m$,

$$\begin{aligned} &|(1-\psi^m)(y_i)\cdot p(y_i)-f_1(y_i)|<1/2 && (i=1, 2, \cdots, n_1) \\ &|(1-\psi^m)(y_i)\cdot p(y_i)|<1/4 && (i=n_1+1,\ n_1+2, \cdots), \\ \text{so}\quad &|(1-\psi^m)(y_i)p(y_i)-f_1(y_i)|<1/2 && (i=n_1+1,\ n_1+2, \cdots). \end{aligned}$$

2) Since $f_0(M)\subset\Gamma$ (Lemma 3.3. Footnote) we can find the function $\psi$ by the similar method as the proof of Lemma 3.3.

Put $(1-\psi^m)p=g_1$, then $g_1\in A$, $\|f_1-g_1\|_{F_0}<1/2$ and $\|g_1\|_X\leqq 2\gamma$. If we $g_1^*=f_1-g_1$, $g_1^*(M)=0$ and $\|g_1^*\|_{F_0}<1/2$. By repeating the same argument, we have a sequence $\{g_n\}\subset A$ such that $\|g_n\|_X<2^{-(n-2)}\cdot\gamma$ and $g_n^*=f_1-g_1-g_2\cdots-g_n$ satisfies that $g_n^*(M)=0$ and $\|g_n^*\|_{F_0}<2^{-n}$ for any $n$. Put $h_n=g_1+g_2+\cdots+g_n$, then $h_n\in A$. If $m>n$, $\|h_m-h_n\|_X=\|g_m+\cdots+g_{n+1}\|_X\leqq\|g_m\|_X+\cdots+\|g_{n+1}\|_X\leqq 2^{-(n-2)}\gamma$, so $h_n$ converges to some $h\in A$, and $\|f_1-h_n\|_{F_0}=\|g_n^*\|_{F_0}<2^{-n}$. This shows that $f_1=h$ on $F_0$ for some $h\in A$.

Before the proof of Theorem 3.2 we need the following lemmas.

**Lemma. 3. 3.** *Let $A$ be a function algebra having a generator $f_0$ and let a sequence of points $\{y_n\}\subset X\sim\partial A$ converges to point $y$ in $X\sim\partial A$. Then*

$$\rho(y_n, y)=\sup_{\substack{f\in A\\ f\neq 0}}|f(y_n)-f(y)|/\|f\| \quad \textit{converges to } 0.$$

Proof. Since $A$ is generated by $f_0$, the set of all polynomials $a_0+a_1f_0+\cdots+a_mf_0^m$ ($a_i$ is a complex number) is dense in $A$. For any polynomial $g$ of $f_0$, $g=a_0+a_1f_0+\cdots+a_mf_0^m$, we put $g'(z)=a_0+a_1z+\cdots+a_mz^m$. We consider the polynomial $g'(z)$ as function on $P$. Put $\|g'\|=\sup_{z\in P}|g'(z)|$. Then there is a complex number $z_0$ ($z_0\in\Gamma$) such that $\|g'\|=|g'(z_0)|$. By the property (*), there is a point $x_0\in X$ such that $z_0=f_0(x_0)$, so $\|g'\|=|a_0+a_1f_0(x_0)+a_2f_0^2(x_0)+\cdots+a_mf_0^m(x_0)|\leqq\|g\|$. Therefore,

$$|g(y_n)-g(y)|/\|g\|\leqq|g'[f_0(y_n)]-g'[f_0(y)]|/\|g'\| \quad\cdots\cdots\cdots(1)$$

Now, we easily see that $f_0(x)\notin\Gamma$ if $x\notin\partial A$. For, let $f_0(x)\in\Gamma$ for a point $x\notin\partial A$. If $f_0(x)=f_0(x')$ for another point $x'$, then $f(x)=f(x')$ for any $f\in A$, since $A$ is generated by $f_0$. This contradiction shows that there exists no point $x'$ (different from $x$) such that $f_0(x)=f_0(x')$. If we put $u_0=f_0(x)$, then $u_0\in\Gamma$. The function algebra of continuous functions of $\Gamma$ which admit a continuous extension to $P$ that is analytic on the interior of $P$ is a Dirichlet algebra ([10]) and the one point set $u_0$ is a closed restriction set. Therefore $\{u_0\}$ is peak set ([4]), so there is a continuous function $\psi$ on $P$ which is analytic on the interior of $P$ such that $\psi(u_0)=1$ and $|\psi(u)|<1$ for any $u\in P$ ($u_0\neq u$). Put $h=\psi\circ f_0$. Then $h\ni A$, since $\psi$ is approximated uniformly on $P$ by polynomials of $z$. We see that $\|h\|=1$ and $x$ is the sole point satisfying $|h(x)|=1$. Since $x\notin\partial A$, this is a contradiction, so $f_0(x)\notin\Gamma$ if $x\notin\partial A$.[3] Coming back argument, let $y_n$, $y\notin\partial A$ and $y_n\to y$. Then $f_0(y_n)\notin\Gamma$ for any $n$, $f_0(y)\notin\Gamma$ and $f_0(y_n)\to f_0(y)$. Since $g'(z)$ is analytic in the interion of $P$,

