COHOMOLOGY RELATIONS IN SPACES WITH A TOPOLOGICAL TRANSFORMATION GROUP[1]

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1. Introduction

Let Q be a topological transformation group operating on the left of a topological space X. Let us denote by B the orbit space and $p: X \to B$ the projection. p is a continuous and open map of X onto B. For an arbitrary abelian coefficient group G, the continuous map p induces homomorphisms

$$p^*: H^n(B, G) \to H^n(X, G), \quad (n \geqq 0),$$

of the Alexander-Wallace cohomology groups [1][2]. These induced homomorphisms are, in general, not onto isomorphisms. They depend on the manner in which the topological transformation group Q operates on X.

To measure the deviation of these induced homomorphisms p^* from the onto isomorphisms, we introduce, in the present paper, the *weakly residual cohomology groups*

$$H^n_w(X, G), \quad (n \geqq 0).$$

They are invariants depending on X, Q, G and the operations of Q on X. By means of these groups, we shall establish an exact sequence

$$H^0(B, G) \xrightarrow{p^*} \dots \to H^n(B, G) \xrightarrow{p^*} H^n(X, G) \to H^n_w(X, G) \to H^{n+1}(B, G) \xrightarrow{p^*} \dots .$$

This indicates that the weakly residual cohomology groups $H^n_w(X, G)$ might play an important role in the further studies of the cohomology structures of the orbit space.

For each point $x \in X$, there is a canonical homomorphism

$$k^*_x: H^n_w(X, G) \to H^n(Q, G), \quad (n \geqq 0).$$

It is proved that if Q is compact and if x and y are two points contained in a compact connected subset of X then $k^*_x = k^*_y$.

2. Preliminaries

Throughout the present paper, let Q be a topological group acting as a

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[2] Numbers in square brackets refer to the bibliography at the end of the paper.

group of transformations on the *left* of a topological space X. By this we mean that, with each element q in Q, there is associated a transformation

$$W_q : X \to x$$

such that, if we use the notation $W_q(x) = qx$, the following conditions are satisfied.

(2.1) qx is continuous in q and x simultaneously;

(2.2) $q_1(q_2x) = (q_1q_2)x, \quad (q_1 \in Q,\ q_2 \in Q,\ x \in X);$

(2.3) $ex = x, \quad (x \in X),$

where e denotes the neutral element of Q. More precisely, the condition (2.1) means that the map

$$M : Q \times X \to X$$

defined by $M(q, x) = qx$ for each $q \in Q$ and each $x \in X$ is continuous. Obviously, W_q is a homeomorphism of X for each $q \in Q$.

Two points x and y are said to be *equivalent* if there exists an element q in Q such that $y = qx$. This equivalence relation divides the points of X into disjoint equivalence classes called the *orbits* of Q in X. The orbit which contains the point $x \in X$ will be denoted by Qx. Hence $Qx = Qy$ if and only if x and y are equivalent. Let B denote the set of all orbits of Q in X. There is a natural map

$$p : X \to B$$

of X onto B defined by $p(x) = Qx$ for each $x \in X$. p will be called the *projection* of X onto B. Let us give B the *identification topology* determined by p. That is to say, a subset V in B is called open if and only if $p^{-1}(V)$ is an open set in X. The topological space B thus obtained will be called the *orbit space* of the transformation group Q. B is a T_1-space if and only if every orbit of Q is a closed subset in X.

The projection $p : X \to B$ is both continuous and open. In fact, the continuity of p follows from the definition of the identification topology in B determined by p. To see that p is open, let U be an arbitrary open set in X and call $V = p(U)$. It suffices to show that $p^{-1}(V)$ is an open set in X. By the definition of p, the set $p^{-1}(V)$ consists of the totality of the points qx in X such that $q \in Q$ and $x \in U$. Hence $p^{-1}(V)$ is the union QU of the sets $W_q(U)$ for all $q \in Q$. For each q in Q, W_q is a homeomorphism of X. This implies that $W_q(U)$ is open and hence, as a union of open sets, $p^{-1}(V)$ is open.

3. The various cohomology groups

For convenience of the reader, we shall briefly recall the definition of the

Alexander-Wallace cohomology groups [1]. Let G be an abelian group used as the coefficient group of the various cohomology groups defined in the sequel.

