ON THE METRICAL THEOREMS OF CLUSTER SETS OF MEROMORPHIC FUNCTIONS

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1. Introduction. Recently the important contributions to the cluster sets theory of the meromorphic functions in the unit-disc have been done by many authors. For its recent development, we refer to K. Noshiro [10]. Roughly speaking, these studies can be divided into two classes; the first one is topological, and the second one is metrical. As far as the author knows, there exist very few results on the metrical theorems on cluster sets of functions meromophic in an arbitrary connected domain, except for the case that its boundary is of logarithmic capacity zero. (K. Noshiro [10] pp. 5-31).

The object of this note is to supply this gap. Our method is based upon the systematic use of both the hyperbolic distance and the normal family in P. Montel's sense. In the case of the unit-disc, this method has been effectively employed by K. Noshiro [11], Lehto-Virtanen [7], [8], Bagemihl-Seidel [2], [3], [15] and C. M. Faust [6] (K. Noshiro [10] pp. 86-87).

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2. Notations and definitions. In the sequel, we use the following notations.

(1) Let $w=f(z)$ be uniform and meromorphic in the connected domain D, whose boundary Γ has *at least three points.*

(2) Let S be the sequence of points $\{z_n\}$ $(z_n \in D)$ tending to the fixed boundary point z_0.

(3) Since Γ has at least three points, we can introduce the element of

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length in the hyperbolic metric[1] of D : $d\sigma_z$ $(z \in D)$ ([12] p. 49). Let $d(z_1, z_2)$ be the hyperbolic distance between two points z_i $(i=1, 2)$ in D:

$$d(z_1, z_2) = \min \int_{z_1}^{z_2} d\sigma_z.$$

(4) $C(a, r)$ is the hyperbolic circle with the centre a and hyperbolic radius r:

$$C(a, r) = E\{z ;\ d(a, z) < r\}.$$

(5) We write $D(S, r)$ for the union of the hyperbolic circles with centres on the sequence S and hyperbolic radius r:

$$D(S, r) = \bigcup_n C(z_n, r).$$

(6) We define $C(r)$ as follows:

$$C(r) = \varlimsup_{\substack{z \to z_0 \\ z \in D(S, r)}} ds_z / d\sigma_z,$$

where ds_z is the spherical element of length at $z \in D$, i.e.

$$|f'(z)|/(1+|f(z)|^2) \cdot |dz|$$

and $d\sigma_z$ is the hyperbolic element of length at $z \in D$. Now we introduce some definitions.

DEFINITION 1. *Since $C(r)$ is the non-decreasing function of r, we can define the normalcy radius $r(S)$ along the sequence S as follows:*

(i) *If $C(r) < +\infty$*[2] *for $0 < r < +\infty$, then we put $r(S) = +\infty$.*

(ii) *If $C(r) < +\infty$ for $0 < r < r^*(< +\infty)$, and $C(r) = +\infty$ for $r^* < r$, then we put $r(S) = r^*$.*

(iii) *If $C(r) = +\infty$ for $0 < r$, then we put $r(S) = 0$.*

If $r(S) = +\infty$ or 0, then we call S the normal or singular sequence respectively.

As usual, we associate with z_0 on Γ the following sets. ([10] pp. 1-2)

DEFINITION 2.

(1) *The cluster set $C_D(f, z_0)$ is defined by*

$$C_D(f, z_0) = \bigcap_{r>0} \overline{\mathscr{D}_r}.$$

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1) For the interesting applications of hyperbolic metric to the value-distribution theory, we refer to C. Constantinescu [5].

2) It does not mean that $C(r)$ is uniformly bounded.

where $\mathscr{D}_r$ *is the set of values of* $w=f(z)$ *in* $D\cap(|z-z_0|<r)$, *and* $\overline{\mathscr{D}_r}$ *is the closure of* $\mathscr{D}_r$. *We define the range of values* $R_D(f, z_0)$ *as follows:*

$$R_D(f, z_0)=\bigcap_{r>0}\mathscr{D}_r.$$

(2) *Let* z_0 *be an accessible boundary point of* D. *If* $f(z)\to\alpha$ *as* $z\to z_0$ *along a path in* D *terminating at* z_0, *then* α *is called the asymptotic value of* $w=f(z)$ *at* z_0. *The asymptotic set* $A_D(f, z_0)$ *is defined as the set of asymptotic values at* z_0.

We classify $\{C(z_n, r)\}$ into following three classes.

DEFINITION 3.

(1) *If* $f(z)\to a$ *as* $z\to z_0$, $z\in D(S, r)$, *then we call* $\{C(z_n, r)\}$ *the asymptotic circles with the asymptotic value* a.

(2) *If* $R_{D(S,r)}(f, z_0)$ *contains the neighbourhood of* a, *then we call* $\{C(z_n, r)\}$ *the covering circles upon* a.

(3) *If* $\mathscr{C}R_{D(S,r)}(f, z_0)$ *consists of at most two points, then we call* $\{C(z_n, r)\}$ *the filling circles.*[3)]

3. Cluster set in $D(S, r)$. With these notations and definitions, we can establish the following metrical theorem on cluster sets in $D(S, r)$.

