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ON THE STRUCTURE OF SPLITTING FIELDS OF STATIONARY GAUSSIAN PROCESSES WITH FINITE MULTIPLE MARKOVIAN PROPERTY

Dedicated to the late Machiko Okabe

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§ 1. Introduction

Let $X = (X(t)\,; t \in \boldsymbol{R})$ be a real stationary mean continuous Gaussian process with expectation zero which is purely nondeterministic. In this paper we shall investigate the structure of splitting fields of X having finite multiple Markovian property using the results in [6]. We follow the notations and terminologies in [6].

We shall remember three kinds of definitions of the N-ple Markovian property ($N \in \boldsymbol{N}$).

DEFINITION 1.1. We say that X has the N-ple Markovian property in the broad sense if the splitting field $\boldsymbol{F}_X^{+/-}(t)$ is generated by N linearly independent random variables in $\boldsymbol{M}$ for any $t \in \boldsymbol{R}$.

It is known that X has the N-ple Markovian property in the broad sense if and only if X has a rational spectral density of degree $2N$ ([1], [5]).

DEFINITION 1.2. We say that X has the N-ple Markovian property in the narrow sense if X has the N-ple Markovian property in the broad sense and $\boldsymbol{F}_X^{+/-}(t)$ is equal to the germ field $\partial \boldsymbol{F}_X(t)$ for any $t \in \boldsymbol{R}$.

It is also known that X has the N-ple Markovian property in the narrow sense if and only if its spectral density is the reciprocal of a polynomial of degree $2N$ ([1], [5], [6]).

The third definition is

DEFINITION 1.3. We say that X has the N-ple Markovian property in the sense of T. Hida if, for any $N + 2$ real numbers $t_0 < t_1 < \cdots < t_{N+1}$,

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$\{E(X(t_n) | F_X^-(t_0)) ; 1 \leq n \leq N\}$ is linearly independent and $\{E(X(t_n) | F_X^-(t_0)) ; 1 \leq n \leq N+1\}$ is linearly dependent.

It is shown in [3] that, if X has the N-ple Markovian property in the sense of T. Hida, X has a rational spectral density of degree $2N$.

In this paper we shall consider the case where X has the N-ple Markovian property in the broad sense.

In § 2 we shall give a formula for the canonical representation kernel of our process X (Theorem 2.1). In the proof of Theorem 2.1 we shall use Theorem 8.1 in [6], which gives a formula for the canonical representation kernel of process X having the Markovian property. By the Markovian property we mean that X satisfies $F_X^{+/-}(t) = \partial F_X(t)$ for any $t \in R$ ([5], [6]).

In § 3 we shall construct an N-dimensional stationary Gaussian process $\mathscr{X} = (\mathscr{X}(t) ; t \in R)$ satisfying

(1.1) {the n-th component of $\mathscr{X}(t) ; 1 \leq n \leq N$} is lineary independent in M and

(1.2) $F_X^{+/-}(t) = F_{\mathscr{X}}^{+/-}(t) = \sigma(\mathscr{X}(t))$ for any $t \in R$ (Theorems 3.2 (ii) and 3.3). We can give an expression of the linear predictor of $X(t)$ $(t > 0)$ using the past $F_X^-(0)$ in terms of the process $\mathscr{X}$ (Theorem 3.2 (i)). The relation (1.2) implies that $\mathscr{X}$ has a simple Markovian property.

In § 4 we shall investigate the structure of $\mathscr{X}$ from the point of view of Markov processes, and show that a Markov process $(\mathscr{X}(t), P(\cdot | \mathscr{X}(0) = x) ; t > 0, x \in R^N)$ is a recurrent Gaussian diffusion process with transition probability density and has a unique invariant measure (Theorem 4.3).

We shall prove in § 5 that the N-dimensional stationary Gaussian process $\mathscr{X}$ satisfying (1.1) and (1.2) is unique up to multiplicative nonsingular $N \times N$-matirices (Theorem 5.1).

In § 6 we shall define a nonsingular $N \times N$-matrix T and an associated N-dimensional stationary Gaussian process $\mathscr{Y} = (\mathscr{Y}(t) ; t \in R)) = (T^{-1}\mathscr{X}(t) ; t \in R)$. We note that the matrix T can be definitely expressed in terms of the spectral density of X. Then we shall prove that the N-th component process of $\mathscr{Y}$ $(= Y)$ has the N-ple Markovian property in the narrow sense and satisfies

$$F_X^{+/-}(t) = F_Y^{+/-}(t) = \partial F_Y(t) \qquad (t \in R) \tag{1.3}$$

(Theorem 6.2). We can also give an alternative expression of the linear predictor of $X(t)$ $(t > 0)$ using the past $F_X^-(0)$ in terms of the process $\mathscr{Y}$ (Theorem 6.3 (i)).

Finally in § 7 we shall give three applications of our results. At first we shall characterize the Markovian property of stationary Gaussian processes from the point of view of representations and then give a necessary and sufficient condition for the N-ple Markovian property in the sense of T. Hida (Theorems 7.1 and 7.2). Next we shall characterize the linear predictor of $X(t)$ $(t > 0)$ using the past $F_X^-(0)$ as a unique solution of an initial value problem of a differential equation, which is derived from the spectral density of $\boldsymbol{X}$. As the third application, we shall give an expression of nonlinear predictors of $X(t)$ $(t > 0)$ using the past $F_X^-(0)$ in terms of the Gaussian diffusion process $(\mathscr{X}(t), P(\cdot\,|\,\mathscr{X}(0) = x)\,;\, t > 0,$ $x \in \boldsymbol{R}^N)$ defined in § 4 (Theorem 7.4).

§ 2. Rational weights

Let N be a positive integer and let $\Delta = \Delta(\lambda)$ be a rational function of degree $2N$ which is nonnegative, symmetric and integrable. Then we have the following decomposition:

$$(2.1)\qquad \begin{cases} \Delta(\lambda) = \left|\dfrac{Q(-\lambda)}{P(-\lambda)}\right|^2 \qquad (\lambda \in \boldsymbol{R})\,, \\ V_P = \boldsymbol{C}^+\,, \quad V_Q \subset \boldsymbol{C}^+ \cup \boldsymbol{R}\,, \quad V_P \cap V_Q = \phi \quad \text{and} \\ Q(z) = \sum\limits_{n=0}^{N-1} b_n(-iz)^n\,, \quad P(z) = \sum\limits_{n=0}^{N} c_n(-iz)^n\,, \quad b_n,\, c_n \in \boldsymbol{R},\, c_N \neq 0\,, \end{cases}$$

where V_S denotes the set of zero points of a polynomial S. Such a decomposition is unique up to multiplicative constants of absolute one.

2.1. We denote by F the Fourier transform of the reciprocal of a function $P(-\cdot)$ in (2.1):

$$(2.2)\qquad F = (P(-\cdot)^{-1})^{\wedge}\,.$$

It is easy to see that $F = 0$ in $(-\infty, 0)$ and $F^{(n)} \in \mathscr{A}((0,\infty)) \cap L^2((0,\infty))$ $(n = 0, 1, 2, \cdots)$. By Lemmas 8.5, 8.6 (ii) and Proposition 8.1 in [6] we have

LEMMA 2.1. (i) $F^{(n)}(0+) = 0$ $(0 \le n \le N-1)$, $F^{(N-1)}(0+) = 2\pi(-1)^N c_N^{-1}$,
(ii) $F^{(n)} \in L^2(\boldsymbol{R})$ $(0 \le n \le N-1)$ *(distribution derivatives),*
(iii) $\{F^{(n)}\,;\, 0 \le n \le N-1\}$ *is linearly independent in* $L^2(\boldsymbol{R})$.