Denote by

$$A^n(X, G), \quad (n \geqq 0),$$

the group of all n-functions $\phi : X^{n+1} \to G$ on X into G and

$$A_0^n(X, G), \quad (n \geqq 0),$$

the subgroup of $A^n(X, G)$ consisting of the n-functions with empty support, where the support $S(\phi)$ of an n-function $\phi : X^{n+1} \to G$ is the closed set of X defined by the following assertion:

(3.1) A point $x \in X$ is not in $S(\phi)$ if and only if there exists an open neighborhood U of x in X such that

$$\phi(x_0, x_1, \ldots, x_n) = 0$$

whenever $x_i \in U$ for all $i = 0, 1, \ldots, n$.

The coboundary homomorphism

$$\delta : A^n(X, G) \to A^{n+1}(X, B) \tag{3.2}$$

is defined as usual, namely[3]

$$(\delta\phi)(x_0, \ldots, x_{n+1}) = \sum_{i=0}^{n+1}(-1)^i \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{n+1})$$

for arbitrary $(x_0, \ldots, x_{n+1}) \in X^{n+2}$. Obviously we have

$$\delta(A_0^n((X, G)) \subset A_0^{n+1}(X, G).$$

Let

$$C^n(X, G) = A^n(X, G)/A_0^n(X, G).$$

Then δ in (3.2) induces a coboundary homomorphism

$$\delta : C^n(X, G) \to C^{n+1}(X, G). \tag{3.3}$$

The elements of $C^n(X, G)$ are called the n-cochains of X over G. For each n-function $\phi \in A^n(X, G)$, we shall denote by $[\phi]$ the n-cochain which contains ϕ, that is,

$$[\phi] = \phi + A_0^n(X, G),$$

We say that ϕ represents $[\phi]$.

Let $Z^n(X, G) \subset C^n(X, G)$ denote the kernel of δ in (3.3), and $B^{n+1}(X, G) = \delta(C^n(X, G))$. Further, we define $B^0(X, G) = 0$. Since $\delta\delta = 0$, we have

$$B^n(X, G) \subset Z^n(X, G), \quad (n \geqq 0).$$

The quotient group

$$H^n(X, G) = Z^n(X, G)/B^n(X, G)$$

[3] The circumflex over x_i indicates that x_i is omitted.

is called the *n-dimensional cohomology group* of X over G.

An n-function $\phi \in A^n(X, G)$ is said to be *strongly invariant under* Q if

$$\phi(q_0 x_0, \ldots, q_n x_n) = \phi(x_0, \ldots, x_n)$$

for all $x_i \in X$ and all $q_i \in Q$, $i = 0, \ldots, n$. An n-cochain $c \in C^n(X, G)$ is said to be *strongly invariant under* Q if c contains an n-function $\phi \in A^n(X, G)$ which is strongly invariant under Q. Obviously the strongly invariant n-cochains of X over G form a subgroup

$$C_s^n(X, G) \subset C^n(X, G)$$

and

$$\delta(C_s^n(X, G)) \subset C_s^{n+1}(X, G),$$

hence the δ in (3.3) defines a coboundary homomorphism

$$\delta : C_s^n(X, G) \to C_s^{n+1}(X, G). \tag{3.4}$$

Let $Z_s^n(X, G) \subset C_s^n(X, G)$ denote the kernel of δ in (3.4) and $B_s^{n+1}(X, G) = \delta(C_s^n(X, G))$. Further, we define $B_s^0(X, G) = 0$. Then evidently we have

$$Z_s^n(X, G) = Z^n(X, G) \cap C_s^n(X, G).$$

The quotient group

$$H_s^n(X, G) = Z_s^n(X, G) / B_s^n(X, G)$$

is called the *n-dimensional strongly invariant cohomology group* of X over G (under the topological transformation group Q).