THEOREM 1. *Under the notations in* 2, *if* $f(z_n)\to a$ *as* $z_n\to z_0$, *then for any given positive* ε, *following two propositions hold:*

(1) $\{C(z_n, r(S)+\varepsilon)\}$ *are the filling circles.*

(2) $\{C(z_n, r(S)-\varepsilon)\}$ *are the asymptotic circles with the asymptotic value* a *or the covering circles upon* a.

Remark. If S is the normal sequence, i.e. $r(S)=+\infty$, then for any positive r, only proposition (2) holds with respect to $\{C(z_n, r)\}$. If S is the singular sequence, i.e. $r(S)=0$, then for any positive ε, only proposition (1) holds with respect to $\{C(z_n, \varepsilon)\}$.

To establish this theorem, we need some lemmas.

LEMMA 1. (F. Marty [9], [1] p. 169). *A family* $\mathfrak{F}$ *of meromorphic functions is normal in a domain* D *if and only if for every compact set* Δ *in* D

3) $\mathscr{C}E$ is the complement of E with respect to the whole w-plane.

there exists a positive constant $M(\Delta)$ depending upon Δ such that

$$\rho(f(z)) < M(\Delta) \qquad \textit{for all } f(z) \in \mathfrak{F} \textit{ and } z \in \Delta,$$

where $\rho(f(z))$ is the spherical derivative of $f(z)$, i.e. $|f'(z)|/(1+|f(z)|^2)$.

LEMMA 2. *Under the notations as in 2, suppose that $z = T_n(t)$ $(n = 1, 2, \ldots)$ is the function mapping $|t| < 1$ conformally onto the universal covering surface of D such that $z_n = T_n(0)$. Then, the family $\mathfrak{F}$: $\{f_n(t)\} = \{f(T_n(t))\}$ is normal in $|t| < R$, and not normal in $|t| < R + \varepsilon$ for any positive ε, where $R = \tanh(r(S))$.*[4]

Proof. Since $d\sigma_z = |dt|/(1 - |t|^2)$, by the simple calculation we have

$$\rho(f_n(t)) = ds_z/d\sigma_z \cdot 1/(1 - |t|^2). \tag{3.1}$$

Suppose that $0 < r(S) \leqq +\infty$. Then $0 < R \leqq 1$. To the Euclidean circle: $|t| \leqq R - \varepsilon$, there corresponds the hyperbolic circle: $C(z_n, r_1)$ $(n = 1, 2, \ldots)$, where $r_1 = \tanh^{-1}(R - \varepsilon)$[4]. Since $r_1 < r(S) = \tanh^{-1}(R)$, by (3.1) and the definition of $r(S)$,

$$\rho(f_n(t)) < (C(r_1) + \varepsilon)/(1 - (R - \varepsilon)^2) < +\infty \text{ for } |t| \leqq R - \varepsilon,\ n \geqq N(\varepsilon),$$

$N(\varepsilon)$ being a sufficiently large integer. Hence, by Lemma 1, the family $\{f_n(t)\}$ is normal in $|t| < R$.

Suppose that $\{f_n(t)\}$ were normal in $|t| < R + \varepsilon$. Then, by Lemma 1, there would exist a constant $M(\varepsilon)$ such that

$$\rho(f_n(t)) < M(\varepsilon) < +\infty \text{ for } |t| \leqq R + \varepsilon/2.$$

Hence, by (3.1)

$$ds_z/d\sigma_z < M(\varepsilon) < +\infty \text{ for } z \in D(S, r_2),$$

where $r_2 = \tanh^{-1}(R + \varepsilon/2)$, so that $C(r_2) \leqq M(\varepsilon)$, which is evidently impossible because of $r_2 > r(S)$. Therefore $\{f_n(t)\}$ is not normal in $|t| < R + \varepsilon$. In the case that $r(S) = 0$, i.e. $R = 0$, by the similar arguments as above, we can prove that $\{f_n(t)\}$ is not normal in $|t| < \varepsilon$. Thus our lemma is completely established.

LEMMA 3. *Let $\{f_n(t)\}$ $(n = 1, 2, \ldots)$ be the normal family of meromorphic functions in $|t| < 1$. Suppose that $\lim_{k \to +\infty} f_{n_k}(t_0) = a$ for t_0 $(|t_0| < 1)$ and a sub-*

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[4] $\tanh x = (e^{2x} - 1)/(e^{2x} + 1)$, $\tanh^{-1} x = 1/2 \cdot \log\{(1+x)/(1-x)\}$.

sequence $\{n_k\}$ *of* $\{n\}$. *Then there exists the subsequence* $\{n_{k_i}\}$ *of* $\{n_k\}$ *such that*

(1) $\lim_{i\to+\infty} f_{n_{k_i}}(t)=a$ *uniformly in every compact set in* $|t|<1$

or

(2) *any value in the neighborhood of* a *is taken infinitely many times by the family* $\{f_{n_{k_i}}(t)\}$ $(i=1, 2, \ldots)$ *in the neighborhood of* t_0.

