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EXISTENCE OF BASE-POINT-FREE PENCILS OF DEGREE $g-1$ ON BI-ELLIPTIC CURVES

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1. Introduction

It has been known that a smooth complex projective irreducible algebraic curve X of genus $g \geq 4$ has a base-point-free and complete pencil of degree $g-1$ unless X is hyperelliptic. One way of proving this result is to reduce the problem into a few special cases and then check the validity of the statement for the following three classes of curves:
(i) X is trigonal.
(ii) X is a smooth plane quintic.
(iii) X is a bi-elliptic curve, i.e. X is a double cover of an elliptic curve E.
As it turned out, the cases (i) and (ii) were relatively easy to handle; cf. [6, Beispiel 3]. However, for the case of bi-elliptic curves, some of the proofs which appeared in the literature does not seem to be complete, which has been already pointed out in [4]. For example, in the proof of [5, Theorem 5] the author obtained a plane model of a bi-elliptic curve of degree $g+1$ with a singular point s of certain high multiplicity. He then proceeded to exhibit the existence of another singular point by using a well-known formula for the geometric genus of a singular plane curve. Unfortunately, the singular point different from s could be infinitely singular points lying over s. Therefore the projection method used in [5] to obtain a complete and base-point-free pencil of degree $g-1$ which is cut out by lines through the other singular point does not work well if the singular point s of high multiplicity is not an ordinary singular point. Incidentally, the same objection applies to the proof of Shokurov [7]. A proof due to J. Harris, which was sketched in [2, Chapter VIII; Exercise D and F], seems to be the only complete proof without a gap which appeared in the literature as far as the author knows. On the other hand, the proof of Harris uses the so-called enumerative method as well as several advanced results in Brill-Noether theory and hence one needs a quite a bit of heavy duty machinery for a proof of this seemingly simple fact; indeed the proof in [2] shows the reducibility of $W^1_{g-1}(C)$, which is a much harder problem and the existence of the base-point-free pencil follows as a corollary.

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The purpose of this paper is to provide a simple and easy geometric proof of the following theorem of J. Harris only using elementary tools using an idea from [4, Appendix].

Theorem 1 (J. Harris [2]). *Let X be a bi-elliptic curve of genus $g \geq 6$. Then X has a base-point-free and complete pencil g^1_{g-1}.*

Using a similar method, we also provide a relatively simpler proof of the fact that there exists a complete base-point-free pencil g^1_{g-2} on a curve X of genus g which is a double cover of a curve of genus 2 if $g \geq 11$. It should be remarked that the result has been known already; cf. [4, Appendix] for $g \geq 13$ and [3] for $g \geq 11$. Note that the proof in [3] uses enumerative method whereas the proof in [4, Appendix] uses only simple and geometric arguments.

Therefore our main purpose is to assure the readers that geometric arguments can be pushed forward even beyond the range of $g \geq 13$ in [4] so that one gets the same genus bound as in [3].

We use standard notation for divisors, linear series and line bundles on algebraic curves following [2]. As usual, g^r_d is an r-dimensional linear series of degree d on X, which may be possibly incomplete. If D is a divisor on X, we write $|D|$ for the associated complete linear series on X. By K we denote a canonical divisor on X, and $|K|$ is the canonical linear series on X. A base-point-free g^r_d on X defines a morphism $f: X \longrightarrow \mathbb{P}^r$ onto a non-degenerate irreducible (possibly singular) curve in $\mathbb{P}^r$. We close this section by recalling the following well-known result which will be used in the next section.

Proposition 1 ([2, Chapter III-Exercise F]). *Let L be a line bundle of degree $d \geq 2g+2$ on a smooth curve X of genus g. Let*

$$\varphi_L : X \to \mathbb{P}^{d-g}$$

be the embedding induced by L. Then $\varphi_L(X)$ is the intersection of quadrics.

2. Proof of Theorem 1

Let $\pi: X \longrightarrow E$ be the two sheeted map onto an elliptic curve E; note that such π is unique (up to isomorphism) by Castelnuovo-Severi inequality [1, Thorem 3.5]. We break up the proof into several steps as follows.

STEP 1. The canonical image of X lies on a cone of degree $g-1$.

Since X is non-hyperelliptic, we may identify X with its canonical image $\varphi_K(X)$ in $\mathbb{P}^{g-1}$.