For each integer $n \geqq 0$, let

$$C_w^n(X, G) = C^n(X, G) / C_s^n(X, G).$$

The elements of $C_w^n(X, G)$ are called the *weakly residual n-cochains* (with respect to Q) of X over G. Since the coboundary homomorphism δ in (3.3) maps $C_s^n(X, G)$ into $C_s^{n+1}(X, G)$, it induces a coboundary homomorphism

$$\delta : C_w^n(X, G) \to C_w^{n+1}(X, G). \tag{3.5}$$

Let $Z_w^n(X, G) \subset C_w^n(X, G)$ denote the kernel of δ in (3.5) and $B_w^{n+1}(X, G) = \delta(C_w^n(X, G))$. Further, we define $B_w^0(X, G) = 0$. The quotient group

$$H_w^n(X, G) = Z_w^n(X, G) / B_w^n(X, G)$$

is called the *n-dimensional weakly residual cohomology group* of X over G (with respect to the topological transformation group Q).

Let us denote respectively by

$$\iota : C_s^n(X, G) \to C^n(X, G),$$
$$\pi : C^n(X, G) \to C_w^n(X, G)$$

the natural inclusion and projection homomorphisms. Since both ι and π commute with the coboundary operator δ, they induce homomorphisms

(3.6) $$\iota^* : H_s^n(X, G) \to H^n(X, G),$$

(3.7) $$\pi^* : H^n(X, G) \to H_w^n(X, G)$$

for each integer $n \geqq 0$. We are going to define a homomorphism

(3.8) $$\delta^* : H_w^n(X, G) \to H_s^{n+1}(X, G)$$

for every $n \geqq 0$ as follows. Let α be an arbitrary element of $H_w^n(X, G)$. Choose a weakly residual n-cocycle $c_w \in C_w^n(X, G)$ which represents α. Since π maps $C^n(X, G)$ onto $C_w^n(X, G)$, there is an n-cochain $c \in C^n(X, G)$ with $\pi c = c_w$. Since $\pi\delta c = \delta\pi c = \delta c_w = 0$, we have $\delta c \in Z_s^{n+1}(X, G)$. Hence δc represents an element β of $H_s^{n+1}(X, G)$. It is not difficult to see that β depends only on α. We define the homomorphism δ^* by taking

$$\delta^*(\alpha) = \beta.$$

The following theorem is a direct consequence of a general theorem of Kelley and Pitcher [2].

THEOREM I. *The sequence of groups and homomorphisms*

$$H_s^0(X, G) \xrightarrow{\iota^*} \cdots \xrightarrow{\delta^*} H_s^n(X, G) \xrightarrow{\iota^*} H^n(X, G) \xrightarrow{\pi^*} H_w^n(X, G) \xrightarrow{\delta^*} H_s^{n+1}(X, G) \xrightarrow{\iota^*} \cdots$$

is exact in the sense that the image of each homomorphism coincides with the kernel of the following one.

4. The isomorphism p_s^*

The projection $p : X \to B$ induces a homomorphism

(4.1) $$p^\# : A^n(B, G) \to A^n(X, G)$$

of the n-functions $A^n(B, G)$ of the orbit space B into the n-functions $A^n(X, G)$ of X as follows. Let $\phi \in A^n(B, G)$ be an arbitrarily given n-functions of the orbit space B into G. The n-function $p^\#\phi \in A^n(X, G)$ is defined by

$$(p^\#\phi)(x_0, \ldots, x_n) = \phi(px_0, \ldots, px_n)$$

for every $(x_0, \ldots, x_n)$ of X^{n+1}. Since

$$p(qx) = p(x)$$

for every $x \in X$ and every $q \in Q$, $p^\#\phi$ is strongly invariant under Q. Let us denote by

$$A_s^n(X, G)$$

the subgroup of $A^n(X, G)$ which consists of the strongly invariant n-functions. Then (4.1) may be written in the following more precise form

(4.2) $$p_s^{\#} : A^n(B, G) \to A_s^n(X, G).$$

(4.3) LEMMA. *$p_s^{\#}$ maps $A^n(B, G)$ isomorphically onto $A_s^n(X, G)$.*

Proof. That $p_s^{\#}$ is an isomorphism is a consequence of the fact that p is onto. In fact, suppose that $\phi \in A^n(B, G)$ and $p_s^{\#}\phi = 0$. Let $(b_0, \ldots, b_n)$ be an arbitrary point of B^{n+1}. Since p maps X onto B, there are $n+1$ points $x_0, \ldots, x_n$ in X such that $px_i = b_i$ for each $i = 0, \ldots, n$. Then we have

$$\phi(b_0, \ldots, b_n) = (p_s^{\#}\phi)(x_0, \ldots, x_n) = 0.$$

Since $(b_0, \ldots, b_n)$ is arbitrary, this proves that $\phi = 0$ and hence $p_s^{\#}$ is an isomorphism.

