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ON TANGENTIAL PRINCIPAL CLUSTER SETS OF NORMAL MEROMORPHIC FUNCTIONS

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1. Introduction

Let $w = f(z)$ be a normal meromorphic function defined in the upper half plane $U = \{Im(z) > 0\}$. We recall that a meromorphic function $f(z)$ is *normal* in U if the family $\{f(S(z))\}$, where $z' = S(z)$ is an arbitrary one-one conformal mapping of U onto U, is normal in the sense of Montel. It is the purpose of this paper to state some results on the behavior of $f(z)$ on curves which approach a point x_0 on the real axis R with a fixed (finite) order of contact q at x_0.

We use as a definition of order of contact the following: A set $A \subset U$ will have *order of contact* $q > 0$ at $x_0 \in R$ if $\bar{A} \cap R = \{x_0\}$ (the bar denoting closure), and if there exists a positive number ρ such that

$$\limsup_{\substack{z=x+iy\to x_0\\ z\in A}} \frac{|x - x_0|^{q+1}}{y} = \rho.$$

We remark that this definition agrees with the usual definition of order of contact of a subset B of $|z| < 1$ at a point $e^{i\theta_0}$ of $\{|z| = 1\}$ (see [5, p. 168]) in the sense that if Φ is a Möbius transformation of $|z| < 1$ onto U, then $\Phi(B)$ will have order of contact q in the above sense at $x_0 = \Phi(e^{i\theta_0})$, if and only if B has order of contact q at $e^{i\theta_0}$. Both the geometry and calculations are simplified by considering U as the domain of definition of $f(z)$. For example, the hyperbolic metric in U has the form

$$\rho(z, z') = 1/2 \log \frac{1 + \chi(z, z')}{1 - \chi(z, z')} \quad z, z' \in U,$$

where $\chi(z, z') = |z - z'|/|z - \bar{z}'|$. We note that if $\{z_n\}$ and $\{z'_n\}$ are sequences of points of U, then $\rho(z_n, z'_n) \to 0$ if and only if $\chi(z_n, z'_n) \to 0$.

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2. Definitions And Notation

We shall say that an arc $A \subseteq U$ is an *admissible q-arc at* x_0 if $A \cap R = \{x_0\}$ and there exists a number ρ_A such that the limit

$$\lim_{\substack{z=x+iy \to x_0 \\ z \in A}} \frac{|x - x_0|^{q+1}}{y}$$

exists and equals ρ_A.

For any triple (α, β, δ) satisfying $0 < \alpha < \beta$, $0 < \delta < 1$, we difine the right q-angle $\nabla^+(\alpha, \beta, \delta, q)(x_0)$ as the open region of U lying between the curves $\{y = \alpha(x - x_0)^{q+1},\ x > x_0\}$, and $\{y = \beta(x - x_0)^{q+1},\ x > x_0\}$, and below the line $y = \delta$. The left q-angle, $\nabla^-(\alpha, \beta, \delta, q)(x_0)$ is the reflection of $\nabla^+(\alpha, \beta, \delta, q)(x_0)$ in the line $x = x_0$.

When we do not care to specify either a q-angle is a right q-angle or a left q-angle we simply write $\nabla(\alpha, \beta, \delta, q)(x_0)$.

We remark that a curve A is admissible if and only if there is a collapsing sequence of right or left q-angles $\{\nabla(\alpha_n, \beta_n, \delta_n, q)(x_0)\}$ having the property that $\alpha_n \nearrow \rho_A$, $\beta_n \searrow \rho_A$ and a terminal portion of A lies in each such q-angle.

We define the cluster sets of $f(z)$ at a point x_0 on the sets $A, \nabla^+(\alpha, \beta, \delta, q)(x_0)$ and $\nabla^-(\alpha, \beta, \delta, q)(x_0)$ in the usual manner and denote them by $C_A(f, x_0)$, $C_{\nabla^+(\alpha,\beta,q)}(f, x_0)$ and $C_{\nabla^-(\alpha,\beta,\delta)}(f, x_0)$.

We let

$$C_{\mathscr{A}_q}(f, x_0) = \bigcup_{0<\alpha<\beta<\infty} C_{\nabla(\alpha,\beta,q)}(f, x_0),$$

the set of all cluster values on sequences of order of contact q, and

$$\Pi_{T_q}(f, x_0) = \bigcap_A C_A(f, x_0),$$

the intersection taken over all admissible q-arcs at x_0. We let $L_q(f) = \{x \mid C_{\nabla(\alpha,\beta,q)}(f, x) = \bigcup_{0 \leqslant p \leqslant q} C_{\mathscr{A}_p}(f, x)$, for any q-angle $\nabla(\alpha, \beta, \delta, q)(x)$ at $x\}$.

Finally, let $K_q(f) = \{x \mid C_{\nabla(\alpha,\beta,q)}(f, x) = C_{\nabla(\alpha',\beta',q)}(f, x)$ for any two (right or left) q-angles at $x\}$.

We say that nearly every point of R belongs to a set A if A is a residual subset of R, i.e. if A^c is a set of Baire category I.

3. Results And Proofs

We have shown [7, Theorem 2] for $w = f(z)$ an arbitrary complex valued function in U the following property of the set $L_q(f)$.

THEOREM 1. *Let $w = f(z)$ be arbitrary in U. Then the complement of $L_q(f)$ is a set of measure* 0 *and Baire category I.*

Since $L_q(f) \subseteq K_q(f)$ for arbitrary $f(z)$ we have:

COROLLARY 1. *Let $w = f(z)$ be arbitrary in U. Then almost every and nearly every point of R belongs to $K_q(f)$.*

By Theorem 1, at a.e. and n.e. $x \in R$, all cluster values of $f(z)$ on sequences of order of contact $\leq q$ are obtained in any q-angle at x. By imposing the condition that $f(z)$ be meromorphic and normal in U we show that all such cluster values are obtained on any admissible q-arc at a.e. and n.e. $x \in R$.