We define for any $n \in \{0, 1, \cdots, N-1\}$ an L^2-function F_n by

$$F_n(t) = \begin{cases} (2\pi)^{-1} \sum_{k=0}^{N-n-1} c_{n+k+1}(-1)^{k+1} F^{(k)}(t) & (t > 0)\,, \\ 0 & (t \leq 0)\,. \end{cases} \tag{2.3}$$

In particular we have

$$F_{N-1} = (-2\pi)^{-1} c_N F\,. \tag{2.4}$$

Then it follows from Lemmas 8.2, 8.3 and Proposition 8.1 in [6] that

LEMMA 2.2. (i) $F_0(0+) = 1$, $F_n(0+) = 0$ $(1 \leq n \leq N-1)$,
(ii) $F_n = (2\pi)^{-1} \sum_{k=0}^{N-n-1} c_{n+k+1}(-1)^{k+1} F^{(k)}$ $(1 \leq n \leq N-1)$,
(iii) $F_0^{(1)} = \delta - (2\pi)^{-1} c_0 F$, $F_n^{(1)} = -F_{n-1} - (2\pi)^{-1} c_n F$ $(1 \leq n \leq N-1)$,
(iv) *$\{F_n ; 0 \leq n \leq N-1\}$ is linearly independent in $L^2(\boldsymbol{R})$.*

Furthermore it follows from Theorem 8.1 in [6] that

LEMMA 2.3. *For any $s \in (-\infty, 0)$, $t \in (0, \infty)$ and $n \in \{0, \cdots, N-1\}$,*
(i) $F(t-s) = \sum_{n=0}^{N-1} (-1)^n F^{(n)}(t) F_n(-s)$,
(ii) $F_n(t-s) = (2\pi)^{-1} \sum_{\ell=0}^{N-1} (-1)^{\ell} (\sum_{m=0}^{N-n-1} c_{n+m+1}(-1)^{m+1} F^{(\ell+m)}(t)) F_\ell(-s)$.

By using Lemmas 2.1 (i), 2.2 (i) and 2.2 (iii), we can show

LEMMA 2.4. $F_n^{(m)}(0+) = (-1)^n \delta_{mn}$ $(0 \leq m, n \leq N-1)$.

Next we shall prove

LEMMA 2.5. *There exist N positive numbers $t_0 < t_1 < \cdots < t_{N-1}$ such that* $\det (F^{(m)}(t_n))_{0 \leq m,n \leq N-1} \neq 0$.

Proof. Assume that $\det (F^{(m)}(t_n)) = 0$ for any N positive numbers $t_0 < t_1 < \cdots < t_{N-1}$. Differentiating it n times with respect to t_n for each $n \in \{0, 1, \cdots, N-1\}$ and then letting $t_0 < t_1 < \cdots < t_{N-1}$ tend to zero, we see from Lemma 2.1 (i) that

$$\det \begin{pmatrix} 0 & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & * \\ 1 & & & \end{pmatrix} = 0\,.$$

This is absurd. Therefore we have the desired result. (Q.E.D.)

Finally in subsection 2.1 we shall show

LEMMA 2.6. *The following* (i) *and* (ii) *are equivalent*:
(i) $\det(F^{(m)}(t_n))_{0\leq m,n\leq N-1} \neq 0$ *for any* N *positive numbers* $t_0 < t_1 < \cdots < t_{N-1}$;
(ii) $V_P \subset \{z \in \boldsymbol{C}^+ ; \operatorname{Re} z = 0\}$.

Proof. We decompose $P(z) = d_1 \prod_{n=0}^{N-1} (\lambda_n + iz)$, where d_1 is a constant and $\operatorname{Re}\lambda_n > 0$ $(0 \leq n \leq N-1)$. Denoting by f_n the Fourier transform of $(\lambda_n - i\cdot)^{-1}$ $(0 \leq n \leq N-1)$, we find that $f_n(t) = 2\pi(\operatorname{Re}(\lambda_n))^{-1}e^{-\lambda_n t} (t > 0)$, $f_n(t) = 0$ $(t < 0)$ and $F = d_2 f_0 * f_1 * \cdots * f_{N-1}$ with some constant d_2. At first we assume that (ii) holds and so $\lambda_n \in \boldsymbol{R}$ $(0 \leq n \leq N-1)$. We define $N+1$ functions v_n in $\mathscr{A}((0,\infty))$ $(0 \leq n \leq N)$ by

$$\begin{cases} v_0(t) = d_1^{-1} e^{\lambda_0 t} , \\ v_n(t) = e^{(\lambda_n - \lambda_{n-1})t} \qquad (1 \leq n \leq N-1) , \\ v_N(t) = e^{-\lambda_{N-1} t} \end{cases}$$

and then N functions G_n in $\mathscr{A}((0,\infty))$ $(1 \leq n \leq N)$ by

$$G_n(t) = v_N(t) \int_0^t v_{N-1}(t_1)dt_1 \int_0^{t_1} v_{N-2}(t_2)dt_2 \cdots \int_0^{t_{N-n}} v_{n-1}(t_{N-n+1})dt_{N-n+1} .$$

It may be easily seen that $P\left(\frac{1}{i}\frac{d}{dt}\right)G_n = 0$ in $(0,\infty)$ $(1 \leq n \leq N)$. Since v_n's are positive, we can apply (II, 30) in [3] to get that $\det(G_m(t_n)) \neq 0$ for any N positive numbers $t_0 < t_1 < \cdots < t_{N-1}$. Since $P\left(\frac{1}{i}\frac{d}{dt}\right)F_n = 0$ in $(0,\infty)$ $(0 \leq n \leq N-1)$, we see from Lemma 2.1 (iii) that there exists a nonsingular $N \times N$-matrix C satisfying $(F^{(m)}(t_n)) = C(G_m(t_n))$ and so (i) holds. Next let's assume that (ii) does not hold. Since $\overline{P(\lambda)} = P(-\lambda)$ $(\lambda \in \boldsymbol{R})$, we then may assume and do that $\lambda_0 \notin \boldsymbol{R}$ and $\lambda_1 = -\overline{\lambda_0}$. By an easy calculation it is shown that $f \equiv f_0 * f_1$ is equal to $d_3 \sin(\operatorname{Re}\lambda_0 \cdot t)e^{-iIm\lambda_0 \cdot t}$ in $(0,\infty)$ for some constant d_3. Since $f_2 * f_3 * \cdots * f_{N-1}$ is a fundamental solution of a differential operator $S\left(\frac{1}{i}\frac{d}{dt}\right)$ of order $N-2$ with constant coefficients, we find that $S\left(\frac{1}{i}\frac{d}{dt}\right)F = d_1 f$. This implies that, for any N positive numbers $t_0 < t_1 < \cdots < t_{N-1}$,

$$\det(F^{(m)}(t_n)) = d_4 \det \begin{bmatrix} F(t_0) \cdots\cdots\cdots & F(t_{N-1}) \\ \vdots & \vdots \\ F^{(N-3)}(t_0) \cdots & F^{(N-3)}(t_{N-1}) \\ f(t_0) \cdots\cdots\cdots & f(t_{N-1}) \\ F^{(N-1)}(t_0) \cdots & F^{(N-1)}(t_{N-1}) \end{bmatrix},$$

where d_4 is a constant. Since $f(n\pi(\mathrm{Re}\,\lambda_0)^{-1}) = 0$ $(n \in \boldsymbol{N})$, we find that (i) does not hold. Thus we have proved Lemma 2.6. (Q.E.D.)

2.2. We denote by E the Fourier transform of a function $P(-\cdot)^{-1}Q(-\cdot)$:

$$E = (P(-\cdot)^{-1}Q(-\cdot))^{\wedge} . \tag{2.5}$$

By (2.2) we have

$$E = Q\left(\frac{1}{i}\frac{d}{dt}\right)F . \tag{2.6}$$

We define for any $n \in \{0, 1, \cdots, N-1\}$ an L^2-function E_n by

$$E_n(t) \equiv \begin{cases} Q\left(\frac{1}{i}\frac{d}{dt}\right)F_n(t) & (t > 0) , \\ 0 & (t \leq 0) . \end{cases} \tag{2.7}$$

In particular we see from (2.4) and (2.6) that

$$E_{N-1} = (-2\pi)^{-1}c_N E . \tag{2.8}$$

Immediately from Lemma 2.3 and (2.7) we have

THEOREM 2.1. *For any $s \in (-\infty, 0)$, $t \in (0, \infty)$ and $n \in \{0, 1, \cdots, N-1\}$,*
(i) $E(t-s) = \sum_{n=0}^{N-1}(-1)^n F^{(n)}(t)E_n(-s)$,
(ii) $E_n(t-s) = (2\pi)^{-1}\sum_{\ell=0}^{N-1}(-1)^{\ell}(\sum_{k=0}^{N-n-1}(-1)^{k+1}c_{n+k+1}F^{(k+\ell)}(t))E_{\ell}(-s)$.

Moreover it follows from Lemmas 2.2 (iii) and 2.4 that

LEMMA 2.7. (i) $E_n(0+) = b_n$ $(0 \leq n \leq N-1)$,
(ii) $E_0'(t) = (-2\pi)^{-1}c_0E(t)$, $E_n'(t) = -E_{n-1}(t) - (2\pi)^{-1}c_nE(t)$ $(t > 0, 1 \leq n \leq N-1)$.