To prove that $p_s^{\#}$ maps $A^n(B, G)$ onto $A_s^n(X, G)$, let

$$\psi : X^{n+1} \to G$$

be an arbitrary strongly invariant n-function. Define an n-function

$$\phi : B^{n+1} \to G$$

as follows. Let $(b_0, \ldots, b_n)$ be any point in B^{n+1}. Choose $n+1$ points $x_0, \ldots, x_n$ in X such that $px_i = b_i$ for each $i = 0, \ldots, n$. Then ϕ is defined by taking

(4.4) $$\phi(b_0, \ldots, b_n) = \psi(x_0, \ldots, x_n).$$

To justify this definition, it suffices to show that $\phi(b_0, \ldots, b_n)$ does not depend on the choice of $x_0, \ldots, x_n$. In fact, let $y_0, \ldots, y_n$ be any $n+1$ points in X with $py_i = b_i$ for each $i = 0, \ldots, n$. Then there are $q_0, \ldots, q_n$ in Q such that

$$y_i = q_i x_i, \qquad (i = 0, \ldots, n).$$

It follows from the strong invariance of ψ that

$$\psi(y_0, \ldots, y_n) = \psi(q_0 x_0, \ldots, q_n x_n) = \psi(x_0, \ldots, x_n).$$

This justifies the definition of ϕ. By (4.4), it is clear that $\psi = p_s^{\#}\phi$. Hence $p_s^{\#}$ maps $A^n(B, G)$ onto $A_s^n(X, G)$. This completes the proof of (4.3).

(4.5) LEMMA. *$p_s^{\#}$ maps $A_0^n(B, G)$ onto $A_s^n(X, G) \cap A_0^n(X, G)$.*

Proof. Let $\phi \in A_0^n(B, G)$ and $x \in X$ be arbitrarily given. Call $b = px$. Since ϕ is of empty support, there is an open neighborhood V of b in B such that

$$\phi(b_0, \ldots, b_n) = 0$$

whenever $b_i \in V$ for each $i = 0, \ldots, n$. It follows from the continuity of p that there exists an open neighborhood U of x in X with

$$p(U) \subset V.$$

Then we have

$$(p_S^{\#}\phi)(x_0, \dots, x_n) = \phi(px_0, \dots, px_n) = 0$$

whenever $x_i \in U$ for each $i = 0, \dots, n$. Hence x is not in the support of $p_S^{\#}\phi$. Since x is arbitrary, $p_S^{\#}\phi$ is of empty support. This and (4.3) prove that

$$p_S^{\#}(A_0^n(B, G)) \subset A_S^n(X, G) \cap A_0^n(X, G).$$

Next, let $\psi \in A_S^n(X, G) \cap A_0^n(X, G)$ be arbitrarily given. By (4.3), there is an n-function $\phi \in A^n(B, G)$ such that $\psi = p_S^{\#}\phi$. It remains to show that the support of ϕ is empty. Let $b \in B$ be any given point. Since p maps X onto B, there is a point $x \in X$ with $px = b$. Since ψ is of empty support, there is an open neighborhood U of x in X such that

$$\psi(x_0, \dots, x_n) = 0$$

whenever $x_i \in U$ for each $i = 0, \dots, n$. Call

$$V = p(U).$$

Since p is an open map, V is an open neighborhood of b in B. Let $(b_0, \dots, b_n)$ be any point in B^{n+1} with $b_i \in V$ for each $i = 0, \dots, n$. Choose $n+1$ points $x_0, \dots, x_n$ in U such that $px_i = b_i$ for each $i = 0, \dots, n$. Then we have

$$\phi(b_0, \dots, b_n) = \psi(x_0, \dots, x_n) = 0.$$

This proves that b is not in the support of ϕ. Since b is arbitrary, the support of ϕ must be empty. This completes the proof of (4.5).