Proof. Considering the family $\{1/f_n(t)\}$ instead of $\{f_n(t)\}$, if necessary, without any loss of generality, we can assume that $a\neq\infty$. By the normalcy of $\{f_n(t)\}$, we can select the subsequence $\{n_{k_i}\}$ of $\{n_k\}$ such that $f_{n_{k_i}}(t)\to f(t)$ uniformly in every compact set in $|t|<1$. Then two cases are possible:

(1) $f(t)\equiv a$,

(2) $f(t)$ is the meromorphic function not reducing to a such that $f(t_0)=a$.

In the case (1), $f_{n_{k_i}}(t)\to a$ uniformly in every compact set in $|t|<1$. In the case (2), since $f(t)$ and $\{f_{n_{k_i}}(t)\}$ $(n_{k_i}\geqq N)$ are regular in the neighborhood of t_0, it is seen by well-known Hurwitz's theorem that $t=t_0$ is the accumulation point of $\{t_i\}$ such that $f_{n_{k_i}}(t_i)=a$. Hence, a is taken infinitely many times by the family $\{f_{n_{k_i}}(t)\}$ in the neighborhood of t_0.

Let $b=f(t_1)$ be an arbitrary but fixed value in the neighborhood of a, where t_1 is the suitable point in the neighborhood of t_0. By the entirely similar arguments, b is taken infinitely many times by the family $\{f_{n_{k_i}}(t)\}$ in the neighborhood of t_1. Hence every value in a neighborhood of a is taken infinitely many times by the family $\{f_{n_{k_i}}(t)\}$ in the neighborhood of t_0, which proves our lemma.

Now we are able to establish Theorem 1.

Proof of Theorem 1. Suppose that $0<r(S)<+\infty$. Then, by Lemma 2, $\{f_n(t)\}$ is not normal in $|t|<\tanh(r(S)+\varepsilon)$, so that every value, except perhaps two, is taken infinitely many times by the family $\{f_n(t)\}$ in $|t|<\tanh(r(S)+\varepsilon)$, which proves that $\{C(z_n, r(S)+\varepsilon)\}$ are the filling circles. Again, by Lemma 2, $\{f_n(t)\}$ is normal in $|t|<\tanh(r(S))$. If $\{f_n(t)\}$ tends uniformly to a in $|t|\leqq\tanh(r(S)-\varepsilon)$, then $C(z_n, r(S)-\varepsilon)$ are the asymptotic circles with the asymptotic value a. On the contrary, if $\{f_n(t)\}$ does not tend uniformly to a in $|t|\leqq\tanh(r(S)-\varepsilon)$, then there exist two sequences $\{n_i\}$ and $\{t(n_i)\}$ such that $|t(n_i)|\leqq\tanh(r(S)-\varepsilon)$ and $f_{n_i}(t(n_i))$ tends to $b\neq a$ as $n_i\to+\infty$.

Since $\{f_n(t)\}$ is normal in $|t|<\tanh(r(S)-\varepsilon/2)$, and $f_{n_i}(0)\to a$, it is verified by Lemma 3 that there exists a subsequence $\{n_{k_i}\}$ of $\{n_i\}$ such that any value in the neighborhood of a is taken infinitely many times by the family $\{f_{n_{k_i}}(t)\}$ in the neighborhood of $t=0$, a fortiori in $|t|<\tanh(r(S)-\varepsilon)$, so that $C(z_n, r(S)-\varepsilon)$ are the covering circles upon a.

If S is the normal or singular sequence i.e. $r(S)=+\infty$ or $r(S)=0$, then by the slight modification of the above arguments we can prove our theorem.

4. $A_D(f, z_0)$. As the first application of Theorem 1, we can prove

THEOREM 2. *By using the notations in 2, if following conditions are satisfied:*

$$(4.1)\qquad \begin{aligned} &(1)\ a\notin \mathscr{J}R_D(f, z_0)^{5)},\\ &(2)\ r(S)>\varlimsup_{n\to+\infty} d(z_n, z_{n+1}), \end{aligned}$$

then $a\in A_D(f, z_0)$, provided that $f(z_n)\to a$.

Remark. In the classical theorem on $A_D(f, z_0)$ ([10] p. 14), the conditions on the boundary cluster sets are always necessary. It should be remarked that in Theorem 2, any conditions on the boundary cluster sets are not assumed.

Since the condition (1) of (4.1) follows immediately from

$$a\notin R_D(f, z_0)\ \text{ or }\ a\in R_D(f, z_0)\cap \mathscr{F}R_D(f, z_0),^{5)}$$

we get

COROLLARY 1. *If following conditions are satisfied:*

(1) $a\notin R_D(f, z_0)$ *or* $a\in R_D(f, z_0)\cap \mathscr{F}R_D(f, z_0)$,

(2) $r(S)>\varlimsup_{n\to+\infty} d(z_n, z_{n+1})$,

then $a\in A_D(f, z_0)$, provided that $f(z_n)\to a$.

