ON THE PRODUCT OF $L(1, \chi)$

TIKAO TATUZAWA

Let $k(\geqq 3)$ be a positive integer and $\varphi(k)$ be the Euler function. We denote by χ one of the $\varphi(k)$ characters formed with modulus k, and by χ_0 the principal character. Let $L(s, \chi)$ be the L-series corresponding to χ. Throughout the paper we use c and $c(\varepsilon)$ to denote respectively an absolute positive constant and a positive constant depending on parameter $\varepsilon(>0)$ alone, not necessarily the same at their various occurrences. We use the symbol $Y = O(X)$ for positive X when there exists a c satisfying $|Y| \leqq cX$ in the full domain of existence of X and Y.

It is well known that

$$L(1, \chi) = O(\log k), \quad \text{for} \quad \chi \neq \chi_0. \tag{1}$$

On the other hand, we know from [7] and [5] (numbers in square brackets refer to the references at the end of the paper) that

$$L(1, \chi)^{-1} = O(\log k), \quad \text{for} \quad \chi \neq \chi_0, \tag{2}$$

with one possible exception, and if such a exceptional character exists, it is a real one. Let us denote it by χ_1. If there exists no exceptional character, we take any non-principal character as χ_1. Then, by Siegel's theorem (see [3] and [8]),

$$c(\varepsilon)k^{-\varepsilon} < |L(1, \chi_1)| \tag{3}$$

for any positive ε.

The object of this paper is to estimate $\prod_{\chi \neq \chi_0} L(1, \chi)$ and $\prod_{\chi(-1) = -1} L(1, \chi)$ as precisely as possible, and make some additions to the results of R. Brauer [2], Ankeny and Chowla [1].

1. In what follows, we denote by p and p_ν the primes.

LEMMA 1. $$\sum_{p \leqq x} p^{-1} = O(\log\log x), \quad \text{for} \quad x \geqq 3.$$

LEMMA 2. $$c(\log x)^{-1} < \prod_{7 \leqq p \leqq x} (1 - p^{-1}) < c(\log x)^{-1}.$$

These are obtained by Theorem 7 of [4].

Let $7 \leqq p_1 < p_2 < \ldots < p_m$. We denote by $F(x; k, l; p_1, p_2, \ldots, p_m)$ the

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number of z satisfying

$$0<z\leqq x,\quad z\equiv l \pmod{k},\quad \text{and}\quad p_\nu \nmid z \quad \text{for}\quad \nu=1, 2, \ldots, m.$$

If we take positive integers $m=m_0, m_1, \ldots, m_{h-1}$ such that

$$m=m_0>m_1>m_2>\ldots>m_{h-1}>m_h=0,$$

$$L_\nu=\prod_{m_{\nu-1}\geqq r>m_\nu}(1-p_r^{-1})\geqq\frac{5}{4},\quad \text{for}\quad \nu=1, 2, \ldots, h,$$

$$(4)\qquad (1-p_{m_\nu}^{-1})L_\nu<\frac{4}{5},\quad \text{for}\quad \nu=1, 2, \ldots, h-1,$$

then we have

LEMMA 3. $F(x; k, l; p_1, p_2, \ldots, p_m)<2xk^{-1}\prod_{\nu=1}^{h}L_\nu+\prod_{\nu=0}^{h-1}(2m_\nu)^2$, for $(k, l)=1$.

The proof is similar to that of Theorem 79 and Theorem 86 of [6].

LEMMA 4. If $x=2\varphi(k)$, then

$$\pi(x; k, l)<c\frac{x}{\varphi(k)\log(x/\varphi(k))},$$

for $(k, l)=1$, where $\pi(x; k, l)$ denotes the number of primes p satisfying $p\leqq x$ and $p\equiv l \pmod{k}$.