Since $p^{\#}$ maps $A_0^n(B, G)$ into $A_0^n(X, G)$ by (4.5), it induces a homomorphism

$$p^{\#} : C^n(B, G) \to C^n(X, G). \tag{4.6}$$

By (4.3), $p^{\#}$ in (4.6) maps $C^n(B, G)$ into $C_S^n(X, G)$. Hence (4.6) may be written in the following more precise form

$$p_S^{\#} : C^n(B, G) \to C_S^n(X, G). \tag{4.7}$$

(4.6) and (4.7) are connected by the following obvious relation

$$\iota p_S^{\#} = p^{\#}, \tag{4.8}$$

where $\iota : C_S^n(X, G) \to C^n(X, G)$ denotes the inclusion homomorphism.

(4.9) LEMMA. *$p_S^{\#}$ maps $C^n(B, G)$ isomorphically onto $C_S^n(X, G)$.*

Proof. To prove that $p_S^{\#}$ maps $C^n(B, G)$ isomorphically into $C_S^n(X, G)$, let $c \in C^n(B, G)$ be any n-cochain of B such that $p_S^{\#}c = 0$. Choose an n-function $\phi : B^{n+1} \to G$ which represents c. $p_S^{\#}c = 0$ implies that $p_S^{\#}\phi$ is of empty support. By (4.3) and (4.5), this implies that the support of ϕ is empty. Hence $c = 0$ and $p_S^{\#}$ is an isomorphism.

To prove that $p_S^{\#}$ maps $C^n(B, G)$ onto $C_S^n(X, G)$, let d be any strongly invariant n-cochain of X over G. Choose a $\psi \in A_S^n(X, G)$ which represents d.

By (4.3), there is a $\phi \in A^n(B, G)$ such that $p_s^{\#}\phi = \psi$. ϕ represents an n-cochain $c \in C^n(B, G)$ and obviously $p_s^{\#} c = d$. This completes the proof of (4.9).

Since both $p^{\#}$ and $p_s^{\#}$ commute with the coboundary operator δ, they induce homomorphisms

$$(4.10) \qquad p^* : H^n(B, G) \to H^n(X, G)$$

$$(4.11) \qquad p_s^* : H^n(B, G) \to H_s^n(X, G)$$

for each integer $n \geqq 0$. The relation (4.8) gives

$$(4.12) \qquad i^* p_s^* = p^*.$$

The following theorem is an immediate consequence of (4.9).

THEOREM II. *p_s^* maps $H^n(B, G)$ isomorphically onto $H_s^n(X, G)$.*

5. The exact sequence

Let us call

$$d^* : H_w^n(X, G) \to H^{n+1}(B, G)$$

the homomorphism defined by

$$(5.1) \qquad d^* = (p_s^*)^{-1}\delta^*.$$

Then the following theorem is a consequence of the theorems I and II together with the relations (4.12) and (5.1).

THEOREM III. *The sequence of groups and homomorphisms*

$$H^0(B, G) \xrightarrow{p^*} \dots \xrightarrow{d^*} H^n(B, G) \xrightarrow{p^*} H^n(X, G) \xrightarrow{\pi^*} H_w^n(X, G) \xrightarrow{d^*} H^{n+1}(B, G) \xrightarrow{p^*} \dots$$

is exact.

6. The canonical homomorphism k_x^*

Let $x \in X$ be a given point. We are going to construct a canonical homomorphism

$$(6.1) \qquad k_x^* : H_w^n(X, G) \to H^n(Q, G)$$

for each integer $n \geqq 0$.

Let $\alpha \in H_w^n(X, G)$ be arbitrarily given. α is represented by a weakly residual n-cocycle $c_w \in Z_w^n(X, G)$ and c_w itself is represented by an n-function $\phi \in A^n(X, G)$ such that

$$(6.2) \qquad \delta\phi = \xi + \eta, \quad \xi \in A_s^{n+1}(X, G), \quad \eta \in A_0^{n+1}(X, G).$$

We may assume that

$$(6.3) \qquad \phi(x, \dots, x) = 0.$$

In fact. if $\phi(x, \ldots, x) = a \neq 0$, we define a strongly invariant n-function $\psi_a \in A_s^n(X, G)$ by taking

$$\psi_a(x_0, \ldots, x_n) = a$$

for each point $(x_0, \ldots, x_n)$ of X^{n+1}. Then, we replacc ϕ by $\phi - \psi_a$ which represents the same weakly residual n-cocycle c_w that ϕ does.