Finally we shall prove

LEMMA 2.8. $\{E_n : 0 \leq n \leq N-1\}$ *is linearly independent in* $L^2(\boldsymbol{R})$.

Proof. Let α_n $(0 \leq n \leq N-1)$ be real constants such that $\sum_{n=0}^{N-1}\alpha_nE_n$

$= 0$. We then see from (2.7) that $Q\left(\frac{1}{i}\frac{d}{dt}\right)(\sum_{n=0}^{N-1}\alpha_n F_n) = 0$ in $\boldsymbol{R} - \{0\}$ in the sense of distributions. Therefore, there exists a polynomial Q_1 such that $Q\left(\frac{1}{i}\frac{d}{dt}\right)(\sum_{n=0}^{N-1}\alpha_n F_n) = Q_1\left(\frac{1}{i}\frac{d}{dt}\right)\delta$. By taking the inverse Fourier transform of both sides, we find that $Q(-\lambda)(\sum_{n=0}^{N-1}\alpha_n\tilde{F}_n(\lambda)) = Q_1(-\lambda)$ $(\lambda \in \boldsymbol{R})$. Since Lemma 2.2 (ii) implies that $\tilde{F}_n(\lambda) = (-2\pi)^{-1}(\sum_{m=0}^{N-n-1} c_{n+m+1}(i\lambda)^m)P(-\lambda)^{-1}(\lambda \in \boldsymbol{R})$, there exists a polynomial Q_2 of at most degree $N-1$ such that $Q(\lambda)Q_2(\lambda)P(\lambda)^{-1} = Q_1(\lambda)$ $(\lambda \in \boldsymbol{R})$. Hence we see from (2.1) that $Q_2 = 0$ and so $Q_1 = 0$. This implies that $\sum_{n=0}^{N-1}\alpha_n F_n = 0$ and so $\alpha_n = 0$ $(0 \leq n \leq N-1)$ by Lemma 2.2 (iv). Thus we have proved Lemma 2.8. Q.E.D.

§3. $F_X^{+/-}(t)$ (I)

In the sequel we shall consider a real stationary Gaussian process $\boldsymbol{X} = (X(t); t \in \boldsymbol{R})$ having the spectral density Δ of the form (2.1). We assume that $\boldsymbol{X}$ has expectation zero. Since $P(-\cdot)^{-1}Q(-\cdot)$ is an outer function of the Hardy weight Δ, we get from (2.5) the following canonical representation:

$$X(t) = \sqrt{2\pi}^{-1}\int_{-\infty}^{t} E(t-s)dB(s)\ , \tag{3.1}$$

where $(B(t); t \in \boldsymbol{R})$ is a standard Brownian motion satisfying

$$F_X^-(t) = \sigma(B(s_1) - B(s_2)\,;\, s_1, s_2 < t) \qquad \text{for any } t \in \boldsymbol{R}\ . \tag{3.2}$$

Using L^2-functions E_n in (2.7) we define random variables $X_n(t)$ $(t \in \boldsymbol{R}, 0 \leq n \leq N-1)$ by

$$X_n(t) = \sqrt{2\pi}^{-1}\int_{-\infty}^{t} E_n(t-s)dB(s) \tag{3.3}$$

and then an N-dimensional stationary Gaussian process $\mathscr{X} = (\mathscr{X}(t); t \in \boldsymbol{R})$ by

$$\mathscr{X}(t) = (X_0(t), \cdots, X_{N-1}(t))^*\ . \tag{3.4}$$

Particularly we see from (2.8) that

$$X_{N-1}(t) = (-2\pi)^{-1}c_N X(t) \qquad (t \in \boldsymbol{R})\ . \tag{3.5}$$

We define an $N \times N$-matrix A and an N-vector $\boldsymbol{b}$ by

(3.6) $$A = \begin{pmatrix} 0 & & & & a_0 \\ -1 & \ddots & & 0 & a_1 \\ & -1 & \ddots & & \vdots \\ & & \ddots & 0 & \vdots \\ & 0 & & \ddots & \vdots \\ & & & -1 & a_{N-1} \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_0 \\ \vdots \\ b_{N-1} \end{pmatrix},$$

where $a_n = c_n c_N^{-1}$ $(0 \leq n \leq N)$.

In the same way as Theorem 9.1 in [6] we can show from (2.8) and Lemma 2.7 that

THEOREM 3.1. *For almost all* ω

$$\mathscr{X}(t) - \mathscr{X}(s) = \sqrt{2\pi}^{-1}\boldsymbol{b}(B(t) - B(s)) + \int_s^t A\mathscr{X}(u)du \qquad (s < t).$$

In particular $\mathscr{X}(t)$ *is continuous in* $t \in \boldsymbol{R}$.

Noting (3.2) we see from Theorem 2.1 (i) and Lemma 2.8 that

THEOREM 3.2. (i) *For any* s *and* $t \in \boldsymbol{R}$, $s < t$,

$$E(X(t) \mid \boldsymbol{F}_X^-(s)) = \sum_{n=0}^{N-1} (-1)^n F^{(n)}(t-s) X_n(s).$$

(ii) $\{X_n(t); 0 \leq n \leq N-1\}$ *is linearly independent in* $\boldsymbol{M}$ *for any* $t \in \boldsymbol{R}$.

We define for any $t \in \boldsymbol{R}$ an $N \times N$-matrix $A(t) = (A(t)_{mn})$ by

(3.7) $$A(t)_{mn} = (2\pi)^{-1} \sum_{k=0}^{N-m-1} (-1)^{n+k+1} c_{m+k+1} F^{(n+k)}(t) \qquad (0 \leq m, n \leq N-1).$$

Then we shall show

LEMMA 3.1. (i) *For any* s *and* $t \in \boldsymbol{R}$, $s < t$,

$$E(\mathscr{X}(t) \mid \boldsymbol{F}_X^-(s)) = A(t-s)\mathscr{X}(s).$$

(ii) $A(t) = e^{tA}$ $(t > 0)$.

Proof. By Theorem 2.1 (ii) we have (i). We particularly see from Lemma 2.8 that $A(s+t) = A(s)A(t)$ $(s > 0, t > 0)$. Since $A(t)$ is continuous in $t \in (0, \infty)$ and $A(0+) = I$, this implies that there exists an $N \times N$-matrix $\tilde{A}$ satisfying $A(t) = e^{t\tilde{A}}$ $(t > 0)$. Since $B(t) - B(0)$ $(t > 0)$ are independent of $\boldsymbol{F}_X^-(0)$ and $\mathscr{X}(0)$ is $\boldsymbol{F}_X^-(0)$-measurable by (3.2), we see from Theorem 3.1 and Lemma 3.1 (i) that

$$E(\mathscr{X}(t)\,|\,\boldsymbol{F}_{\boldsymbol{X}}^{-}(0)) = \left(I + \int_0^t Ae^{u\tilde{A}}du\right)\mathscr{X}(0)$$
$$= e^{t\tilde{A}}\mathscr{X}(0) \qquad (t > 0)\ .$$

By Theorem 3.2 (ii) we get

$$e^{t\tilde{A}} = I + \int_0^t Ae^{u\tilde{A}}du \qquad (t > 0)\ .$$

Differentiating both sides at $t = 0$, we find that $\tilde{A} = A$. Thus we have proved Lemma 3.1. (Q.E.D.)

In the same way as in the case of $\boldsymbol{X}$, we shall consider the past fields $\boldsymbol{F}_{\mathscr{X}}^{-}(t)$, the future fields $\boldsymbol{F}_{\mathscr{X}}^{+}(t)$ and the splitting fields $\boldsymbol{F}_{\mathscr{X}}^{+/-}(t)\ (t \in \boldsymbol{R})$ associated with $\mathscr{X}$ (Definition 9.1 in [6]). We then see from (3.2), (3.3) and (3.4) that

$$\boldsymbol{F}_{\boldsymbol{X}}^{-}(t) = \boldsymbol{F}_{\mathscr{X}}^{-}(t) \qquad (t \in \boldsymbol{R})\ . \tag{3.8}$$

Now we shall prove the following main theorem.