Suppose that $x\geqq 7^a$, $a(>1)$ being a positive number to be determined later. We arrange all primes between 7 and $\sqrt[a]{x}$ except the prime factors of k such that

$$7\leqq p_1<p_2<\ldots<p_m\leqq\sqrt[a]{x}.$$

If we write

$$D(w, k)=\prod_{\substack{7\leqq p\leqq w\\ p\nmid k}}(1-p^{-1})^{-1},$$

then

$$(5)\qquad \prod_{\nu=1}^{h}L_\nu=\prod_{r=1}^{m}(1-p_r^{-1})=\prod_{7\leqq p\leqq\sqrt[a]{x}}(1-p^{-1})D(\sqrt[a]{x}, k).$$

By Lemma 2,

$$\prod_{r=1}^{s}(1-p_r^{-1})<c(\log p_s)^{-1}D(p_s, k)<c(\log 2s)^{-1}D(\sqrt[a]{x}, k)$$

for $s=1, 2, \ldots, m$, it follows therefore that

$$\log(2m_\nu)<cD(\sqrt[a]{x}, k)\prod_{r=1}^{m_\nu}(1-p_r^{-1})^{-1}$$

$$= cD(\sqrt[a]{x}, k)\prod_{r=1}^{m}(1-p_r^{-1})^{-1}L_1L_2 \ldots L_\nu$$

$$= c \prod_{7 \leqq p \leqq \sqrt[a]{x}} (1-p^{-1})^{-1}L_1L_2 \ldots L_\nu$$

$$(6) \qquad < (c/a)(\log x)L_1L_2 \ldots L_\nu < (c/a)(14/15)^\nu \log x$$

by (4), whence follows

$$(7) \qquad \prod_{\nu=1}^{h-1}(2\, m_\nu)^2 < x^{c/a}.$$

On the other hand

$$(8) \qquad \prod_{\nu=1}^{h} L_\nu < ca(\log x)^{-1}D(\sqrt[a]{x}, k) \leqq ca(\log x)^{-1}\prod_{p \mid k}(1-p^{-1})^{-1}$$

$$= ca(\log x)^{-1}k\varphi(k)^{-1},$$

by (5) and Lemma 2.

Inserting (7) and (8) for the right of Lemma 3,
we obtain

$$F(x\,;\, k,\, l\,;\, p_1, p_2, \ldots, p_m) < ca\frac{x}{\varphi(k)\log x} + x^{c/a}$$

for $(k, l) = 1$, provided that $x \geqq 7^a$. This, together with the inequality

$$\pi(x\,;\, k,\, l) \leqq F(x\,;\, k,\, l\,;\, p_1, p_2, \ldots, p_m) + m,$$

gives

$$(9) \qquad \pi(x\,;\, k,\, l) < ca\frac{x}{\varphi(k)\log x} + 2\, x^{c/a}$$

by (6), where we may suppose that the constants c in both terms of the right are the same and $c > 1$.

Suppose first that $x \geqq 7^{2c}\varphi(k)$. Then we have

$$(10) \qquad \Delta = \frac{x}{\varphi(k)} \Big/ \log\left(\frac{x}{\varphi(k)}\right) \geqq \sqrt{\frac{x}{\varphi(k)}} \geqq 7^c.$$

we can easily verify from (10) that the restriction $x \geqq 7^a$ is satisfied, if we put

$$a = \frac{c\log x}{\log \Delta}.$$

Inserting this in (9), and using the first inequality of (10), we get

$$\pi(x\,;\, k,\, l) < c^2\frac{x}{\varphi(k)\log \Delta} + 2\,\Delta < 2\,(c^2+1)\Delta,$$

which is just what the Lemma claims.

Next we consider the trivial case, $2\,\varphi(k) \leqq x < 7^{2c}\varphi(k)$. Then

$$\pi(x;\,k,\,l) < \left[\frac{x}{k}\right] + 1 \leqq \frac{x}{k} + 1 < \frac{2x}{\varphi(k)} < 4c(\log 7)\Delta.$$

Thus the Lemma is completely proved.