Now let us define an n-function $k^{\#}\phi \in A^n(Q, G)$ of Q over G by taking

$$(k^{\#}\phi)(q_0, \ldots, q_n) = \phi(q_0x, \ldots, q_nx)$$

for each point $(q_0, \ldots, q_n)$ of Q^{n+1}.

(6.4) LEMMA. *The coboundary $\delta k^{\#}\phi$ of $k^{\#}\phi$ is of empty support.*

Proof. Let q be an arbitrary point in Q. It suffices to show that q is not in the support of $\delta k^{\#}\phi$. By (6.2), we have

$$\delta\phi = \xi + \eta,$$

where $\xi \in A_s^{n+1}(X, G)$ and $\eta \in A_0^{n+1}(X, G)$. Since η is of empty support, there is an open neighborhood U of the point qx ih X such that

$$\eta(x_0, \ldots, x_n) = 0$$

whenever $x_i \in U$ for all $i = 0, \ldots, n$. Then there exists an open neighborhood V of q in Q such that

$$Vx \subset U.$$

On the other hand, we have $\eta(x, \ldots, x) = 0$. It follows that, for any point $(q_0, \ldots, q_{n+1})$ of Q^{n+2} such that $q_i \in V$ for all $i = 0, \ldots, n+1$, we have

$$\begin{aligned}
\delta k^{\#}\phi(q_0, \ldots, q_{n+1}) &= \sum_{i=0}^{n+1}(-1)^i k^{\#}\phi(q_0, \ldots, \hat{q}_i, \ldots, q_{n+1}) \\
&= \sum_{i=0}^{n+1}(-1)^i\phi(q_0x, \ldots, \widehat{q_ix}, \ldots, q_{n+1}x) = \delta\phi(q_0x, \ldots, q_{n+1}x) \\
&= \xi(q_0x, \ldots, q_{n+1}x) + \eta(q_0x, \ldots, q_{n+1}x) = \xi(x, \ldots, x) \\
&= \delta\phi(x, \ldots, x) - \eta(x, \ldots, x) = 0.
\end{aligned}$$

This proves that q is not in the support of $\delta k^{\#}\phi$ and hence completes the proof of (6.4).

By (6.4), the n-cochain $[k^{\#}\phi] \in C^n(Q, G)$ which contains the n-function $k^{\#}\phi$ defined above is an n-cocycie of Q over G and hence it represents an element $k_x^*(\alpha)$ of $H^n(Q, G)$.

(6.5) LEMMA. *The element $k_x^*(\alpha)$ does not depend on the choice of the n-function $\phi \in A^n(X, G)$ which represents the given element $\alpha \in H_w^n(X, G)$.*

Proof. First assume $n > 0$. Let ϕ' be any n-function which represents α and such that $\phi'(x, \ldots, x) = 0$. Then

$$\phi' - \phi = \delta\psi + \theta + \tau$$

where $\psi \in A^{n-1}(X, G)$, $\theta \in A^n_s(X, G)$ and $\tau \in A^n_0(X, G)$. Define an $(n-1)$-function $\zeta \in A^{n+1}(Q, G)$ of Q over G by taking

$$\zeta(q_0, \ldots, q_{n-1}) = \psi(q_0x, \ldots, q_{n-1}x) - \psi(x, \ldots, x)$$

for each point $(q_0, \ldots, q_{n-1})$ of Q^n. In order to prove (6.5) for $n>0$, it suffices to show that

$$k^{\#}\phi' - k^{\#}\phi - \delta\zeta$$

has empty support. Let q be an arbitrary point in Q. Since the support of τ is empty, there is an open neighborhood U of the point qx in X such that

$$\tau(x_0, \ldots, x_n) = 0$$

whenever $x_i \in U$ for all $i = 0, \ldots, n$. Let V be an open neighborhood of q in Q such that

$$Vx \subset U.$$

Then, for each point $(q_0, \ldots, q_n)$ of Q^{n+1} with $q_i \in V$ for all $i = 0, \ldots, n$, we have

$$\begin{aligned}
&(k^{\#}\phi' - k^{\#}\phi - \delta\zeta)(q_0, \ldots, q_n) \\
&\quad = (\phi' - \phi)(q_0x, \ldots, q_nx) - \delta\psi(q_0x, \ldots, q_nx) + \delta\psi(x, \ldots, x) \\
&\quad = \theta(q_0x, \ldots, q_nx) + \tau(q_0x, \ldots, q_nx) + \delta\psi(x, \ldots, x) \\
&\quad = \theta(x, \ldots, x) + \delta\psi(x, \ldots, x) \\
&\quad = \phi'(x, \ldots, x) - \phi(x, \ldots, x) = 0.
\end{aligned}$$