THEOREM 3.3. *$\boldsymbol{F}_{\boldsymbol{X}}^{+/-}(t) = \boldsymbol{F}_{\mathscr{X}}^{+/-}(t) = \sigma(\mathscr{X}(t))$ for any $t \in \boldsymbol{R}$.*

Proof. By virtue of Lemma 2.5, we see from Theorem 3.2 that $\boldsymbol{M}^{+/-}(t)$ is equal to the closed linear hull of $\{X_n(t)\,;\, 0 \le n \le N-1\}\ (t \in \boldsymbol{R})$. This implies by Lemma 2.1 (iii) in [6] that $\boldsymbol{F}_{\boldsymbol{X}}^{+/-}(t) = \sigma(\mathscr{X}(t))$ for any $t \in \boldsymbol{R}$. It is clear that $\sigma(\mathscr{X}(t)) \subset \boldsymbol{F}_{\mathscr{X}}^{-}(t) \cap \boldsymbol{F}_{\mathscr{X}}^{+}(t) \subset \boldsymbol{F}_{\mathscr{X}}^{+/-}(t)$ since $\mathscr{X}(t)$ is continuous in $t \in \boldsymbol{R}$. On the other hand, it follows from Lemma 3.1 that, for any $t \in \boldsymbol{R}$ and any $h > 0$,

$$X_n(t+h) = A(h)\mathscr{X}(t)_n + \sqrt{2\pi}^{-1}\int_t^{t+h} E_n(t+h-s)dB(s) \qquad (0 \le n \le N-1)\ .$$

Since $B(t+z) - B(t)\ (z > 0)$ are independent of $\boldsymbol{F}_{\mathscr{X}}^{-}(t)$ for any $t \in \boldsymbol{R}$ by (3.2) and (3.8), we can see that $\boldsymbol{F}_{\mathscr{X}}^{-}(t)$ is independent of $\boldsymbol{F}_{\mathscr{X}}^{+}(t)$ under the condition that $\sigma(\mathscr{X}(t))$ is known, and so that $\boldsymbol{F}_{\mathscr{X}}^{+/-}(t) \subset \sigma(\mathscr{X}(t))$. Thus we have proved Theorem 3.3. (Q.E.D.)

§ 4. A Gaussian diffusion process

From Theorem 3.3 we find that a Gaussian process $(\mathscr{X}(t), P(\cdot\,|\,\mathscr{X}(0) = x)\,;\, t > 0, x \in \boldsymbol{R}^N)$ has the usual Markovian property. In this section we shall investigate several properties of such a Gaussian Markov process.

By (3.2) and Lemma 3.1 we have

LEMMA 4.1. (i) $E_n(t) = \sqrt{2\pi}^{-1}(e^{tA}\boldsymbol{b})_n$ $(t > 0, 0 \le n \le N-1)$,

(ii) $\mathscr{X}(t) = e^{(t-s)A}\mathscr{X}(s) + \sqrt{2\pi}^{-1}\int_s^t e^{(t-u)A}\boldsymbol{b}dB(u)$ $(s < t)$.

We denote by $\mu(t,x)$ and $R(t,x)$ the mean vector and the covariance matrix, respectively, under the condition that $\mathscr{X}(0) = x$ $(t > 0, x \in \boldsymbol{R}^N)$:

$$\begin{cases} \mu(t,x) = E(\mathscr{X}(t) \mid \mathscr{X}(0) = x) , \\ R(t,x) = E(\mathscr{X}(t)\mathscr{X}(0)^* \mid \mathscr{X}(0) = x) . \end{cases}$$

It then follows from Lemma 4.1 that

$$\begin{cases} \mu(t,x) = e^{tA}x , \\ R(t,x) = R(t) = \left((2\pi)^{-1}\int_0^t e^{sA}\boldsymbol{b}_m e^{sA}\boldsymbol{b}_n ds\right)_{0 \le m,n \le N-1} . \end{cases} \tag{4.1}$$

We shall prove

THEOREM 4.1. *$\{A^n\boldsymbol{b}; 0 \le n \le N-1\}$ is linearly independent.*

As an application of Theorem 4.1 we find that $R(t)$ is a positive definite matrix for each $t > 0$. Before the proof of Theorem 4.1, we shall prepare several lemmas.

LEMMA 4.2. *For any $n \in \{0, 1, \cdots, N-1\}$ we set*

$$G_n(t) = \begin{cases} \sum_{m=0}^{N-1}(-1)^m b_m F^{(n+m)}(t) & (t > 0) , \\ 0 & (t \le 0) . \end{cases}$$

Then

$$\{G_n; 0 \le n \le N-1\} \qquad \text{is linearly independent in } L^2(\boldsymbol{R}) .$$

Proof. Let α_n $(0 \le n \le N-1)$ be real constants such that $\sum_{n=0}^{N-1}\alpha_n G_n = 0$. We define a polynomial $S(z) = \sum_{n=0}^{N-1}\alpha_n(iz)^n$. Since $G_m(t) = G_0^{(m)}(t)$ for any $t \in \boldsymbol{R} - \{0\}$, we find that $S\left(\frac{1}{i}\frac{d}{dt}\right)G_0 = 0$ in $\boldsymbol{R} - \{0\}$ in the sense of distributions. Therefore, there exists a polynomial Q_1 such that $S\left(\frac{1}{i}\frac{d}{dt}\right)G_0 = Q_1\left(\frac{1}{i}\frac{d}{dt}\right)\delta$ in $\boldsymbol{R}$. Noting that $G_0 \in L^2(\boldsymbol{R})$ and taking the inverse Fourier transform of both sides, we find that $S(-\lambda)\tilde{G}_0(\lambda) = Q_1(-\lambda)$ $(\lambda \in \boldsymbol{R})$. On the other hand, we see that $\tilde{G}_0(\lambda) = Q(-\lambda)\tilde{F}(\lambda)$, since $G_0 =$

$Q\left(\frac{1}{i}\frac{d}{dt}\right)F$. Hence, it follows from (2.2) that $S(\lambda)Q(\lambda) = Q_1(\lambda)P(\lambda)$ $(\lambda \in \boldsymbol{R})$. Since S is a polynomial of at most degree $N-1$, this implies by (2.1) that $S = 0$ and so $\alpha_n = 0$ $(0 \leq n \leq N-1)$. Thus we have proved Lemma 4.2. (Q.E.D.)

LEMMA 4.3. *For any* $m, n \in \{0, 1, \cdots, N-1\}$ *we set*

$$\gamma_{mn} = \sum_{\ell=0}^{N-1} (-1)^\ell b_\ell F_n^{(m+\ell)}(0+) .$$

Then the $N \times N$*-matrix* $(\gamma_{mn})_{0 \leq m,n \leq N-1}$ *is nonsingular.*

Proof. Differentiating (i) in Lemma 2.3 $\ell + m$ times at $s = 0$, we have

$$F^{(\ell+m)}(t) = \sum_{n=0}^{N-1} (-1)^n F^{(n)}(t) F_n^{(\ell+m)}(0+) \qquad (t > 0, 0 \leq \ell, m \leq N-1) .$$

Multiplying it by $(-1)^\ell b_\ell$ and then summing up with respect to ℓ, we get

$$\sum_{\ell=0}^{N-1} (-1)^\ell b_\ell F^{(\ell+m)}(t) = \sum_{n=0}^{N-1} (-1)^n \gamma_{mn} F^{(n)}(t) \qquad (t > 0) .$$

Therefore, by Lemmas 2.1 (iii) and 4.2, we obtain the desired result. (Q.E.D.)

LEMMA 4.4. *The* $N \times N$*-matrix* $(E_n^{(m)}(0+))_{0 \leq m,n \leq N-1}$ *is nonsingular.*

Proof. Differentiating (ii) in Theorem 2.1 m times at $t = 0$ and then letting s tend to zero, we have

$$E_n^{(m)}(0+) = (2\pi)^{-1} \sum_{\ell=0}^{N-1} (-1)^\ell \left(\sum_{k=0}^{N-n-1} (-1)^{k+1} c_{k+n+1} F^{(m+k+\ell)}(0+) \right) E_\ell(0+) .$$

On the other hand, differentiating (i) in Lemma 2.3 m times and $k + \ell$ times at $t = 0$ and $s = 0$, respectively, we get

$$F^{(m+k+\ell)}(0+) = \sum_{j=0}^{N-1} (-1)^j F^{(m+j)}(0+) F_j^{(k+\ell)}(0+) .$$

Therefore it follows from Lemma 2.7 (i) that

$$E_n^{(m)}(0+) = (2\pi)^{-1} \sum_{j=0}^{N-1} \sum_{k=0}^{N-n-1} F^{(m+j)}(0+)(-1)^j \gamma_{kj} (-1)^{k+1} c_{k+n+1} .$$

By Lemma 2.1 (i), the matrix $(F^{(m+j)}(0+))_{0\le m,j\le N-1}$ must be nonsingular. Therefore, we obtain the desired result noting that c_N is not zero and using Lemma 4.3. (Q.E.D.)