2. We write for simplicity

$$Q(x) = \sum_{\chi \neq \chi_0,\, \chi_1} \sum_{n \leqq x} \chi(n)\Lambda(n),$$

where $\Lambda(n)$ is $\log p$ if n is a positive power of a prime p and is 0 otherwise.

LEMMA 5. If $k \leqq \exp(\sqrt{\log x})$ and $\chi \neq \chi_0, \chi_1$, then

$$\sum_{n \leqq x} \chi(n)\Lambda(n) = O(x \exp(-c\sqrt{\log x})).$$

This is the result of Page [7].

LEMMA 6. If $x \geqq \exp(\log k)^3$, then $Q(x) = O(x(\log x)^{-1})$.

By Lemma 5,

$$\begin{aligned} Q(x) &= O(kx \exp(-c\sqrt{\log x})) = O(x \exp(\log k - c\sqrt{\log x})) \\ &= O(x \exp((\log x)^{\frac{1}{3}} - c(\log x)^{\frac{1}{2}})) = O(x(\log x)^{-1}), \end{aligned}$$

since $\log k \leqq \sqrt[3]{(\log x)}$.

3. Let a, b and n be positive integers.

LEMMA 7. If $(a, k) = 1$, then the number of solutions of

$$x^n \equiv a \pmod{k}$$

is at most $n^{\omega(k)+1}$ where $\omega(k)$ denotes the number of prime factors of k.

This follows from the fact that the number of solutions of $x^n \equiv a \pmod{p^b}$, $(a, p) = 1$, is at most n if p is an odd prime, and n^2 if $p = 2$.

LEMMA 8. $$\omega(k) = O(\log k(\log\log k)^{-1}).$$

Suppose that

$$\omega(k) = r,\ k = p_1^{b_1} p_2^{b_2} \ldots p_r^{b_r} \quad \text{and} \quad p_1 < p_2 < \ldots < p_r.$$

By the prime number theorem,

$$cr \log p_r' < \sum_{\nu=1}^{r} \log p_\nu' \leqq \sum_{\nu=1}^{r} \log p_\nu \leqq \log k, \tag{11}$$

where p_r' denotes the r-th prime. If $r > \log k(\log\log k)^{-1}$, then

$$p_r' > cr \log r > c \log k.$$

This combined with (11) gives $r < c(\log k)(\log\log k)^{-1}$.

LEMMA 9. $$\sum_{n \geqq 2} n^{-1} \sum_{\substack{p \leqq k \\ p^n \equiv 1 \pmod{k}}} p^{-n} = O(\omega(k)k^{-1}).$$

By Lemma 7

$$\sum_{n\geqq 2} n^{-1}\sum_{\substack{p\leqq k\\ p^n\equiv 1 \pmod k}} p^{-n} = O\Big(\sum_{n\geqq 2} n^{-1} \sum_{(n^{\omega(k)}+1)} (km+1)^{-1}\Big), \tag{12}$$

where $\sum_{(*)}$ means that the number of terms of the sum is $\leqq *$. If we put

$$a_0 = 0, \quad a_n = \sum_{\nu=1}^{n} \nu^{\omega(k)+1}$$

for $n = 1, 2, \ldots$. then

$$\sum_{n\geqq 2} n^{-1} \sum_{(n^{\omega(k)}+1)} m^{-1} = O\Big(\sum_{n\geqq 1} n^{-1} \sum_{a_{n-1}<m\leqq a_n} m^{-1}\Big).$$

Since

$$\begin{aligned}\sum_{a_{n-1}<m\leqq a_n} m^{-1} &< \log(a_n/a_{n-1})\\ &< \log\Big(\int_0^{n+1} x^{\omega(k)+1}dx \Big/ \int_0^{n-1} x^{\omega(k)+1}dx\Big)\\ &= (\omega(k)+2)\log((n+1)/(n-1)) = O(\omega(k)n^{-1})\end{aligned}$$

for $n\geqq 2$, it follows from (12) that

$$\sum_{n\geqq 2} n^{-1}\sum_{\substack{p\leqq k\\ p^n\equiv 1 \pmod k}} p^{-n} = O\Big(\omega(k)k^{-1}\sum_{n\geqq 1} n^{-2}\Big) = O(\omega(k)k^{-1}).$$