Hence q is not in the support of $k^{\#}\phi' - k^{\#}\phi - \delta\zeta$. Since q is arbitrary, this proves that the support of $k^{\#}\phi' - k^{\#}\phi - \delta\zeta$ is empty.

It remains to dispose of the trivial case $n=0$. Let ϕ and ϕ' be any two 0-functins which represent the same element $\alpha \in H^0_w(X, G)$ and such that $\phi(x) = 0 = \phi'(x)$. Since $A^0_0(X, G) = 0$, we have $\phi' - \phi \in A^0_s(X, G)$. In order to prove (6.5) for $n=0$, it suffices to show that $k^{\#}\phi' - k^{\#}\phi = 0$. Let q be an arbitrary point in Q. Then we have

$$(k^{\#}\phi' - k^{\#}\phi)(q) = (\phi' - \phi)(qx) = (\phi' - \phi)(x) = 0.$$

Since q is arbitrary, we have $k^{\#}\phi' - k^{\#}\phi = 0$. This completes the proof of (6.5).

The correspondence $\alpha \to k^*_x(\alpha)$ obviously defines a homomorphism of $H^n_w(X, G)$ into $H^n(Q, G)$. This completes the construction of the canonical homomorpism (6.1).

7. Relations between the canonical homomorphisms

THEOREM IV. *If Q is compact and if x and y are two points contained in a compact connected subset K of X, then $k^*_x = k^*_y$.*

Proof. Let $n \geqq 0$ be an arbitrary integer and $\alpha \in h_w^n(X, G)$ be an arbitrary element. It is required to prove that

$$k_x^*(\alpha) = k_y^*(\alpha).$$

The element α is represented by an n-function $\phi \in A^n(X, G)$ such that

$$\delta\phi = \xi + \eta, \quad \xi \in A_s^{n+1}(X, G), \quad \eta \in A_0^{n+1}(X, G).$$

According to the construction of the canonical homomorphism k_z^* for an arbitrary point $z \in X$. the element $k_z^*(\alpha)$ of $H^n(Q, G)$ is represented by the n-function

$$k_z^\#\phi : Q^{n+1} \to G$$

defined by

$$(k_z^\#\phi)(q_0, \ldots, q_n) = \phi(q_0 z, \ldots, q_n z) - \phi(z, \ldots, z)$$

for each point $(q_0, \ldots, q_n)$ of Q^{n+1}.

Now, for any two points a and b of X and any $(n+1)$-function $\psi \in A^{n+1}(X, G)$, let us define an n-function

$$D_{a,b}\psi : Q^{n+1} \to G$$

of Q by taking

$$(D_{a,b}\psi)(q_0, \ldots, q_n) = \sum_{i=0}^{n}(-1)^i\psi(q_0 a, \ldots, q_i a, q_i b, \ldots, q_n b)$$

for each point $(q_0, \ldots, q_n)$ of Q^{n+1}. Let $E_{a,b}\psi$ denote the constant n-function of Q defined by

$$(E_{a,b}\psi)(q_0, \ldots, q_n) = (D_{a,b}\psi)(e, \ldots, e)$$

for each point $(q_0, \ldots, q_n)$ of Q^{n+1}, where e denotes the neutral element of Q. Since $\xi \in A_s^{n+1}(X, G)$. clearly we have

$$D_{a,b}\,\xi = E_{a,b}\,\xi.$$

If $n > 0$. direct calculation shows that

$$\begin{aligned} k_b^\#\phi - k_a^\#\phi &= (\delta D_{a,b}\,\phi + D_{a,b}\,\delta\phi) - (\delta E_{a,b}\,\phi + E_{a,b}\,\delta\phi) \\ &= \delta(D_{a,b}\,\phi - E_{a,b}\,\phi) + D_{a,b}\,\eta - E_{a,b}\,\eta \end{aligned} \tag{7.1}$$

since $\delta\phi = \xi + \eta$ and $D_{a,b}\,\xi = E_{a,b}\,\xi$. If $n = 0$, then we have

$$k_b^\#\phi - k_a^\#\phi = D_{a,b}\,\delta\phi - E_{a,b}\,\delta\phi = D_{a,b}\,\eta - E_{a,b}\,\eta. \tag{7.2}$$