THEOREM 2. *Let $w = f(z)$ be a normal meromorphic function in U. Then for every $x_0 \in L_q(f)$ we have*

$$\Pi_{T_q}(f, x_0) = \bigcup_{0 \leq p \leq q} C_{\mathscr{A}_p}(f, x_0).$$

Proof. The inclusion $\Pi_{T_q}(f, x) \subseteq \bigcup_{0 \leq p \leq q} C_{\mathscr{A}_p}(f, x_0)$ is trivial.

To show that $\bigcup_{0 \leq p \leq q} C_{\mathscr{A}_p}(f, x_0) \subseteq \Pi_{T_q}(f, x_0)$ we let ω be a point of $\bigcup_{0 \leq p \leq q} C_{\mathscr{A}_p}(f, x_0)$ and Λ an admissible q-arc at x_0. Since Λ is an admissible q-arc at x_0, there exists a positive constant ρ_Λ so that for any sequence $\{z_n^*\}$ of points of Λ which tends to x_0, we have, with $z_n^* = x_n^* + i y_n^*$

$$\lim_{n \to \infty} \frac{|x_n^* - x_0|^{q+1}}{y_n^*} = \rho_\Lambda. \tag{1}$$

By Theorem 1, we have for any q-angle $\nabla(\alpha, \beta, \delta, q)(x_0)$,

$$\omega \in C_{\nabla(\alpha, \beta, q)}(f, x_0).$$

Thus, we can select a sequence of points $\{z_n\}$ of U satisfying

1. $z_n \to x_0$
2. $z_n \in \nabla\left(\rho_\Lambda - \frac{1}{n},\ \rho_\Lambda + \frac{1}{n},\ \frac{1}{n},\ q\right)(x_0)$
3. $f(z_n) \to \omega$.

For this sequence $\{z_n\}$ as well, we have with $z_n = x_n + iy_n$,

$$\lim_{n\to\infty} \frac{|x_n - x_0|^{q+1}}{y_n} = \rho_A. \tag{2}$$

It remains for us to pick a sequence $\{z_n^*\}$ of points of Λ on which $f(z_n^*) \to \omega$. To this end, let

$$z_n^* = \Lambda \cap \{x = x_n\},$$

the last intersection being taken if there is more than one.

Now $\chi(z_n, z_n^*) = \dfrac{|z_n - z_n^*|}{|z_n - \bar{z}_n^*|} = \dfrac{|y_n - y_n^*|}{y_n + y_n^*}$

From (1) and (2) it follows that

$$\lim_{n\to\infty} \chi(z_n, z_n^*) = 0$$

so that $\rho(z_n, z_n^*) \to 0$. It follows from [4, Lemma 1, p. 10] that $f(z_n^*) \to \omega$ so that $\omega \in C_\Lambda(f, x_0)$ and our proof is complete.

There exist functions normal and holomorphic in U for which $K_q(f) - L_q(f) \neq \emptyset$. Such a function is $w = f(z) = e^{-i/z}$. For this function $0 \in K_q(f) - L_q(f)$ for $q > 1$. For such q, $\Pi_{T_q}(f, 0) = \{|w| = 1\}$ and $\bigcup_{0 \leq p \leq q} C_{\mathscr{A}_p}(f, 0) = \{|w| \leq 1\}$, so that Theorem 2 does not hold at $x_0 = 0$. However, by essentially the same methods used in Theorem 2 we can prove

THEOREM 3. *Let $w = f(z)$ be a normal meromorphic function in U. Then for every $x \in K_q(f)$ we have*

$$\Pi_{T_q}(f, x) = C_{\mathscr{A}_q}(f, x)$$

The following theorem is an extension of a theorem of Bagemihl [3, Theorem 9, p. 17], who proved it for the case $q = 1$. Here Ω denotes the extended w-plane.

THEOREM 4. *Let $w = f(z)$ be a nonconstant, normal meromorphic function in U, and assume that the set $A(f)$ of asymptotic values of $f(z)$ has harmonic measure zero. Then, at almost every and nearly every point x of R,*

$$\Pi_{T_q}(f, x) = \Omega.$$

Proof. By Tsuji's extension of Privaloff's Theorem [6, p. 72], the set of Fatou points has measure zero. Thus, by Plessner's Theorem, almost every point of R is a Plessner point of $f(z)$. By Theorem 1, and the Bagemihl approximation theorem [1], almost every and nearly every point x of R is both a Plessner point and a point of $L_q(f)$, so that by Theorem 2,

$$\Pi_{T_q}(f, x) \supseteq C_{\mathscr{A}_0}(f, x) = \Omega, \text{ a.e. and n.e.}$$

In conclusion, we remark that the condition of normality cannot be removed in Theorem 2 or Theorem 3. F. Bagemihl [3, p. 12] has constructed a holomorphic function $w = f(z)$ for which

$$\Pi_{T_1}(f, x) \subset C_{\mathscr{A}_0}(f, x) \text{ a.e. and n.e.}$$

This function clearly fails to satisfy the conclusion of Theorem 2. Since, by Theorem 1, for arbitrary $f(z)$ we have $C_{\mathscr{A}_0}(f, x) \subseteq C_{\mathscr{A}_1}(f, x)$ a.e. and n.e., we also have

$$\Pi_{T_1}(f, x) \subset C_{\mathscr{A}_1}(f, x) \text{ a.e. and n.e.,}$$

so that the conclusion of Theorem 3 is violated as well.

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