3) We can prove that $f_0(\partial A)=\Gamma$. For, $f_0$ is a homeomorphism of $X$ onto $f_0(X)$ and $\Gamma\subset f_0(X)\subset P$.

$$|g'[f_0(y_n)]-g'[f_0(y)]|/\|g'\| \leqq M|f_0(y_n)-f_0(y)| \qquad \cdots\cdots\cdots (2)$$

, where $M$ is a constant number which is independent of $n$. By (1), (2),

$$|g(y_n)-g(y)|/\|g\| \leqq M\eta_n, \qquad \text{where} \quad \eta_n \to 0 \text{ for } n \to \infty .$$

Since the set of all such functions $g$ is dense in $A$, $\rho(y_n, y) \to 0$.

This lemma implies the following

**Lemma 3.4.** *Let $A$ be a function algebra on a compact metric space $X$ having a generator $f_0$. Then any equibounded sequence of functions in $A$ is equicontinuous on any compact subset $K$ in $X \sim \partial A$.*

Proof of Theorem 3.2. In order to prove (ii′), let $\{c_1, c_2, \cdots, c_n, \cdots\}$ be a bounded sequence of complex numbers. If $F_0 \sim \partial A = \{y_1, y_2, \cdots, y_n, \cdots\}$, by Theorem 3.1 (ii), there is a $g_n \in A$ for any $n$ such that $g_n(y_i) = c_i$ $(i=1, 2, \cdots, n)$ and $\|g_n\|_X \leqq \gamma \sup_i |c_i|$, so $\{g_n\}$ is equibounded. By Lemma 3.4 $\{g_n\}$ is equicontinuous on any compact subset in $X \sim \partial A$. Therefore, by the diagonal argument, there is a subsequence $\{g_{n_i}\}$ of $\{g_n\}$ such that $g_{n_i}$ converges uniformly to some $h$ on any compact subset in $X \sim \partial A$. By definition, $h \in H^\infty$ and $h(y_i) = c_i$ $(1, 2, 3, \cdots)$. Conversely, let $H^\infty$ have the property (D). Then since $H^\infty$ is a Banach algebra and since the sequence of points $\{y_1, y_2, \cdots, y_n, \cdots\}$ $(=F_0 \sim \partial A)$ is an $H^\infty$-interpolating sequence, for any bounded function $f$ on $\{y_1, y_2, \cdots, y_n, \cdots\}$ there is an $h \in H^\infty$ such that $f(y_i) = h(y_i)$ $(i=1, 2, 3, \cdots)$ and $\|h\|_\infty \leqq \gamma \sup |f(y_i)|$ ($\gamma$ is independent of $f$) by the same argument as [7] (P. 196). Therefore, we can prove that $F_0$ is an interpolation set of $A$ by the similar method as Theorem 3.1, since $H^\infty$ has the property (D).

TOKYO JOSHI DAIGAKU

---

**References**

[1] S. B. Bear: *Complex function algebras,* Trans. Amer. Math. Soc. **90** (1959), 383–393.

[2] ————: *A strong maximum modulus theorem for maximal function algebras,* Trans. Amer. Math. Soc. **92** (1959), 365–469.

[3] E. Bishop: *A minimal boundary for function algebras,* Pacific J. Math. **9** (1959), 629–642.

[4] I. Glicksberg: *Measures orthogonal to algebras and sets of antisymmetry,* Trans. Amer. Math. Soc. **105** (1962), 415–435.

[5] ————: *Function algebras with closed restrictions,* Proc. Amer. Math. Soc. **14** (1963), 158–161.

[6] K. Hoffman and I. M. Singer: *Maximal algebras of continuous functions*, Acta Math. **103** (1960), 218–241.
[7] K. Hoffman: Banach spaces of analytic functions, Prentice-Hall, 1962.
[8] J. Tomiyama: *Some remarks on antisymmetric decompositions of function algebras*, to appear, Tôhoku Math. J. **16** No. 4, (1964).
[9] J. Wada: *On the Šilov boundaries of function algebras*, Proc. Japan Acad. **39** (1963), 425–428.
[10] J. Wermer: *Dirichlet algebras*, Duke Math. J. **27** (1960), 373–382.

*Added in proof.* M. Hasumi also proved Theorem 3.2. without the property (A) by use of the maximal ideal space of $A$. Some theorems of this paper can be extended to the case which $A$ is a more general algebra.