LEMMA 4.5. *The* $N\times N$*-matrix* $(E^{(m+n)}(0+))_{0\le m,n\le N-1}$ *is nonsingular.*

Proof. Differentiating (i) in Theorem 2.1 ℓ times and m times at $t=0$ and $s=0$, respectively, we have

$$E^{(\ell+m)}(0+)=\sum_{n=0}^{N-1}(-1)^n F^{(\ell+n)}(0+)E_n^{(m)}(0+)\ .$$

Therefore, by Lemma 4.4, we get the result. (Q.E.D.)

LEMMA 4.6. $\{A^n\boldsymbol{a};0\le n\le N-1\}$ *is linearly independent, where* $\boldsymbol{a}=(a_0\cdots a_{N-1})^*$.

Proof. Since $A\boldsymbol{a}=-(0a_0\cdots a_{N-2})^*+a_{N-1}\boldsymbol{a}$, we have the result noting that a_0 is not zero. (Q.E.D.)

LEMMA 4.7. *For any* ℓ, m *and* $n\in\{0,1,\cdots,N-1\}$,

$$(A^n)_{\ell m}=(2\pi)^{-1}\sum_{k=0}^{N-\ell-1}c_{\ell+k+1}(-1)^{m+k+1}F^{(m+k+n)}(0+)\ .$$

Proof. Differentiating e^{tA} k times at $t=0$, we obtain the result from (3.7) and Lemma 3.1 (ii). (Q.E.D.)

LEMMA 4.8. $\sum_{n=0}^{N-1}(-1)^n b_n A^n$ *is nonsingular.*

Proof. We denote by $\boldsymbol{a}_\ell$ the $\ell+1$ row of the matrix $\sum_{n=0}^{N-1}(-1)^n b_n A^n$ and set $\boldsymbol{e}_\ell=(\cdots(-1)^{n+1}c_{\ell+n+1}\cdots)^*$ $(0\le\ell\le N-1)$, where $c_m=0$ for $m\ge N+1$. By (2.6) and Lemma 4.7 we have

$$\begin{aligned}\boldsymbol{a}_\ell&=(2\pi)^{-1}(-1)^\ell\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}(-1)^k b_k F^{(\ell+k+n)}(0+)\boldsymbol{e}_n\\&=(2\pi)^{-1}(-1)^\ell\sum_{n=0}^{N-1}\left(Q\left(\frac{1}{i}\frac{d}{dt}\right)F(t)\right)^{(\ell+n)}\Bigg|_{t=0}\boldsymbol{e}_n\\&=(2\pi)^{-1}(-1)^\ell\sum_{n=0}^{N-1}E^{(\ell+n)}(0+)\boldsymbol{e}_n\ .\end{aligned}$$

Therefore, since $\det(\boldsymbol{e}_0\cdots\boldsymbol{e}_{N-1})=((-1)^N c_N)^N$ is not zero, we have the desired result from Lemma 4.5. (Q.E.D.)

After these preparations, we are in a position to prove Theorem 4.1.

Proof of Theorem 4.1.: Let α_n $(0\le n\le N-1)$ be real constants such that $\sum_{n=0}^{N-1}\alpha_n A^n \boldsymbol{b}=0$. Since $A\boldsymbol{b}=-(0b_0\cdots b_{N-2})^*+b_{N-1}\boldsymbol{a}$, we have

$$A^{N+n}\boldsymbol{b}=(-1)^{N-1}\sum_{m=0}^{N-1}(-1)^m b_m A^{m+n}\boldsymbol{a} \qquad (0\le n\le N-1)\,.$$

Then operating the matrix A^N to both sides, we get

$$\left(\sum_{m=0}^{N-1}(-1)^m b_m A^m\right)\left(\sum_{n=0}^{N-1}\alpha_n A^n\boldsymbol{a}\right)=\sum_{n=0}^{N-1}\alpha_n A^{N+n}\boldsymbol{b}=0\,.$$

and so $\alpha_n=0$ $(0\le n\le N-1)$ by Lemmas 4.8 and 4.6. This completes the proof of Theorem 4.1. (Q.E.D.)

As an application of Lemma 4.4 we shall show the following

THEOREM 4.2. (i) *There exist N positive numbers $t_0<t_1<\cdots<t_{N-1}$ such that the matrix $(E^{(m)}(t_n))_{0\le n\le N-1}$ is nonsingular.*

(ii) *In order that for any N positive numbers $t_0<t_1<\cdots<t_{N-1}$ the matrix $(E^{(m)}(t_n))_{0\le m,n\le N-1}$ is nonsingular, it is a necessary and sufficient condition that the zero points of P are located in the positive imaginary axis.*

Proof. Differentiating (i) in Theorem 2.1 m times at $s=0$, we have

$$E^{(m)}(t)=\sum_{n=0}^{N-1}(-1)^n F^{(n)}(t)E_n^{(m)}(0+) \qquad (t>0)\,.$$

Therefore, combining Lemmas 2.5, 2.6 and 4.4, we obtain the result. (Q.E.D.)

Now we shall apply Theorem 4.1 to get several properties of the Gaussian Markov process $(\mathscr{X}(t), P(\cdot\,|\,\mathscr{X}(0)=x)\,;\,t>0, x\in\boldsymbol{R}^N)$. It is easy to see from (4.1) that the covariance matrices $R(t)$ $(t>0)$ are positive definite. Therefore it follows from (4.1) that the Gaussian Markov process $(\mathscr{X}(t), P(\cdot\,|\,\mathscr{X}(0)=x)\,;\,t>0, x\in\boldsymbol{R}^N)$ has a transition probability density $P(t,x,y)$;

$$\text{(4.2)}\qquad \begin{cases} P(\mathscr{X}(t)\in dy\,|\,\mathscr{X}(0)=x)=P(t,x,y)dy\,, \\ P(t,x,y)=(2\pi)^{-N/2}(\det R(t))^{-1/2}e^{-1/2(y-e^{tA}x,\,R^{-1}(t)(y-e^{tA}x))}\,. \end{cases}$$

Since $\boldsymbol{b}$ is not zero, it follows from Theorem 3.1 that

$$\text{(4.3)}\qquad \sigma(B(s)-B(t)\,;\,s,t\in D)\subset \boldsymbol{F}_{\mathfrak{x}}(D) \qquad \text{for any open set } D \text{ in } \boldsymbol{R}\,.$$

Therefore, by (3.2), (3.8) and (4.3), we can apply K. Ito's formula to the stochastic differential equation in Theorem 3.1 and find that the Gaussian

Markov process $(\mathscr{X}(t), P(\cdot\,|\,\mathscr{X}(0)=x);\, t>0, x\in \boldsymbol{R}^N)$ becomes a diffusion process whose infinitesimal generator $\mathscr{G}_{\mathfrak{X}}$ is given by

$$\mathscr{G}_{\mathfrak{X}} = \tfrac{1}{2}(\sqrt{2\pi}^{-1}\boldsymbol{b}\cdot\nabla)^2 + (Ax)\cdot\nabla\ . \tag{4.4}$$

From Theorem 4.1 we find that this differential operator $\mathscr{G}_{\mathfrak{X}}$ is hypoelliptic ([4]).

It is easy to see from (2.1) and (3.6) that the characteristic equation of the matrix A is equal to $(-1)^N c_N^{-1}P(i^{-1}\lambda)$:

$$\det(\lambda - A) = (-1)^N c_N^{-1}P(i^{-1}\lambda) = (-1)^N \sum_{n=0}^{N} a_n(-\lambda)^n\ . \tag{4.5}$$

This particulary implies that the real part of all eigenvalues of A is negative. Noting this fact and applying Theorems 4.1, 6.1 and 7.1 in [2] to our Gaussian diffusion process, we have

THEOREM 4.3. *The Gaussian diffusion process* $(\mathscr{X}(t), P(\cdot\,|\,\mathscr{X}(0)=x);$ $t>0, x\in\boldsymbol{R}^N)$ *is recurrent and there uniquely exists an invariant measure* $\mu(dy)$:

$$\begin{cases}\mu(dy) = \varphi(y)dy\ ,\\ \varphi(y) = e^{-\frac{1}{2}(y, R^{-1}(\infty)y)}\ ,\end{cases} \tag{4.6}$$

where $R^{-1}(\infty)$ *is the inverse matrix of a positive definite matrix* $R(\infty)$ $=\lim_{t\to\infty} R(t)$.