For $r_i \in E$, let $\pi^*(r_i) = p_i + \overline{p}_i\,; i = 1, 2$. Then for the effective divisor

$$D = \pi^*(r_1 + r_2) = p_1 + \overline{p}_1 + p_2 + \overline{p}_2 \in g^1_4 = \pi^*(g^1_2) = \pi^*(|r_1 + r_2|),$$

$\dim \overline{D} = 2$ by the geometric version of Riemann-Roch theorem; i.e. D spans a 2-plane in $\mathbb{P}^{g-1}$. Therefore for any $r, r' \in E$, the two lines spanned by $\pi^*(r)$ and $\pi^*(r')$ must intersect. Since X is non-degenerate in $\mathbb{P}^{g-1}$, all the lines spanned by $\pi^*(r)$, $r \in E$ pass through a point $v \in \mathbb{P}^{g-1}$. Let

$$S_{g-1} = \bigcup_{r \in E} \overline{\pi^*(r)},$$

which is a cone with vertex v containing the canonical image of X. Furthermore, one sees easily that $v \notin \varphi_K(X)$; if $v \in \varphi_K(X)$, then the divisor $v + \pi^*(r_1)$ with $\pi(v) \neq r_1$ is a trisecant line hence $v + \pi^*(r_1)$ moves in a pencil which is contradictory to the Castelnuovo-Severi inequality.

Let $H \simeq \mathbb{P}^{g-2}$ be a hyperplane in $\mathbb{P}^{g-1}$ not passing through v and φ be the projection away from v to H. By our construction, E is isomorphic to the hyperplane section $H \cap S_{g-1}$, which we use the same symbol E for simplicity. A hyperplane section $H_E = E \cap \mathbb{P}^{g-3} \subset H = \mathbb{P}^{g-2}$ of E is the image under φ of the intersection $\varphi_K(X) \cap \langle H_E, v\rangle$, where $\langle H_E, v\rangle$ is the hyperplane in $\mathbb{P}^{g-1}$ spanned by H_E and v. Since the projection φ is indeed the degree two morphism $\pi\colon X \longrightarrow E$,

$$\deg E = \deg(H_E) = \frac{1}{2}\deg(\varphi_K(X) \cap \langle H_E, v\rangle) = \frac{1}{2}(2g - 2) = g - 1,$$

and hence $\deg S_{g-1} = g - 1$.

STEP 2. There is a sequence of birational maps $\{\varphi_i\}_{0 \le i \le g-4}$ with the following properties.
(1) $\varphi_0 = \varphi_K\colon X \longrightarrow \mathbb{P}^{g-1}$ is the canonical map of X.
(2) For $1 \le i \le g - 4$, $\varphi_i\colon \mathbb{P}^{g-i} \dashrightarrow \mathbb{P}^{g-1-i}$ is a projection away from a point and restricted to $(\varphi_{i-1} \circ \cdots \circ \varphi_0)(X)$, $\varphi_i\colon (\varphi_{i-1} \circ \cdots \circ \varphi_0)(X) \dashrightarrow \mathbb{P}^{g-1-i}$ is still birational onto its image.
(3) $(\varphi_i \circ \cdots \circ \varphi_0)(X)$ has only one singular point for every $2 \le i \le g - 4$.
(4) $(\varphi_{g-4} \circ \cdots \circ \varphi_0)(X)$ lies on a cubic cone in $\mathbb{P}^3$.

Choose a point $p_1 \in X_{g-1} := \varphi_0(X)$ and let φ_1 be the projection away from p_1 onto $\mathbb{P}^{g-2}$. Let $q_{g-1} := \overline{p}_1$ be the conjugate point of p_1 with respect to π and take $X_{g-2} := \varphi_1(X_{g-1})$. The image of S_{g-1} under the projection φ_1 is also a cone of degree $g - 2$ with vertex $q_{g-2} := \varphi_1(q_{g-1}) = \varphi_1(v)$, which is denoted by S_{g-2}. Now we take a general point p_2 in X_{g-2} and let φ_2 be the projection away from p_2 onto $\mathbb{P}^{g-3}$. Applying this process repeatedly, we can obtain $\{(\varphi_i, S_{g-1-i}, X_{g-1-i}, p_i, q_{g-1-i})\colon i =$