COROLLARY 2. *If following conditions are satisfied:*

(1) $a\notin R_D(f, z_0)$,

(2) S: *the normal sequence (i.e. $r(S)=+\infty$),*

(3) $\varlimsup_{n\to+\infty} d(z_n, z_{n+1})<+\infty$,

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5) We write $\mathscr{J}E$ and $\mathscr{F}E$ for the set of inner points of E and the frontier of E respectively.

then $a \in A_D(f, z_0)$, *provided that* $f(z_n) \to a$.

Bagemihl-Seidel's theorem ([2] p. 4, Theorem 1) is contained in Corollary 2. If three distinct values are omitted by $w = f(z)$ in a neighborhood of z_0, in view of Lemma 2 we have easily $r(S) = +\infty$. Hence we have

COROLLARY 3. *Suppose that $f(z)$ is uniform and meromorphic in D, and that three distinct values a, b, c are omitted by $f(z)$ in the neighborhood of z_0 on Γ. If $f(z_n) \to a$ as $z_n \to z_0$, $z_n \in D$ and $\varlimsup_{n\to+\infty} d(z_n, z_{n+1}) < +\infty$, then $a \in A_D(f, z_0)$.*

It contains W. Seidel's theorem ([14] p. 169, Corollary 4).

Proof of Theorem 2. In view of Theorem 1 and $a \notin \mathscr{J}R_D(f, z_0)$, for $r = r(S) - \varepsilon > \varlimsup_{n\to+\infty} d(z_n, z_{n+1})$, $\{C(z_n, r)\}$ are the asymptotic circles with the asymptotic value a. Since $r > \varlimsup_{n\to+\infty} d(z_n, z_{n+1})$, $C(z_n, r) \cap C(z_{n+1}, r)$ is not empty for $n \geqq N$, where N is a sufficiently large integer. Hence we can connect $\{z_n\}$ by the Jordan arc contained in $D(S, r)$ and terminating at $z_0 \in \Gamma$, so that $a \in A_D(f, z_0)$, which proves our theorem.

If we replace (2) of (4.1) by $\lim_{n\to+\infty} d(z_n, z_{n+1}) = 0$, then without (1) of (4.1) we can establish

THEOREM 3. *Under the notations in* 2, *if* $\lim_{n\to+\infty} d(z_n, z_{n+1}) = 0$, *then for any* $\varepsilon > 0$, $\{C(z_n, \varepsilon\}$ *are the filling circles or* $a \in A_D(f, z_0)$, *provided that* $f(z_n) \to a$.

To prove this theorem, we need the following lemma.

LEMMA 4. *Let S be not the singular sequence, i.e.* $r(S) > 0$. *If* $f(z_n) \to a$ *and there exists another sequence* S' *of points* $\{z'_n\}$ $(z'_n \in D)$ *tending to* $z_0 \in \Gamma$ *such that* $\lim_{n\to+\infty} d(z_n, z'_n) = 0$, *then we have also*

$$\lim_{n\to+\infty} f(z'_n) = a.$$

Remark. This lemma was proved by Bagemihl-Seidel ([2] p. 10, Lemma 1) in the special case that D is the unit disc and $r(S) = +\infty$. Our proof is entirely different from theirs.

Proof. We write $\chi(a, b)$ for the chordal distance between a and b:

$$\chi(a, b) = |a - b| / \sqrt{1+|a|^2}\sqrt{1+|b|^2}.$$

We have easily

$$\chi(w_n, w_n') < \int_{w_n}^{w_n'} |dw|/(1+|w|^2) = \int_{z_n}^{z_n'} ds_z \tag{4.2}$$

for any integration path, where $w_n = f(z_n)$, $w_n' = f(z_n')$ and ds_z is the spherical element of length at $z \in D$, i.e. $ds_z = |f'(z)|/(1+|f(z)|^2) \cdot |dz|$. Since $r(S) > 0$ and $\lim_{n\to+\infty} d(z_n, z_n') = 0$, we can choose $r(0<r<r(S))$ such that $z_n' \in D(S, r)$ for $n \geqq N$, and $\overline{\lim}_{\left(\substack{z\to z_0 \\ z\in D(S,r)}\right)} ds_z/d\sigma_z = C(r) < +\infty$. Then, taking account of (4.2), we see that for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$\chi(w_n, w_n') \leqq (C(r)+\varepsilon) \cdot \int_{z_n}^{z_n'} d\sigma_z \qquad \text{for } n \geqq N(\varepsilon).$$

If we choose a suitable integration path, we can put

$$\int_{z^n}^{z_n'} d\sigma_z = d(z_n, z_n').$$

Hence $\chi(w_n, w_n') \leqq (C(r)+\varepsilon)\, d(z_n, z_n')$ for $n \geqq N(\varepsilon)$, from which

$$\lim_{n\to+\infty} \chi(w_n, w_n') = 0.$$

By the inequality $\chi(w_n', a) \leqq \chi(w_n, a) + \chi(w_n, w_n')$, our lemma is completely established.