Remark 4.1. It follows from (4.1) that

$$R(\infty) = \left((2\pi)^{-1}\int_0^\infty e^{tA}\boldsymbol{b}_m e^{tA}\boldsymbol{b}_n dt\right)_{0\le m,n\le N-1}\ . \tag{4.7}$$

§ 5. $F_X^{+/-}(t)$ (II)

We have constructed in § 3 an example $\mathscr{X}$ of N-dimensional stationary Gaussian processes $\mathscr{Y} = (\mathscr{Y}(t);\, t\in\boldsymbol{R})$ satisfying the following conditions:

(5.1) $\mathscr{Y}(t)$ is continuous in the mean;

(5.2) For any $t\in\boldsymbol{R}$, each component of $\mathscr{Y}(t)$ belongs to $\boldsymbol{M}$ and {the n-th component of $\mathscr{Y}(t);\, 1\le n\le N$} is linearly independent;

(5.3) $F_X^{+/-}(t) = \sigma(\mathscr{Y}(t))$ for any $t\in\boldsymbol{R}$.

In this section we shall show the next theorem about the uniqueness of such a process.

THEOREM 5.1. *For any N-dimensional stationary Gaussian process* $\mathscr{Y} = (\mathscr{Y}(t)\,;\, t \in \boldsymbol{R})$ *satisfying* (5.1), (5.2) *and* (5.3), *there uniquely exists a constant nonsingular* $N \times N$*-matrix* T *such that* $\mathscr{Y}(t) = T\mathscr{X}(t)$ *for any* $t \in \boldsymbol{R}$.

Before proving this theorem, we shall prepare three lemmas. We define for any $t \in \boldsymbol{R}$ an $N \times N$-matrix $K_{\mathfrak{x}}(t)$ by

$$K_{\mathfrak{x}}(t) = E(\mathscr{X}(t)\mathscr{X}(0)^*) \,. \tag{5.4}$$

By Theorem 3.2 (ii) and Lemma 3.1 we have

LEMMA 5.1. (i) $K_{\mathfrak{x}}(0)$ *is symmetric and positive definite,*

(ii)
$$K_{\mathfrak{x}}(t) = \begin{cases} e^{tA}K_{\mathfrak{x}}(0) & (t \geq 0)\,, \\ K_{\mathfrak{x}}(0)e^{-tA} & (t < 0)\,. \end{cases}$$

We define a symmetric $N \times N$-matrix B by

$$B = (b_m b_n)_{0 \leq m,n \leq N-1} \,. \tag{5.5}$$

Then we shall prove

LEMMA 5.2. $AK_{\mathfrak{x}}(0) + K_{\mathfrak{x}}(0)A^* = -(2\pi)^{-1}B$.

Proof. Since $\mathscr{X} = (\mathscr{X}(t)\,;\, t \in \boldsymbol{R})$ is stationary, it follows from (3.2), (3.8) and Lemma 4.1 (ii) that

$$K_{\mathfrak{x}}(0) = e^{tA}K_{\mathfrak{x}}(0)e^{tA^*} + (2\pi)^{-1}\int_0^t e^{sA}Be^{sA^*}ds \qquad (t > 0)\,.$$

Differentiating it at $t = 0$, we obtain the result. (Q.E.D.)

Next we shall show the following general statement.

LEMMA 5.3. *Let* A, B *and* K *be real* $N \times N$*-matrices such that*

(i) $B = (b_m b_n)_{0 \leq m,n \leq N-1}$, $\boldsymbol{b} = (b_0 \cdots b_{N-1})^* \neq 0$,

(ii) K *is symmetric and positive definite,*

(iii) $AK + KA^* = -B$

and

(iv) $\{A^n\boldsymbol{b}\,;\, 0 \leq n \leq N-1\}$ *is linearly independent.*

If an $N \times N$*-matrix* $\tilde{A}$ *satisfies*

$$e^{t\tilde{A}}Ke^{t\tilde{A}^*} = e^{tA}Ke^{tA^*} \qquad \textit{for any } t \in \boldsymbol{R}\,,$$

then

$$\tilde{A} = A\,.$$

Proof. Since K has a symmetric and positive definite root $K^{\frac{1}{2}}$, we can define $A_1 = K^{-\frac{1}{2}}AK^{\frac{1}{2}}$, $\tilde{A}_1 = K^{-\frac{1}{2}}\tilde{A}K^{\frac{1}{2}}$ and $B_1 = K^{-\frac{1}{2}}BK^{-\frac{1}{2}}$. It then follows that

$$\begin{cases} A_1 + A_1^* = -B_1 , \\ e^{t\tilde{A}_1}e^{t\tilde{A}_1^*} = e^{tA_1}e^{tA_1^*} \qquad \text{for any } t \in \boldsymbol{R} . \end{cases} \tag{5.6}$$

Since B_1 is a symmetric, nonnegative definite matrix of rank one, there exist an orthogonal matrix P_1 and a positive number ε such that $B_1 = P_1 \begin{pmatrix} \varepsilon_0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} P_1^{-1}$ and so $\varepsilon^{-1} \sum_{n=0}^{N-1} (B_1)_{nn} = 1$. Therefore we can find another orthogonal matrix P_2 such that $(P_2)_{n0} = \sqrt{\varepsilon^{-1}}(K^{-\frac{1}{2}}\boldsymbol{b})_n$ $(0 \leq n \leq N-1)$, because $(B_1)_{nn} = (K^{-\frac{1}{2}}\boldsymbol{b})_n^2$. It is then easy to see that $P_2 \begin{pmatrix} \varepsilon_0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} P_2^{-1} = B_1$. Hence, setting $A_2 = P_2^{-1}A_1P_2$, $\tilde{A}_2 = P_2^{-1}\tilde{A}_1P_2$ and $T = \begin{pmatrix} -\varepsilon_0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$, we see from (5.6) and Theorem 4.1 that

$$A_2 + A_2^* = T , \tag{5.7}$$

$$e^{t\tilde{A}_2}e^{t\tilde{A}_2^*} = e^{tA_2}e^{tA_2^*} \qquad \text{for any } t \in \boldsymbol{R} \tag{5.8}$$

and

$$\{((A_2^nT)_{00}, (A_2^nT)_{10}, \cdots, (A_2^nT)_{N-1\,0})^* \,;\, 0 \leq n \leq N-1\} \tag{5.9}$$

is linearly independent.

We define a sequence $(D_p)_{p=0}^{\infty}$ of $N \times N$-matrices by

$$D_p = A_2D_{p-1} + D_{p-1}A_2^* \qquad (p = 1, 2, \cdots), D_0 = I . \tag{5.10}$$

Since $D_1 = T$ by (5.7), we have

$$D_{p+1} = \sum_{k=0}^{p} \binom{p}{k} A_2^kTA_2^{*p-k} \qquad (p = 0, 1, 2, \cdots) . \tag{5.11}$$

Setting $L = \tilde{A}_2 - A_2$ and then differentiating (5.8) at $t = 0$, we get

$$LD_p + D_pL^* = 0 \qquad (p = 0, 1, 2, \cdots) . \tag{5.12}$$

Therefore, putting $S=[L,A_2]$ $(=LA_2-A_2L)$, we see from (5.10) and (5.12) that

$$SD_p+D_pS^*=0 \qquad (p=0,1,2,\cdots)\,. \tag{5.13}$$

From (5.12) in the case of $p=1$ we have

$$L+L^*=0\,. \tag{5.14}$$

Furthermore, applying (5.12) in the case of $p=1$, we find that $[L,T]$ $=0$. Therefore, since $T=\begin{bmatrix}-\varepsilon_0 & & 0\\ & \ddots & \\ 0 & & 0\end{bmatrix}$, we get

$$LT=TL=0\,. \tag{5.15}$$

Similarly it follows from (5.13) in the case of $p=0$ and $p=1$ that

$$S+S^*=0 \tag{5.16}$$

and

$$ST=TS=0\,. \tag{5.17}$$

Fixing any $p_0\in\{0,1,2,\cdots\}$ we shall assume that $SA_2^pT=TA_2^pS=0$ for any $p\in\{0,\cdots,p_0\}$. By (5.7), (5.11), (5.13), (5.16) and (5.17), we find that $SA_2^{p_0+1}T=TA_2^{p_0+1}S$. Since $T=\begin{bmatrix}-\varepsilon_0 & & 0\\ & \ddots & \\ 0 & & 0\end{bmatrix}$, this implies that $(SA_2^{p_0+1})_{n0}$ $=0$ for any $n\in\{1,2,\cdots,N-1\}$. Moreover we see that $(SA_2^{p_0+1})_{00}=0$ because S_{0n} for any $n\in\{0,1,\cdots,N-1\}$ by (5.17). For this reason it follows that $SA_2^{p_0+1}T=TA_2^{p_0+1}S=0$. By mathematical induction on p_0, we conclude that $SA_2^pT=0$ for any $p\in\{0,1,2,\cdots\}$. Therefore, using (5.9), we find that $S=0$. Since this conclusion implies that L commutes with A_2, it follows from (5.15) that $LA_2^pT=0$ for any $p\in\{0,1,\cdots\}$. Consequently, using (5.9) again, we see that $L=0$ and so $\tilde{A}=A$. Now we complete the proof of Lemma 5.3. (Q.E.D.)