$1, \dots, g-4\}$ as follows;

$$\begin{array}{ccccccc}
 & \varphi_1 & & \varphi_2 & & \varphi_{g-4} & \\
\mathbb{P}^{g-1} & \dashrightarrow & \mathbb{P}^{g-2} & \dashrightarrow & \cdots & \dashrightarrow & \mathbb{P}^3 \\
\cup & & \cup & & & & \cup \\
S_{g-1} & \dashrightarrow & S_{g-2} & \dashrightarrow & \cdots & \dashrightarrow & S_3 \\
\cup & & \cup & & & & \cup \\
X_{g-1} & \dashrightarrow & X_{g-2} & \dashrightarrow & \cdots & \dashrightarrow & X_3
\end{array}$$

where φ_i is the projection away from a general point $p_i \in X_{g-i}$ onto a hyperplane, $q_{g-2-i} = \varphi_i(q_{g-1-i})$, $S_{g-2-i} = \varphi_i(S_{g-1-i})$ and $X_{g-2-i} = \varphi_i(X_{g-1-i})$. Note that S_{g-1-i} is a cone with vertex q_{g-1-i} of degree $g-1-i$, X_{g-1-i} is a curve of degree $2g-2-i$ and $\operatorname{mult}_{q_{g-1-i}} X_{g-1-i} = i$; X_{g-1-i} is the image of the morphism induced by $|K - p_1 - \cdots - p_i|$, $\dim |K - p_1 - \cdots - p_i - \overline{p}_1 - \cdots - \overline{p}_i| = \dim |K - p_1 - \cdots - p_i| - 1$ and $|K - p_1 - \cdots - p_i - \overline{p}_1 - \cdots - \overline{p}_i|$ is base-point-free. In particular the image of X_4 in $\mathbb{P}^3$ under φ_{g-4} lies on a cubic cone S_3.

Let $E_k := S_k \cap H$ where $H \cong \mathbb{P}^{k-1}$ is a hyperplane not passing through the vertex q_k of $S_k \subset \mathbb{P}^k$. Since S_{g-1} is a cone over the elliptic curve $E \subset \mathbb{P}^{g-2}$ of degree $g-1$ and S_k is obtained by successive projections, we easily see that $\deg E_k = \deg S_k = k$ and $E_k \cong E$, i.e. $g(E_k) = 1$. Applying Proposition 1 to the hyperplane bundle on $E_k \subset \mathbb{P}^{k-1}$, E_k is cut out by quadrics in H and hence S_k is also cut out by quadrics in $\mathbb{P}^k$ for $k \geq 4$.

Note that, for $k \geq 3$, any singular point of X_k different from q_k may only arise from a trisecant line of $X_{k+1} \subset S_{k+1} \subset \mathbb{P}^{k+1}$ other than rulings of the cone S_{k+1}. Since S_{k+1} is cut out by quadrics for $k \geq 3$, we see that there is no trisecant line of X_{k+1} other than rulings of S_{k+1}. Therefore X_k has no singular point other than q_k for $k = 3, \dots, g-3$.

STEP 3. X is birational to a plane curve $X_2 \subset \mathbb{P}^2$ of degree $g+1$ with ordinary singular point of multiplicity $g-3$.

The projection away from a general point $p_{g-3} \in X_3$, denoted by φ_{g-3}, gives a birational map from X_3 onto $X_2 := \varphi_{g-3}(X_3)$ in $\mathbb{P}^2$. Note that $\deg \varphi_{g-3}(X_3) = \deg X_3 - 1 = g+1$ and the point $q_2 := \varphi_{g-3}(q_3)$ is singular point with multiplicity $g-3$. We observe that q_2 being an ordinary singular point is equivalent to

$$(1) \qquad |K - p_1 - \cdots - p_{g-3} - \overline{p}_1 - \cdots - \overline{p}_{g-3} - \overline{p_i} - \overline{p_j}| = \emptyset$$

for all distinct $i, j \in \{1, 2, \dots, g-3\}$. Therefore in order to show that q_2 is an ordinary singular point, we need to choose the points $p_1, \dots, p_{g-4} \in X$ properly in Step 2 as well as p_{g-3} which satisfy the condition (1). We now set

$$T_{ij} := \{(p_1, \dots, p_{g-3}) \in X^{g-3} \colon \dim |K - p_1 - \cdots - p_{g-3} - \overline{p_1} - \cdots - \overline{p}_{g-3} - \overline{p}_i - \overline{p}_j| \geq 0\}$$

for distinct $i,\ j \in \{1, 2, \ldots, g-3\}$ and $T := \bigcup T_{ij}$. Since T_{ij} is closed in the $(g-3)$-fold product X^{g-3}, so is T. Therefore it is sufficient to show that each T_{ij} is a proper closed subset in X^{g-3}; then any $(p_1, \ldots, p_{g-3}) \in X^{g-3} \backslash T$ satisfies the condition (1). Accordingly, without loss of generality, we assume $(i, j) = (1, 2)$ and proceed as follows.

CLAIM. For distinct $p_1,\ p_2 \in X$ such that $\pi(p_1) \neq \pi(p_2)$, $|p_1 + 2\overline{p}_1 + p_2 + 2\overline{p}_2| = g^1_6$.