4. We write

$$\begin{aligned}\prod_{\chi\neq\chi_0,\,\chi_1} L(1,\chi) &= \exp\Big(\sum_{\chi\neq\chi_0,\,\chi_1}\sum_{n\geqq 2}\chi(n)\Lambda(n)(n\log n)^{-1}\Big)\\ &= \exp\big(\textstyle\sum_1+\sum_2+\sum_3+\sum_4+\sum_5+\sum_6\big), \text{ say,}\end{aligned}$$

where

$$\begin{aligned}\textstyle\sum_1 &= \sum_{p<A} p^{-1}\Big(\sum_{\chi}\chi(p)\Big), \quad \textstyle\sum_2 = \sum_{A\leqq p<B} p^{-1}\Big(\sum_{\chi}\chi(p)\Big),\\ \textstyle\sum_3 &= -\sum_{p<B} p^{-1}(\chi_0(p)+\chi_1(p)),\\ \textstyle\sum_4 &= \sum_{p^n<B}\sum_{n\geqq 2}(np^n)^{-1}\Big(\sum_{\chi}\chi(p^n)\Big),\\ \textstyle\sum_5 &= -\sum_{p^n<B}\sum_{n\geqq 2}(np^n)^{-1}(\chi_0(p^n)+\chi_1(p^n)),\\ \textstyle\sum_6 &= \sum_{B\leqq n}\sum_{\chi\neq\chi_0,\,\chi_1}\chi(n)\Lambda(n)(n\log n)^{-1}\end{aligned}$$

and

$$A = 2k, \quad B = [\exp(\log k)^3].$$

Now we shall estimate $\sum_\nu$ using the above lemmas.

$$\textstyle\sum_1 = \varphi(k)\sum_{\substack{p<A\\ p\equiv 1 \pmod k}} p^{-1} \leqq \varphi(k)\sum_{mk+1<A}(mk+1)^{-1} = O(1).$$

$$\sum\nolimits_2 = \varphi(k) \sum_{\substack{A \leqq p < B \\ p \equiv 1 \pmod k}} p^{-1} = \varphi(k) \sum_{A \leqq n < B} (\pi(n;\ k,1) - \pi(n-1;\ k,1))n^{-1}$$

$$= \varphi(k)\Big(\sum_{A \leqq n < B} \pi(n;\ k,1)(n^{-1} - (n+1)^{-1}) - \pi(A-1;\ k,1)A^{-1}$$

$$+ \pi(B-1;\ k,1)B^{-1}\Big)$$

$$= O\Big(\sum_{A \leqq n < B} (n+1)^{-1}\Big(\log \frac{n}{\varphi(k)}\Big)^{-1} + O(1)\Big) \quad \text{(by Lemma 4)}$$

$$= O\Big(\int_A^B \frac{du}{u \log(u/\varphi(k))}\Big) + O(1) = O(\log\log k).$$

$$\sum\nolimits_3 = O\big(\sum_{p<B} p^{-1}\big) = O(\log\log B) = O(\log\log k). \quad \text{(by Lemma 1)}$$

$$\sum\nolimits_4 = \varphi(k) \sum_{\substack{p^n < B \\ p^n \equiv 1 \pmod k}} \sum_{n \geqq 2} (np^n)^{-1} \leqq \varphi(k) \sum_{\substack{p \\ p^n \equiv 1 \pmod k}} \sum_{n \geqq 2} (np^n)^{-1}$$

$$= \varphi(k) \sum_{\substack{p>k \\ p^n \equiv 1 \pmod k}} \sum_{n \geqq 2} (np^n)^{-1} + \varphi(k) \sum_{n \geqq 2} n^{-1} \sum_{\substack{p \leqq k \\ p^n \equiv 1 \pmod k}} p^{-n}$$