Proof of Theorem 3. Suppose first that $r(S) = 0$. Then, by Theorem 1, for any $\varepsilon > 0$, $\{C(z_n, \varepsilon)\}$ are the filling circles. Next suppose that $r(S) > 0$. If we choose r such that $0 < r < r(S)$, then by $\lim_{n\to+\infty} d(z_n, z_{n+1}) = 0$, we can connect z_n and z_{n+1} by the hyperbolic segment l_n contained in $C(z_n, r) \cup C(z_{n+1}, r)$. By putting $l = \bigcup_n l_n$, l is a Jordan arc contained in $D(S, r)$ and terminating at z_0. Let S' be an arbitrary sequence of points $\{z_n'\}$ such that $z_n' \in l_n$. Then, by the inequality $d(z_n, z_n') \leqq d(z_n, z_{n+1})$ and Lemma 4, we have

$$\lim_{n\to+\infty} f(z_n') = a.$$

Since S' is arbitrary, we have $\lim f(z) = a$ as $z \to z_0$ along l. Hence $a \in A_D(f, z_0)$. Thus, our theorem is completely established.

As its immediate corollary, we obtain

COROLLARY 4. *Under the notations in* 2, *if* $r(S) > 0$ *and* $\lim_{n\to+\infty} d(z_n, z_{n+1})$

$=0$, *then* $a \in A_D(f, z_0)$, *provided that* $f(z_n) \to a$.

Bagemihl-Seidel's theorem ([2] p. 10, Theorem 2) is contained in this corollary.

As the second application of Theorem 1, we can prove

THEOREM 4. *Let* $f(z)$ *be uniform and meromorphic in* D. *Suppose that there exist a value* a *and two sequences of points* $\{z_n\}$, $\{z'_n\}$ $(z_n, z'_n \in D)$ *tending to* z_0 *on* Γ *such that*

$$
(4.3)\qquad
\begin{aligned}
&(1)\ \lim_{n\to+\infty} f(z_n) = a,\\
&(2)\ \varliminf_{n\to+\infty} \chi(f(z'_n),\ a) > 0,^{6)}\\
&(3)\ a \notin \mathscr{J}R_D(f,\ z_0),\\
&(4)\ \varlimsup_{n\to+\infty} d(z_n,\ z'_n) = r < +\infty.
\end{aligned}
$$

Under these conditions, for any $\varepsilon > 0$, $\{C(z_n, r+\varepsilon)\}$ *are the filling circles.*

Proof. We have

$$
(4.4)\qquad r(S) \leqq r,
$$

where S is the sequence of points $\{z_n\}$. On the contrary, we can choose $\varepsilon(>0)$ such that $r+\varepsilon < r(S)$. In view of Theorem 1 and (1), (3) of (4.3), $\{C(z_n, r+\varepsilon)\}$ are the asymptotic circles with the asymptotic value a, which is contrary to (2) of (4.3). Hence (4.4) is proved. Agagin by Theorem 1 and (4.4), $\{C(z_n, r+\varepsilon)\}$ are the filling circles, which is to be proved.

5. Theorems of P. Montel type [I]. Let L be the Jordan arc lying in D and terminating at *the accessible boundary point* $z_0 \in \Gamma$. We represent L parametrically by

$$
z = z(x)\ \ (0 \leqq x < +\infty),\ \lim_{x\to+\infty} z(x) = z_0.
$$

Similary as in 2, we use the following notations:

$$
\begin{aligned}
D(L,\ r) &= \bigcup_{0\leqq x<+\infty} C(z(x),\ r),\\
C^*(r) &= \varlimsup_{\substack{z\to z_0\\ z\in D(L,\ r)}} ds_z/d\sigma_z.
\end{aligned}
$$

6) $\chi(a, b)$ is the chordal distance between a and b.

In Definition 1, using $C^*(r)$ instead of $C(r)$, we can define *the normalcy radius $r(L)$ along L*. With these notations and definition, we can establish

THEOREM 5. *Suppose that $f(z)$ is uniform and meromorphic in D, and that it has the asymptotic value a along L. Then following propositions hold:*

(1) *For $0<r<r(L)$, $f(z)\to a$ uniformly as $z\to z_0$ inside $D(L, r)$.*

(2) *For $r(L)<r$, $\mathscr{C}R_{D(L,r)}(f, z_0)$ consists of at most two points.*

Remark. (1) If $r(L)=0$, the case (1) does not occur, and if $r(L)=+\infty$, the case (2) does not happen.

(2) W. Seidel's theorem ([14] p. 170, Corollary 5) is contained in this theorem.

(3) Theorem 5 is also a generalization of Lehto-Virtanen's theorem. ([7] pp. 49-50, Theorem 1; p. 53, Theorem 2, K. Noshiro [10] p. 86)

To prove this theorem, we need the following lemma.

LEMMA 5. *Let $z=T_x(t)$ $(0\leqq x<+\infty)$ be the function mapping $|t|<1$ conformally onto the universal covering surface of D such that $z(x)=T_x(0)$. Then, the family $\mathfrak{F}$: $\{f_x(t)\}=\{f(T_x(t))\}$ is normal in $|t|<R$, but not normal in $|t|<R+\varepsilon$ for any positive ε, where $R=\tanh(r(L))$ and $r(L)$ is the normalcy radius along L.*

Its proof is entirely similar to that of Lemma 2, so that we omit its proof.