For this we consider $|2p_1 + 2p_2 + 2\overline{p}_1 + 2\overline{p}_2|$ and let $\pi(p_i) = r_i \in E$, $i = 1, 2$. Since X cannot be hyperelliptic $\dim|2p_1 + 2p_2 + 2\overline{p}_1 + 2\overline{p}_2| = \dim|\pi^*(2r_1 + 2r_2)| = 3$ by Clifford's theorem. Note that the linear series $|2p_1 + 2p_2 + 2\overline{p}_1 + 2\overline{p}_2|$ induces the double covering $\pi\colon X \longrightarrow E$. Therefore, $\dim|p_1 + 2\overline{p}_1 + p_2 + 2\overline{p}_2| = \dim|2p_1 + 2p_2 + 2\overline{p}_1 + 2\overline{p}_2| - 2 = 1$ since $\pi(p_1) \neq \pi(p_2)$ and this finishes the proof of the claim.

By the claim, $\dim|K - p_1 - p_2 - 2\overline{p}_1 - 2\overline{p}_2| = g - 6$ and therefore we may choose general points $p_3, \ldots, p_{g-4} \in X$ so that $\dim|K - p_1 - p_2 - 2\overline{p}_1 - 2\overline{p}_2 - p_3 - \cdots - p_{g-4}| = 0$. Finally we take a point $p_{g-3} \in X$ such that $p_{g-3} \notin |K - p_1 - p_2 - 2\overline{p}_1 - 2\overline{p}_2 - p_3 - \cdots - p_{g-4}|$ and p_{g-3} is not a conjugate point of p_i for any $i = 1, \ldots, g-4$. Then $(p_1, \ldots, p_{g-3}) \notin T_{12}$ and this shows that T_{12} is a proper closed subset of X^{g-3}.

STEP 4. The plane curve X_2 constructed in Step 3 has another singular point with multiplicity 2.

Since q_2 is a singular point of multiplicity $g-3$, we have

$$g \leq p_a(X_2) - \frac{(g-3)(g-4)}{2} = \frac{g(g-1)}{2} - \frac{(g-3)(g-4)}{2} = 3g - 6.$$

Note that $g < 3g - 6$ for $g \geq 6$. Since q_2 is an ordinary singular point, it follows that there exist another singular point, say $q_0 \in X_2$ besides q_2. Suppose that $\operatorname{mult}_{q_0} X_2 \geq 3$. Recall that X_k has only one singular point q_k for every $k = 3, \ldots, g-3$. Therefore the singular point $q_0 \in X_2$ with $\operatorname{mult}_{q_0} X_2 \geq 3$ arises from at least a 4-secant line passing through p_{g-3} other than ruling of the cone S_3. Since S_3 is a cubic cone, this is impossible. Therefore we have $\operatorname{mult}_{q_0} X_2 = 2$ and the pencil of lines through q_0 cuts out base-point-free and complete g^1_{g-1} on X. $\square$

3. Double covering of a curve of genus 2

We will provide a simpler proof of a result in [3] by using a similar argument we used in the previous section. This will also improve the genus bound in [4, Appendix] ($g \geq 11$ compared with the bound $g \geq 13$ in [4]). The proof given in [4, Appendix] consists of two parts. In the first part it is shown that there exists a plane model of degree g with a singular point s of multiplicity $g-6$, where everything works well even with the assumption $g \geq 11$. In the second part it is shown that s is an ordinary singularity and the restricted assumption $g \geq 13$ is required when monodromy argument is used. Accordingly, we only need to argue that s is still an ordinary singularity under a slightly wider range $g \geq 11$.

Theorem 2. *Let X be a double cover of a genus-2-curve C of genus $g \geq 11$. Then X has a complete and base-point-free pencil g^1_{g-2} of degree $g-2$.*

Proof. Let $f\colon X \to C$ be the double covering over a curve C of genus 2. We note that such a covering is unique by the Castelnuovo-Severi inequality and the assumption $g \geq 11$. We briefly recall several facts which were already shown in the first part of the proof in [4, Appendix]. The series $|K - g^1_4|$ is very ample for the unique $g^1_4 = f^*(|K_C|) = f^*(g^1_2)$. For a general choice of $p_1, \dots, p_{g-6} \in X$, the series $|K - g^1_4 - p_1 - \cdots - p_{g-6}|$ induces a singular plane model Γ of X of degree g. Denoting the conjugate points of $p_1, \dots, p_{g-6}$ by $\overline{p}_1, \dots, \overline{p}_{g-6}$, the series $|K - g^1_4 - p_1 - \cdots - p_{g-6} - \overline{p}_1 - \cdots - \overline{p}_{g-6}|$ is a base-point-free g^1_6 and hence there is a singularity $s \in \Gamma$ with multiplicity $g-6$. To show that s is an ordinary singularity, it is enough to prove that

$$(1) \qquad |K - g^1_4 - p_1 - \cdots - p_{g-6} - \overline{p}_1 - \cdots - \overline{p}_{g-6} - \overline{p}_i - \overline{p}_j| = \emptyset$$

for $1 \leq i < j \leq g-6$.