After these preparations, we are in a position to prove Theorem 5.1.

Proof of Theorem 5.1: Since the subspace of $\boldsymbol{M}$ whose elements are $\boldsymbol{F}_X^{+/-}(t)$-measurable is equal to the space $\boldsymbol{M}^{+/-}(t)$ with the algebraic dimension N, it follows from (5.2) and (5.3) that there exists a non-singular $N\times N$-matrix $T(t)$ satisfying $\mathscr{Y}(t)=T(t)\mathscr{X}(t)$ $(t\in\boldsymbol{R})$. For any

s and $t \in \boldsymbol{R}$, $s < t$, we define an $N \times N$-matrix $C(t, s)$ by

$$C(t, s) = T(t)e^{(t-s)A}T(s)^{-1}.$$

Then it follows from Lemma 3.1 and (5.2) that

$$C(u, s) = C(u, t)C(t, s) \qquad (s < t < u) \tag{5.18}$$

and

$$E(\mathscr{Y}(t) \mid \boldsymbol{F}_{\boldsymbol{X}}^{-}(s)) = C(t, s)\mathscr{Y}(s) \qquad (s < t). \tag{5.19}$$

Since $\mathscr{Y} = (\mathscr{Y}(t); t \in \boldsymbol{R})$ is stationary, we see from (5.2) and (5.19) that $C(t, s) = C(t - s, 0)$ $(s < t)$. Setting $C(t) = C(t, 0)$ $(t > 0)$, we can show from (5.1), (5.2) and (5.18) that $C(t)$ is continuous in $t \in [0, \infty)$, $C(0) = I$ and $C(s + t) = C(s)C(t)$ $(s, t \in [0, \infty))$. Therefore, there exists an $N \times N$-matrix $\tilde{A}$ such that $C(t) = e^{tT(0)\tilde{A}T(0)^{-1}}$ $(t \geq 0)$. Since it is easily seen that $T(t)$ is real analytic in $t \in \boldsymbol{R}$, we obtain

$$T(t) = T(0)e^{t\tilde{A}}e^{-tA} \qquad \text{for any } t \in \boldsymbol{R}. \tag{5.20}$$

On the other hand, by Lemma 5.1 and (5.19), we have

$$C(t - s)T(0)K_{\mathfrak{X}}(0)T(0)^* = T(t)e^{(t-s)A}K_{\mathfrak{X}}(0)T(s)^* \qquad (s < t).$$

Combining this with (5.20), we get

$$e^{t\tilde{A}}K_{\mathfrak{X}}(0)e^{t\tilde{A}^*} = e^{tA}K_{\mathfrak{X}}(0)e^{tA^*} \qquad (t \in \boldsymbol{R}).$$

Therefore, by Theorem 4.1, Lemmas 5.1 (i) and 5.2, we can apply Lemma 5.3. to obtain the conclusion. (Q.E.D.)

EXAMPLE 6.1. Using N positive numbers t_n in Lemma 2.5, we define a nonsingular $N \times N$-matrix $T = ((-1)^n F^{(m)}(t_n))_{0 \leq m, n \leq N-1}$ and a stationary Gaussian process $\mathscr{Y} = (\mathscr{Y}(t); t \in \boldsymbol{R}) = (T\mathscr{X}(t); t \in \boldsymbol{R})$. It follows from Theorem 3.2 (i) that the $n + 1$-th component of $\mathscr{Y}(t)$ is equal to $E(X(t + t_n) \mid \boldsymbol{F}_{\boldsymbol{X}}^{-}(t))$ $(t \in \boldsymbol{R}, 0 \leq n \leq N - 1)$.

§ 6. $\boldsymbol{F}_{\boldsymbol{X}}^{+/-}(t)$ (III)

Using the L^2-function F in (2.2) and the Brownian motion B in (3.1), we define a real stationary Gaussian process $\boldsymbol{Y} = (Y(t); t \in \boldsymbol{R})$ such that

$$Y(t) = \sqrt{2\pi}^{-1} \int_{-\infty}^{t} F(t - s)dB(s) \qquad (t \in \boldsymbol{R}). \tag{6.1}$$

It is easy to see that this representation is canonical and $\boldsymbol{Y}$ has the N-ple Markovian property in the narrow sense. Since Q is a polynomial of at most degree $N-1$, we see from Lemma 2.1 (i), (2.6) and (3.1) that

$$X(t) = Q\Big(\frac{1}{i}\frac{d}{dt}\Big)Y(t) \qquad (t \in \boldsymbol{R}) . \tag{6.2}$$

Now we define an $N \times N$-matrix T by

$$T = (\boldsymbol{b}(-A)\boldsymbol{b} \cdots (-A)^{N-1}\boldsymbol{b}) , \tag{6.3}$$

which is nonsingular by virtue of Theorem 4.1. Since the characteristic polynomial of A is $(-1)^N c_N^{-1}P(i^{-1}\lambda)$, it follows from Caley-Hamilton's theorem that $\sum_{n=0}^{N} a_n(-A)^n = 0$ ((4.5)). Therefore we can easily see that

$$T^{-1}\boldsymbol{b} = (10 \cdots 0)^* \tag{6.4}$$

and

$$T^{-1}AT = A . \tag{6.5}$$

Using this matrix T we define an N-dimensional stationary Gaussian process $\mathscr{Y} = (\mathscr{Y}(t)\,;\, t \in \boldsymbol{R})$ satisfying (5.1), (5.2) and (5.3) as follows:

$$\mathscr{Y}(t) = T^{-1}\mathscr{X}(t) \qquad (t \in \boldsymbol{R}) . \tag{6.6}$$

We denote by $Y_n(t)$ the $n+1$-th component of $\mathscr{Y}(t)$ $(0 \le n \le N-1, t \in \boldsymbol{R})$. By (2.3), (3.3), (3.7), Lemma 3.1 (ii) and 4.1 (i), we can show that

$$\mathscr{Y}(t) = \sqrt{2\pi}^{-1}\int_{-\infty}^{t} e^{(t-s)A}\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} dB(s) \qquad (t \in \boldsymbol{R}) \tag{6.7}$$

and

$$\left(e^{tA}\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}\right)_n = F_n(t) \qquad (t > 0, 0 \le n \le N-1) . \tag{6.8}$$

By (2.4), we particularly find

(6.9) $$Y_{N-1}(t) = (-2\pi)^{-1} c_N Y(t) \qquad (t \in \boldsymbol{R}) .$$

By (3.8) and (6.6) we note

(6.10) $$\boldsymbol{F}_X^-(t) = \boldsymbol{F}_{\mathscr{Y}}^-(t) .$$

Using Theorem 3.1, Lemmas 3.1 and 4.1 (ii), we see from (6.4) and (6.5) that

THEOREM 6.1. *For almost all* ω

(i) $\mathscr{Y}(t) - \mathscr{Y}(s) = \sqrt{2\pi}^{-1}(B(t) - B(s), 0, \cdots, 0)^* + \int_s^t A\mathscr{Y}(u)du \ (s < t)$,

(ii) $\mathscr{Y}(t) = e^{(t-s)A}\mathscr{Y}(s) + \sqrt{2\pi}^{-1}\int_s^t e^{(t-s)A}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} dB(u) \ (s < t)$,

(iii) $E(\mathscr{Y}(t) \mid \boldsymbol{F}_X^-(s)) = e^{(t-s)A}\mathscr{Y}(s) \ (s < t)$.