Proof of Theorem 5. Contrary to the assertion, suppose that $f(z)$ does not tend uniformly to a as $z\to z_0$ inside $D(L, r)$ for $0<r<r(L)$. Then there exists a sequence of points $\{z_n\}$ such that

$$(5.1)\qquad \begin{array}{l} \text{(i) } z_n\to z_0,\ z_n\in D(L, r), \\ \text{(ii) } f(z_n) \text{ does not tend to } a. \end{array}$$

We may suppose that there exist a positive ε and a sequence $\{x_n\}$ $(x_1<x_2<\cdots<x_n\to+\infty)$ such that

$$z_n\in C(z(x_n), r+\varepsilon),\ r+\varepsilon<r(L).$$

Since $\{f_x(t)\}$ is normal in $|t|<R=\tanh(r(L))$ by Lemma 5, we can select the subsequence $\{f_{x_{n_i}}(t)\}$ which tends uniformly to the meromorphic function $F(t)$ in $|t|\leqq\tanh(r+\varepsilon)<R$. By the inverse transformation $t=T_x^{-1}(z)(0=T_x^{-1}(z(x)))$, the image of the part of L contained in $C(z(x_{n_i}), r+\varepsilon)$ is mapped on the curve

passing through $t=0$, and contained in $|t| \leqq \tanh(r+\varepsilon) < R$, which accumulates to the continuum γ passing through 0. On γ, we have easily $F(t)=a$, so that $F(t) \equiv a$, which is contrary to (ii) of (5.1). Thus, the part (1) is proved.

On account of Lemma 5, there exists at least one not-normal point t_1 on $|t|=R$. Hence the family $\{f_x(t)\}$ takes every value, except at most two values, infinitely many times in $|t-t_1|<\varepsilon$ for any $\varepsilon>0$. Therefore, $\mathscr{C}R_{D(L,r)}(f, z_0)$ consists of at most two points for $r=\tanh^{-1}(R+\varepsilon)$. Since ε is arbitrary, the part (2) is proved.

We shall now study the case that there exists the sequence S : $\{z_n\}$ on L such that $f(z_n) \to a$ as $z_n \to z_0$.

THEOREM 6. *Suppose that $f(z)$ is uniform and meromorphic in D, and that there exists the sequence S : $\{z_n\}$ on L such that*

(5.4)
(i) $f(z_n) \to a$ *as* $z_n \to z_0$,
(ii) $l(z_n, z_{n+1}) < M < +\infty$ $(n=1, 2, \ldots)$,
where $l(z_n, z_{n+1})$ is the hyperbolic arc-length between z_n and z_{n+1}.[7]

If $a \overline{\in} \mathscr{J}R(f, z_0)$, then following propositions hold:

(1) *For any r satisfying $r(L)>r$, $f(z)$ tends uniformly to a as $z \to z_0$ inside $D(L, r)$.*

(2) *For any $r>r(L)$, $\mathscr{C}R_{D(L,r)}(f, z_0)$ consists of at most two points.*

Similar remarks as in Theorem 5 should also be mentioned here. As its immediate consequence of Theorem 6, we get

COROLLARY 5. *Under the same conditions as in Theorem 6, if*

$$a \overline{\in} R(f, z_0) \text{ or } a \in R(f, z_0) \cap \mathscr{T}R(f, z_0),$$

then $f(z)$ tends uniformly to a as $z \to z_0$ inside $D(L, r)$ for any r satisfying $r(L)>r>0$.

This corollary contains C. M. Faust's theorem ([6] p. 96).

Proof of Theorem 6. We assume first that $r(L)>0$. Since we have easily $r(S) \geqq r(L)$, we get $r(S)>r>0$ for any r satisfying $0<r<r(L)$. Hence, by

7) $l(z_n, z_{n+1}) = \int_{\widehat{z_n z_{n+1}}} d\sigma_z$

Theorem 1 and $a \overline{\in} \mathscr{J}R(f, z_0)$, $\{C(z_n, r)\}$ are the asymptotic circles with the asymptotic value a. On the arc $\widehat{z_n z_{n+1}}$ we select $z_n^{(1)}$ such that $z_n^{(1)} \in C(z_n, r)$. Denoting by $S^{(1)}$ the sequence $\{z_n^{(1)}\}$, we can see $r(S^{(1)}) \geqq r(L)$, so that we have also $r(S^{(1)}) > r > 0$. By the similar arguments as above, $\{C(z_n^{(1)}, r)\}$ are the asymptotic circles with the asymptotic value a. On the arc $\widehat{z_n^{(1)} z_{n+1}}$, we select $z_n^{(2)}$ such that $z_n^{(2)} \in C(z_n^{(1)}, r)$. Then as above $\{C(z_n^{(2)}, r)\}$ are the asymptotic circles with the asymptotic value a. By (ii) of (5.4), after finite k-steps, we have

$$\begin{cases} \text{(i)} \ z_{n+1} \in C(z_n^{(k)}, r) \\ \text{(ii)} \ C(z_n^{(k)}, r): \text{ the asymptotic circles with the asymptotic value } a. \end{cases}$$

Hence, L is the asymptotic path with the asymptotic value a, so that, by (1) of Theorem 5, the part (1) is proved.