Keeping these in mind, we now proceed as follows.
We let $T_{ij} := \{(p_1, \dots, p_{g-6}) \in X^{g-6} : \dim|K - g^1_4 - p_1 - \cdots - p_{g-6} - \overline{p}_1 - \cdots - \overline{p}_{g-6} - \overline{p}_i - \overline{p}_j| \geq 0\} \subset X^{g-6}$ for distinct i, j and $T := \bigcup T_{ij}$. Since T_{ij} is closed in the $(g-6)$-fold product X^{g-6}, so is T. Therefore it is enough to show that each T_{ij} is a proper closed subset in X^{g-6}; then any $(p_1, \dots, p_{g-6}) \in X^{g-6} \backslash T$ satisfies the condition (1). Accordingly, without loss of generality, we assume $(i, j) = (1, 2)$.

CLAIM. For any p_1 and $p_2 \in X$ with $f(p_1) \neq f(p_2)$, $|g^1_4 + p_1 + 2\overline{p}_1 + p_2 + 2\overline{p}_2| = g^2_{10}$.

To demonstrate the validity of the claim, we recall the well-known Riemann-Hurwitz relation for double coverings. Let C be a curve of genus h and let $f\colon X \to C$ be a double covering. Let $R \subset C$ be a branch locus of f. Then we have

$$(2) \qquad f_*(\mathcal{O}_X) \cong \mathcal{O}_C \oplus \mathcal{S} \quad \text{and} \quad \mathcal{S}^{\otimes 2} \cong \mathcal{O}_C(-R).$$

In our case, $h = 2$ and $\deg \mathcal{S} = 3 - g \leq -8$. Let $\pi(p_i) = \pi(\overline{p}_i) = r_i \in C$, $i = 1, 2$ and we consider $\mathcal{O}_X(g^1_4 + 2p_1 + 2p_2 + 2\overline{p}_1 + 2\overline{p}_2)$. By (2) and the projection formula, we have

$$\begin{aligned} h^0(X, \mathcal{O}_X(g^1_4 + 2p_1 + 2p_2 + 2\overline{p}_1 + 2\overline{p}_2)) &= h^0(X, \mathcal{O}_X(f^*(g^1_2 + 2r_1 + 2r_2))) \\ &= h^0(C, f_*\mathcal{O}_X(f^*(g^1_2 + 2r_1 + 2r_2))) \\ &= h^0(C, \mathcal{O}_C(g^1_2 + 2r_1 + 2r_2)) + h^0(C, \mathcal{O}_C(g^1_2 + 2r_1 + 2r_2) \otimes \mathcal{S}) = 5. \end{aligned}$$

Note that the linear series $|g^1_4 + 2p_1 + 2p_2 + 2\overline{p}_1 + 2\overline{p}_2|$ induces the double covering

$f : X \longrightarrow C$. Therefore,

$$\dim |g_4^1 + p_1 + 2\overline{p}_1 + p_2 + 2\overline{p}_2| = \dim |g_4^1 + 2p_1 + 2p_2 + 2\overline{p}_1 + 2\overline{p}_2| - 2 = 2$$

since $f(p_1) \neq f(p_2)$ and this finishes the proof of the claim.

By the claim, $|K - g_4^1 - p_1 - p_2 - 2\overline{p}_1 - 2\overline{p}_2| = g_{2g-12}^{g-9}$ and hence we can choose $p_3, \dots, p_{g-7} \in X$ such that $\dim |K - g_4^1 - p_1 - 2\overline{p}_1 - p_2 - 2\overline{p}_2 - p_3 - \cdots - p_{g-7}| = 0$. Finally, we take a point $p_{g-6} \in X$ such that $p_{g-6} \notin |K - g_4^1 - p_1 - 2\overline{p}_1 - p_2 - 2\overline{p}_2 - p_3 - \cdots - p_{g-7}|$ and p_{g-6} is not conjugate to p_i, for any $i = 1, \dots, g-7$. Therefore $(p_1, \dots, p_{g-6}) \notin T_{12}$ and it follows that T_{12} is proper and closed in X^{g-6}. $\square$

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