Since η is of empty support, there exists for each point z in X, an open neighborhood U_z of z in X such that

$$\eta(x_0, \ldots, x_{n+1}) = 0$$

whenever $x_i \in U_z$ for each $i = 0, \ldots, n+1$. It follows from the simultaneous continuity of the operations of Q on X that, for each $z \in X$ and $w \in Q$ there exist an open neighborhood V_w of z in X and an open neighborhood W_w of w in Q such that

$$W_w V_w \subset U_{wz}.$$

Since Q is compact. there are a finite number of points $w_1, \ldots, w_m$ such that the open sets

$$\mathfrak{W}_z = \{W_{w_1}, \ldots, W_{w_m}\}$$

form an open covering of Q. Call

$$V_z = V_{w_1} \cap \ldots \cap V_{w_m}.$$

Then V_z is an open neighborhood of z in X.

Now let a and b be any two points in V_z. We are going to show that both $D_{a,a}\eta$ and $E_{a,b}\eta$ are of empty supports. Let q be an arbitrary point in Q. Choose an open set W_{w_j} from the covering $\mathfrak{W}_z$ which contains q. Then we have

$$W_{w_j} V_z \subset U_{w_j} z.$$

Let $(q_0, \ldots, q_n]$ be any point of Q^{n+1} such that $q_i \in W_{w_j}$ for each $i = 0, \ldots, n$. Then the points

$$q_0 a, \ldots, q_n a, q_0 b, \ldots, q_n b$$

are all contained in $U_{w_j} z$. Hence we have

$$(D_{a,b}\eta)(q_0, \ldots, q_n) = \sum_{i=0}^{n}(-1)^i \eta(q_0 a, \ldots, q_i a, q_i b, \ldots, q_n b) = 0.$$

This proves that q is not in the support of $D_{a,b}\eta$. Since q is arbitrary, the support of $D_{a,b}\eta$ must be empty. This implies that

$$(E_{a,b}\eta)(q_0, \ldots, q_n) = (D_{a,b}\eta)(e, \ldots, e) = 0$$

for every point $(q_0, \ldots, q_n)$ of Q^{n+1}. That is to say, $E_{a,b}\eta = 0$ and hence $E_{a,b}\eta$ is of empty support. Then it follows from (7.1) and (7.2) that

$$k_a^*(\alpha) = k_b^*(\alpha). \tag{7.3}$$

Since x and y are contained in a compact connected subset K of X, there exist a finite number of points $z_1, \ldots, z_r$ of X such that $x \in V_{z_1}$, $y \in V_{z_r}$, and the intersection $V_{z_i} \cap V_{z_{i+1}}$ is nonvoid for every $i = 1, \ldots, r-1$. Choose a point t_i from $V_{z_i} \cap V_{z_{i+1}}$ for each $i = 1, \ldots, r-1$ and call $t_0 = x$ $t_r = y$. Thus we obtain a finite sequence of points

$$x = t_0, t_1, \ldots, t_{r-1}, t_r = y$$

such that V_{z_i} contains t_{i-1} and t_i for each $i = 1, \ldots, r$. By (7.3), this implies that

$$k_{t_{i-1}}^*(\alpha) = k_{t_i}^*(\alpha)$$

for each $i=1, \ldots, r$. Hence we obtain $k_x^*(\alpha)=k_y^*(\alpha)$. This completes the proof of Theorem IV.

A topological space X is said to be *compactly connected* if every pair of points x and y *of* X are contained in some compact connected subset of X. Compact connected spaces and arcwise connected spaces are examples of compactly connected spaces.

The following theorem is an immediate consequence of Theorem IV.

THEOREM V. *If a compact transformation group Q operates on a compactly connected topolcgical space X, then the canonical homomorphism k_x^* does not depend on the choice of the basic point $x \in X$ and hence it may be denoted by*

$$k^*: H_w^n(X, G) \to H^n(Q, G).$$

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