By Lemma 5, $\{f_x(t)\}$ is not normal in $|t| < \tanh(r)$ for $r > r(L)$. Hence every value, except for two values, is taken infinitely many times by the family $\{f_x(t)\}$ in $|t| < \tanh(r)$ $(r > r(L))$, so that part (2) is proved.

Next suppose that $r(L) = 0$. Then the same argument as above proves the part (2).

6. Theorems of P. Montel type [II]. In this section, we shall study the case that $f(z)$ tends to a as $z \to z_0$ along *the general pointset E contained in D.* For this purpose, we shall introduce some definitions.

DEFINITION 4. *Let e be the point set completely contained in D. In the definition of Carathéodory's linear measure* ([14] p. 53), *using the hyperbolic distance instead of the Euclidean distance, we can define the hyperbolic Carathéodory's linear measure of e, which we denote by $\Lambda_h(e)$.*

DEFINITION 5. *We write E for the point set in D, whose closure contains the accessible boundary point $z_0 \in \Gamma$. Let L be the Jordan arc lying in D, and terminating at z_0. We define the mean hyperbolic linear mesure of E with respect to $D(L, r)$ as follows:*

$$M(E, L, r) = \varliminf_{\substack{z \to z_0 \\ z \in L}} \Lambda_h(E \cap C(z, r)).$$

Now we can state our theorem.

THEOREM 7. *Suppose that $f(z)$ is uniform and meromorphic in D, and that $f(z)$ tends to a as z tends to the accessible boundary point z_0 along E, where E is the point set in D, whose closure contains z_0. Let L be the Jordan arc lying in D, and terminating at z_0. If $r(L)>0$, and $M(E, L, r_0)>0$ for fixed r_0 satisfying $0<r_0<r(L)$, then $f(z)$ tends to a uniformly as $z\to z_0$ inside $D(L, r)$ for any $r<r(L)$.*

Remark. Generalizations of P. Montel's theorem such as Theorem 7 are discussed by M. L. Cartwright [4] and Ohtsuka [13].

To prove this theorem, we prepare the following lemma.

LEMMA 6. *Suppose that the family $\{f_n(t)\}$ of the meromorphic functions is normal in $|t|<1$. Let $\{E_n\}$ be the sequence of point sets such that*

$$(6.1)\qquad \begin{aligned} &(1)\ E_n \text{ is contained in } |t|\leqq 1-\delta(0<\delta<1)\quad (n=1,2,\ldots),\\ &(2)\ \Lambda_h(E_n)>d>0\ (n=1,2,\ldots),\\ &(3)\ \lim_{n\to+\infty} f_n(E_n)=a.^{8)}\end{aligned}$$

Then $f_n(t)$ tends uniformly to a in the wider sense in $|t|<1$.[8]

Proof. Considering $f_n(t)-a$ (if $a=\infty$, $1/f_n(t)$) instead of $f_n(t)$, if necessary, we can assume without any loss of generality that $a=0$.

Assume that $\{f_n(t)\}$ does not tend uniformly to 0 in the wider sense in $|t|<1$. Then we can select out a subsequence $\{f_{n_i}(t)\}$ such that $\{f_{n_i}(t)\}$ tends uniformly to the non-constant meromorphic function in the wider sense in $|t|<1$. Let $\{t_1(n_i)\}$ be a seuqence of points such that $t_1(n_i)\in E_{n_i}$. If necessary, selecting a suitable subsequence of $\{t_1(n_i)\}$, we can assume that $\{t_1(n_i)\}$ tends to t_1 ($|t_1|\leqq 1-\delta$). Since $\Lambda_h(E_{n_i})>d>d/2$ ($i=1,2,\ldots$), there exists a sequence of points $\{t_2(n_i)\}$ such that

(i) $t_2(n_i)$ is outside of $C(t_1, d/2^2)$,
(ii) $t_2(n_i)\in E_{n_i}$.

Then we can assume that $\{t_2(n_i)\}$ tends to t_2 distinct from t_1. Since $\Lambda_h(E_{n_i})>d>d/2+d/2^2$, we can find a sequence $\{t_3(n_i)\}$ such that

(i) $t_3(n_i)$ is outside of $C(t_1, d/2^2)\cup C(t_2, d/2^3)$,
(ii) $t_3(n_i)\in E_{n_i}$.

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[8] The convergence is spherical one.