Noting (3.6) we can show from (6.6), (6.9) and Theorem 6.1 (i) that

(6.11) $$\boldsymbol{F}_{\mathscr{X}}(D) = \boldsymbol{F}_{\mathscr{Y}}(D) = \boldsymbol{F}_Y(D) \qquad \text{for any open set } D \text{ in } \boldsymbol{R}$$

and

(6.12) $$\boldsymbol{F}_Y^{+/-}(t) = \partial \boldsymbol{F}_Y(t) \qquad \text{for any } t \in \boldsymbol{R} .$$

Therefore, combining these with Theorem 3.3, we get

THEOREM 6.2.

$$\boldsymbol{F}_X^{+/-}(t) = \boldsymbol{F}_{\mathscr{Y}}^{+/-}(t) = \sigma(\mathscr{Y}(t)) = \boldsymbol{F}_Y^{+/-}(t) = \partial \boldsymbol{F}_Y(t) \qquad \textit{for any } t \in \boldsymbol{R} .$$

Finally we shall give an alternative expression of the linear predictor by using the process $\mathscr{Y}$.

THEOREM 6.3. (i) *For any* s *and* $t \in \boldsymbol{R}$, $s < t$,

$$E(X(t) \mid \boldsymbol{F}_X^-(s)) = \sum_{n=0}^{N-1} (-1)^n E^{(n)}(t - s) Y_n(s) .$$

(ii) $\{Y_n(t); 0 \leq n \leq N - 1\}$ *is linearly independent in* $\boldsymbol{M}$ *for any* $t \in \boldsymbol{R}$.

Proof. By Theorem 3.2 (i) and (6.6) we have (ii). It follows from Theorem 2.1 (i) and Lemma 4.1 (i) that

$$E(t - s) = \sum_{\ell=0}^{N-1} (-1)^\ell F^{(\ell)}(t)(e^{-sA}\boldsymbol{b})_\ell \qquad (s < 0, t > 0) .$$

Differentiating both sides n times at $s=0$, we get

$$E(t)=\sum_{\ell=0}^{N-1}(-1)^{\ell}F^{(\ell)}(t)(A^{n}\boldsymbol{b})_{\ell}\qquad(0\le n\le N-1)\,.$$

Therefore, by Theorem 3.2 (i) and (6.6), we obtain (i). (Q.E.D.)

§ 7. Applications

7.1. Markovian property.

At first we shall characterize the Markovian property of stationary Gaussian processes from the point of view of representations. In [6] we have proved

THEOREM 7.1. ([6]) *In order that a real mean continuous, purely nondeterministic stationary Gaussian process* X *has the Markovian property:*

(7.1) $$\boldsymbol{F}_X^{+/-}(t)=\partial\boldsymbol{F}_X(t)\qquad \textit{for any } t\in\boldsymbol{R}\,,$$

it is a necessary and sufficient condition that there exists a canonical representation $(\sqrt{2\pi}^{-1}E(t),B(t))$ *possesing*

(7.2) $$\sigma(B(s)-B(t)\,;\,s,t\in D)\subset\boldsymbol{F}_X(D)\qquad \textit{for any open set } D \textit{ in } \boldsymbol{R}\,.$$

We shall give another proof of Theorem 7.1 in case X has a rational spectral density Δ of the form (2.1). Now let's assume (7.2). It then follows from (3.5), (3.6) and Theorem 3.1 that $\mathscr{X}(t)$ is $\partial\boldsymbol{F}_X(t)$-measurable for any $t\in\boldsymbol{R}$. Therefore, by Theorem 3.2 (i), we find that $E(X(u)\,|\,\boldsymbol{F}_X^-(t))$ is $\partial\boldsymbol{F}_X(t)$-measurable $(t<u)$ and so that (7.1) holds. Conversely let's assume (7.1). It then follows from Lemma 2.5 and Theorem 3.2 (i) that $\mathscr{X}(t)$ is $\partial\boldsymbol{F}_X(t)$-measurable for any $t\in\boldsymbol{R}$. Therefore, by (3.6) and Theorem 3.1, we obtain (7.2) since $\boldsymbol{b}$ is not zero. (Q.E.D.)

Next we shall characterize the N-ple Markovian property in the sense of T. Hida ([3]). Immediately from Lemma 2.6 and Theorem 3.2 (i) we can show

THEOREM 7.2. *In order that a real mean continuous, purely nondeterministic stationary Gaussian process* X *has the* N*-ple Markovian property in the sense of T. Hida, it is a necessary and sufficient condition that* X *has a rational spectral density* Δ *of the form* (2.1) *with an additional property*

$$V_p{}' \subset \{z \in \boldsymbol{C}^+ ; \operatorname{Re} z = 0\} . \tag{7.3}$$

7.2. **Initial value problem.**

We shall characterize the linear predictor using the past as a unique solution of an initial value problem. We define an $N \times N$-matrix $D = (D_{mn})_{0 \le m,n \le N-1}$ by

$$D_{mn} = (-1)^n E^{(m+n)}(0+) , \tag{7.4}$$

which is nonsingular by Lemma 4.5.

THEOREM 7.3. *We denote by $Z(t, \omega)$ the linear predictor of $X(t)$ using the whole past;*

$$Z(t, \omega) = E(X(t) \,|\, \boldsymbol{F}_X^-(0)) \qquad (t > 0) .$$

Then, for almost all $\omega \in \Omega$, $Z(t, \omega)$ $(t > 0)$ is a unique solution of the following initial value problem (7.5):

$$\begin{cases} Z(\cdot, \omega) \in \mathscr{A}((0, \infty)) \cap L^2((0, \infty)) , \\ P\left(\dfrac{1}{i} \dfrac{d}{dt}\right) Z(t, \omega) = 0 \qquad \textit{in } (0, \infty) , \\ Z^{(n)}(0+, \omega) = (D\mathscr{Y}(0))_n \qquad (0 \le n \le N-1) . \end{cases} \tag{7.5}$$

Proof. Since $F^{(n)} \in \mathscr{A}((0, \infty)) \cap L^2((0, \infty))$ $(n = 0, 1, 2, \cdots)$ and $P\left(\frac{1}{i} \frac{d}{dt}\right) F = 0$ in $(0, \infty)$, it follows from Theorem 2.1 (i) that $E^{(n)} \in \mathscr{A}((0, \infty)) \cap L^2((0, \infty))$ and $P\left(\frac{1}{i} \frac{d}{dt}\right) E^{(n)} = 0$ in $(0, \infty)$ $(n = 0, 1, 2, \cdots)$. Therefore, by Theorem 6.3 (i), we have (7.5). It is clear that $Z(\cdot, \omega)$ is a unique solution of (7.5), because P is a polynomial of degree N.

(Q.E.D.)

Remark 7.1. By Theorem 6.3 (ii) we note that $\{(D\mathscr{Y}(0))_n ; 0 \le n \le N-1\}$ is linearly independent.

7.3. **Nonlinear prediction.**

As the last application, we shall give an expression of nonlinear predictors of $X(t)$ using the past $\boldsymbol{F}_X^-(0)$ in terms of the transition probability density $P(t, x, y)$ of the Gaussian diffusion process $(\mathscr{X}(t), P(\cdot \,|\, \mathscr{X}(0) = x) ; t > 0, x \in \boldsymbol{R}^N)$. Immediately from (3.5), Theorem 3.3 and (4,2) we have

THEOREM 7.4. *For any bounded measurable function f (or any polynomial) on $\boldsymbol{R}$ and any $t > 0$,*

$$E(f(X(t))\,|\,F_X^-(0)) = \int_{R^N} f(-2\pi c_N^{-1} y_{N-1}) P(t, \mathscr{X}(0), y) dy_0 \cdots dy_{N-1} .$$

REFERENCES

[1] J. L. Doob: The elementary Gaussian processes, Ann. Math. Statist. **15** (1944), 229–282.

[2] H. Dym: Stationary measures for the flow of a linear differential equation driven by white noise, Trans. Amer. Math. Soc. **123** (1966), 130–164.

[3] T. Hida: Canonical representations of Gaussian processes and their applications, Mem. Coll. Sci. Univ. Kyoto, Ser. A, Math. **34** (1960), 109–155.

[4] L. Hörmander: Hypoelliptic second order differential equations, Acta Math. **119** (1967), 141–171.

[5] N. Levinson and H. P. McKean, Jr.: Weighted trigonometrical approximation on $\boldsymbol{R}^1$ with application to the germ field of a stationary Gaussian noise, Acta Math. **112** (1964), 99–143.

[6] Y. Okabe: Stationary Gaussian processes with Markovian property and M. Sato's hyperfunctions, to appear in Japanese J. of Math., **41** (1973), 69–122.

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