$$= O\big(\varphi(k) \sum_{p>k} (p(p-1))^{-1}\big) + O(\omega(k)) \quad \text{(by Lemma 9)}$$

$$= O(\omega(k)).$$

$$\sum\nolimits_5 = O\big(\sum_p \sum_{n \geqq 2} (np^n)^{-1}\big) = O\big(\sum_p (p(p-1))^{-1}\big) = O(1).$$

$$\sum\nolimits_6 = \sum_{B \leqq n} (Q(n) - Q(n-1))(n \log n)^{-1}$$

$$= \sum_{B \leqq n} Q(n)((n\log n)^{-1} - ((n+1)\log(n+1))^{-1})$$

$$- Q(B-1)(B \log B)^{-1}$$

$$= O\Big(\sum_{B \leqq n} \frac{n}{\log n} \int_n^{n+1} \frac{\log x + 1}{x^2 \log^2 x} dx + \frac{1}{\log^2 B}\Big) \quad \text{(by Lemma 6)}$$

$$= O\Big(\int_B^\infty \frac{dx}{x \log^2 x}\Big) = O(1).$$

Collecting these results, we obtain

$$\exp(-c(\log\log k + \omega(k))) < \prod_{\chi \neq \chi_0, \chi_1} L(1,\chi) < \exp(c(\log\log k + \omega(k))).$$

This, combined with (1), (3) and Lemma 8, gives the following

THEOREM 1. $c(\varepsilon)k^{-\varepsilon} < |\prod_{\chi \neq \chi_0} L(1,\chi)| < \exp(c(\log\log k + \omega(k)))$.

In a similar way, we have

THEOREM 2. $c(\varepsilon)k^{-\varepsilon} < |\prod_{\chi(-1)=-1} L(1,\chi)| < \exp(c(\log\log k + \omega(k)))$.

5. For the cyclotomic field $\Omega = P(\zeta)$, $\Omega_0 = P(\zeta + \zeta^{-1})$, where ζ is a l-th root of unity, l being an odd prime, it is well known that

$$\prod_{\chi \neq \chi_0} L(1, \chi) = \frac{(2\pi)^m Rh}{2\, l\sqrt{|d|}} \tag{13}$$

$$\prod_{\chi \neq \chi_0,\ \chi(-1)=1} L(1, \chi) = \frac{2^m R_0 h_0}{2\sqrt{|d_0|}} \tag{14}$$

where h, h_0, are respectively the class numbers of Ω and Ω_0 and R, R_0 the regulators of them, and further $m = \frac{l-1}{2}$, $|d| = l^{l-2}$, $|d_0| = l^{m-1}$ and $R = R_0 2^{m-1}$.

Combined (13) with Theorem 1, we can infer the following

THEOREM 3. For any positive ε,

$$2\left(\frac{l}{\sqrt{2}\,\pi}\right)^{l-1} \frac{c(\varepsilon)}{l^{\varepsilon}} < Rh < 2\left(\frac{l}{\sqrt{2}\,\pi}\right)^{l-1} (\log l)^c.$$

In this special case, the result is sharper than the one obtained by R. Brauer (see [2]) for the general finite algebraic extension.

Let h_1 be the so-called first factor of the class number of Ω. Then

$$h_1 = h/h_0 = (l\sqrt{l}^{\,m}/2^{m-1}\pi^m) \prod_{\chi(-1)=-1} L(1, \chi).$$

Combined this with Theorem 2, we can infer the following

THEOREM 4. For any positive ε,

$$2\, l\left(\frac{\sqrt{l}}{2\pi}\right)^{\frac{l-1}{2}} \frac{c(\varepsilon)}{l^{\varepsilon}} < h_1 < 2\, l\left(\frac{\sqrt{l}}{2\pi}\right)^{\frac{l-1}{2}} (\log l)^c.$$

The second inequality is better than the result obtained by Ankeny and Chowla (see [1]) on the extended Riemann hypothesis.

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Gakushuin University, Tokyo