Then we can assume that $\{t_3(n_i)\}$ tends to t_3 distinct from t_1 and t_2. Repeating the similar arguments as above, we can find the sequence of points $\{t_n\}(t_i \neq t_j$ for $i \neq j)$ such that

$$(6.2) \quad \begin{array}{l} \text{(i) } \{t_n\} \text{ tends to a point } t_0 \ (|t_0| \leqq 1-\delta), \\ \text{(ii) } t_m \text{ is the limiting point of } \{t_m(n_i)\} \text{ such that } t_m(n_i) \in E_{n_i}. \end{array}$$

Now we shall prove that

$$(6.3) \qquad \lim_{i\to+\infty} f_{n_i}(t_m) = 0 \text{ for any fixed } m \ (m=1, 2, \ldots).$$

By (3) of (6.1)

$$(6.4) \qquad \lim_{i\to+\infty} f_{n_i}(t_m(n_i)) = 0.$$

It is well-known that $\{f_n(t)\}$ is equi-continuous in $|t| \leqq 1-\delta$, provided that $\{f_n(t)\}$ is normal in $|t|<1$. Hence, by (ii) of (6.2)

$$f_{n_i}(t_m) - f_{n_i}(t_m(n_i)) \to 0$$

as $i \to +\infty$, so that, by the inequality

$$|f_{n_i}(t_m)| \leqq |f_{n_i}(t_m(n_i))| + |f_{n_i}(t_m) - f_{n_i}(t_m(n_i))|$$

and (6.4),

$$\lim_{i\to+\infty} f_{n_i}(t_m) = 0,$$

which proves (6.3), Therefore, by (i) of (6.2), (6.3) and Vitali's theorem, $\{f_{n_i}(t)\}$ tends uniformly to zero in the wider sense in $|t|<1$, which is contrary to the assumption. Thus our lemma is completely established.

Now we can prove Theorem 7.

Proof of Theorem 7. Let L be defined by

$$z = z(x) \ (0 \leqq x < +\infty), \ \lim_{x\to+\infty} z(x) = z_0.$$

By Lemma 5, $\{f_x(t)\}$ is normal in $|t|<R=\tanh(r(L))$. Let us denote by E_x the image of $E \cap C(z(x), r_0)$ by $t = T_x^{-1}(z)$ $(0 = T_x^{-1}(z(x)))$. Then E_x is contained in $|t| \leqq \tanh(r_0) < R$. Since hyperbolic metric is conformally invariant, we have

$$(6.5) \qquad \Lambda_h(E_x) > \rho/2 > 0 \qquad \text{for } x_\rho \leqq x < +\infty,$$

where $M(E, L, r_0) = \rho > 0$ and x_ρ is a constant depending upon ρ. Because $f(z) \to a$ as $z \to z_0$ along E, we have

$$\lim_{x \to +\infty} f_x(E_x) = a. \tag{6.6}$$

Hence, by (6.5), (6.6) and Lemma 6, $f_x(t)$ tends uniformly to a in the wider sense in $|t| < R$, which proves that $f(z) \to a$ uniformly as $z \to z_0$ inside $D(L, r)$ for $r < r(L)$. Thus our theorem is established.

References

[1] Ahlfors, L., Complex analysis. McGraw-Hill (1935).

[2] Bagemihl, F. and Seidel, W., Sequential and continuous limits of meromorphic functions. Ann. Acad. Sci. Fenn A. I. 280, 1–16 (1960).

[3] Bagemihl, F., Koebe arcs and Fatou points of normal functions. Comment. Math. Helvet. **36**, 9–18 (1961).

[4] Cartwright, M. L., Some generalizations of Montel's theorem. Proc. Cambridge Philos. Soc. **31**, 26–30 (1935).

[5] Constantinescu, C., Einige Anwendungen des hyperbolischen Masses. Math. Nachr. **15**, 155–172 (1956).

[6] Faust, C. M., On the boundary behavior of holomorphic functions in the unit disk. Nagoya Math. J. **20**, 95–103 (1962).

[7] Lehto, O. and Virtanen, K. I., Boundary behaviour and normal meromorphic functions. Acta Math. **97**, 47–56 (1957).

[8] Lehto, O., The spherical derivative of meromorphic functions in the neighbourhood of an isolated singularity. Comment. Math. Helvet. **33**, 196-205 (1959).

[9] Marty, F., Recherches sur la répartition des valeurs d'une fonction méromorphe. Ann. Fac. Sci. Univ. Toulouse (3) **23** (1931), 183–261.

[10] Noshiro, K., Cluster sets. Springer-Verlag. (1960).

[11] Noshiro, K., Contributions to the theory of meromorphic functions in the unit circle. J. Fac. Sci. Hokkaido Univ. **7** (1938), 149–159.

[12] Nevanlinna, R., Eindeutige analytische Funktionen. Zweite Auflage (1953).

[13] Ohtsuka, M., Generalizations of Montel-Lindelöf's theorem on asymptotic values, Nagoya Math. J. **10** (1956), 129–163.

[14] Saks, S., Theory of the integral. Warszawa (1937).

[15] Seidel, W., Holomorphic functions with spiral asymptotic paths. Nagoya Math. J. **14** (1959), 159–171.

[16] Tanaka, C., Note on the cluster sets of the meromorphic functions. Proc. Japan Acad. **35** (1959), 167-168.

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